International Journal of Mechanical, Industrial and Aerospace Sciences
ISSN: 2517-9950
Vol:5, No:4, 2011

Inventory Control for a Joint Replenishment Problem with Stochastic Demand

Bassem Roushdy, Nahed Sobhy, Abdelrhim Abdelhamid, Ahmed Mahmoud

Abstract**—**Most papers model Joint Replenishment Problem (JRP) as a (kT, S) where kT is a multiple value for a common review period T,and S is a predefined order up to level. In general the (T, S) policy is characterized by a long out of control period which requires a large amount of safety stock compared to the (R, Q) policy. In this paper a probabilistic model is built where an item, call it item(i), with the shortest order time between interval (T) is modeled under (R, Q) policy and its inventory is continuously reviewed, while the rest of items (j) are periodically reviewed at a definite time corresponding to item (i). An order up to level policy for items (j) is applied in synchronization with item(i). For the sake of inventory out of control period reduction for items (j), an inventory review is done on the inventory position for items (j) one period before replenishment, at $(k_j - 1)$ period. A lower control value (s_j) is determined using an iterative method, if the inventory position is above this value then an order is done at the $k_j$period, otherwise it is made at $(k_j - 1)$ period. Another iterative method is used to find the optimum order up to level S_j for this policy.

Keywords**—**Inventory management,Joint replenishment, policy evaluation, stochastic process

I. INTRODUCTION

MANY companies order several items simultaneously, rather than individually, this is known as joint replenishment, the principle concept behind these calculations is that the marginal cost of adding one line item to an exciting order is much less than the marginal cost of ordering the items individually. For example a set of items that need to be shipped from a same vendor, each item has a setup cost resulting from processing the required order, then all items have a common mode to be shipped jointly (containers, tanks, ..,etc).

For a single node in a supply chain, joint replenishment requires decisions concerning, the aggregate value, order quantity of each item, the order interval for individual items within a group, and the timing of order releases.

The JRP with deterministic demand was first introduced and solved using an iterative heuristic suggested by Goyal [8] where he searched for the near optimum common time between order intervals (T) between an upper and lower bounds. He determined an integer multiple k_i for each item indicating the time of its order. Other major contributions in this area is presented by Silver [15], Van Eijs [16], Hariga [9], Viswanathan [17,18], Wildmen [19], Fung and Ma [5].

For the JRP under stochastic stationary demand, certain policies are proposed for the coordination between different items, Atkins and Iyogun [1] developed the (k_iT, S_i) policy, where at each review period T an item is raised to a predefined level S_i and the rest of items are raised according to their k_iT periods. In Pantumsinchai [13] proposed the(Q, S_i)policy, where all items are replenished to a predefined level S_i when the aggregate inventory level (stock on-hand plus on-order minus backorder) for all items reaches a certain level Q. An extension for Pantumsinchai work was carried by Nilsen and Larsen [11], they analytically evaluated the optimal (Q, s_i, S_i) policy, it works as when the aggregate inventory level reaches a certain level, items with inventory level less than s_i are replenished in the upcoming order to a predefined level s_i, the total cost is calculated using a recursive procedure based on Zheng and Federgruen method [20], the optimization is carried in two loops, an outer loop where Q is varied and inner loop where each item cost is computed based on (s, S) policy. Larsen [11] extended his work with Nielsen and developed an algorithm to compute an optimal (Q, s_i, S_i) for JRP when demand follows a compound Poisson process. Can- order policy (s_i, c_i, S_i) is suggested and modeled by Johansen and Melchiors [10] based on Markov decision policy, when any item drops to the must level s_i all items with the can levelc_i are replenished to the level S_i jointly with this item. Other contributions are done by Eynan and Kropp [4] using an iterative procedure they showed that the optimum time between order intervals (T) for a normally distributed demand is smaller than the optimal cycle of the deterministic model and hence found the near optimum T and the integer value k_i for each item. In order to get more accurate results, Eynan and Kropp (2007) approximated a part in the stochastic total inventory cost equation as a Taylor's expansion and hence resulted in a simple cost function structure which is similar to that of the deterministic models, they applied this technique for multiple items with joint orders.

In this paper, the (k_iT, S_i) policy is used as an initial solution, this policy is a periodic review policy which characterizes by a long out of control period $(k_iT + \mathrm{LT}$ Lead Time), the inventory is reviewed at the time k_iT and raised up to the level S_i. In the proposed solution, a continuous review inventory monitoring is carried on a single item (call it item i) with $k_i = 1$, and(R_i, Q_i)policy is applied on it, the rest of items (items j) are ordered in synchronization with this item. The main idea here is to review the inventories for each item $j$one period earlier than its order at $(k_j - 1)$, if the inventory if less than a specified value s_i then an early order is incurred,

B.R. Author, is with the Arab Academy for Science and Technology Cairo , Egypt.email:eng.bassem.roushdy@gmail.com
N.S.. Author, is with the department of Industrial Engineering in Ain Shams University.
A.A is with the Arab Academy for Science and Technology Cairo, Egypt.

else the order will be shipped automatically at the pre defined time determined by (k_j). This procedure helps to reduce the out of control period and hence we can reduce the amount of safety stock required to cover this period.

In Section 2 a detailed problem formulation with its notation are shown, in Section 3 the suggested probabilistic evaluation technique is described and the solution procedure is presented, the numerical results in Section 4 for the suggested policy are compared to the other policies, and the cost reduction form this policy is shown with sensitivity analysis. Finally a conclusion and future work recommendations are suggested in Section 5.

II. Problem Formulation

2.1 Assumptions and notations

i. In the developing model we assume two cases; we have n items, in the first case each item has a pure Poisson demand with parameters $(\lambda_i \& \lambda_j)$, while in the second case the demand for items j is compound Poisson, customer arrives according to a Poisson process with rate λ_j, and the order size of each arrival is geometrically distributed with parameter β. Appendix A for details. The Lead Times (LT) for each item is equal.$[i = 1,\ j = (2,3, \dots n)]$

ii. There is a major order cost (A) incurred whenever an orders is depleted, this cost represents the transportation cost per order and cost of dispatching the order.

iii. In addition to the major order cost, there is a minor setup cost $(a_i \& a_j)$, for each item included in order. $[i = 1,\ j = (2,3, \dots n)]$

iv. Holding cost per item per unit time $(h_i \& h_j)$, backorders are permitted and shortage cost per item $(P_{u_i} \& P_{u_j})$. $[i = 1,\ j = (2,3, \dots n)]$

v. Continuous review is carried on an item with shortest order interval (T) while the rest of items a periodic review is carried.

vi. The item with continuous review (item *i*) has an initial inventory position equals its reorder point $R_i$plus quantity ordered Q_i. for simplicity $Q_i$equals to its demand rate times order interval.

vii. A periodic review is carried on the on the rest of items (items j) one period before replenishment after $(k_j - 1)$ orders are done by item (*i*) since last replenishment.

viii. The Total inventory Cost (TC) is calculated based on the Inventory Position(IP) probability for each item at the beginning of the cycle and at the time of replenishment.

An inventory cost formula similar to (Axsäter, 2004) is used to evaluate an approximate Poisson cost function. We shall now determine the inventory position probability at the time of order.

Let us define now:

$P\big(IP\big(t = (k_j - 1)\mathrm{T}\big) = u\big)$: It is the probability that the inventory position equals u after $k_j - 1$ order for item (*i*), which is equal to the initial inventory amount for item (j) minus demand in $k_j - 1$ cycles.

$P\big(IP\big(t = k_j\mathrm{T}\big) = w\big)$: It is the probability that the inventory position equals w after k_j order for item i, which is equal to the initial inventory amount minus demand in k_j cycles.

We assume that the initial inventory position for items j equal S_j and decreases with demand occurrence, so the probability of inventory position equals (c) at any time period (t) is$(IP(t) = c) = P\big(D(t) = S_j - c\big)$.

2.2 System dynamics for the proposed policy

The policy proposed aims to reduce the initial inventory position for items j, this is achieved by reviewing the inventory position for each item one period before ordering, by this procedure more control is acquired and hence less amount of safety stock is required.

In this policy as we call it(R_i, s_j, S_j), R_i represents the reorder level for item$i$, while $s_j$represents the order level for any item j when R_i is reached at its review period, andS_j is the target stock level for item$j$. Since the demand for item i is Poisson or compound Poisson therefore the review periods for items j are stochastic gamma distributed with parameters (Q, λ_i) and density function of

$$f(T) = \frac{\lambda_i e^{-\lambda_i T} (\lambda_i T)^{\,Q_i - 1}}{\Gamma Q_i} \qquad (1)$$

Proposition: The sum of two or more gamma distribution is gammas distributed, if X and Y are two random gamma distributed variables with respective parameters (s, λ) and (t, λ) then $X + Y$ is gamma random variable with parameters $(s + t, \lambda)$. The proof is shown in appendix B.

In order to relate the pure Poisson demand probability for items (j) to the stochastic gamma distributed function for the time between order for item i, the average of the probability of$x$ demand occurrence for item (j) with parameter λ_j in time (T) gamma distributed with parameters (Q,λ_i) is taken;

$$P(x) = \int_{-\infty}^{\infty} \frac{(\lambda_j T)^x}{x!} e^{-\lambda_j T} \frac{\lambda_i e^{-\lambda T} (\lambda_i T)^{\,Q_i - 1}}{\Gamma Q_i} dT \qquad (2)$$

$$= \frac{(\lambda_j)^x (\lambda_i)^{\,Q_i}}{x! \Gamma Q_i} \int_{-\infty}^{\infty} T^{x + Q_i - 1} e^{-(\lambda_j + \lambda_i) T} dT \qquad (3)$$

Since $\Gamma\alpha = \int_{-\infty}^{\infty} T^{\alpha - 1} e^{-T} dT$ is the gamma function, therefore when differentiating by substitution and the simplification $t = (\lambda_j + \lambda_i)T$ the probability of demand is

$$P(x) = \frac{(\lambda_j)^x (\lambda_i)^{\,Q_i} (\Gamma Q_i + x)}{x! \Gamma Q_i (\lambda_i + \lambda_j)^{x + Q_i}} = \binom{x + Q_i - 1}{x} \left(\frac{\lambda_i}{\lambda_i + \lambda_j}\right)^{Q_i} \left(\frac{\lambda_j}{\lambda_i + \lambda_j}\right)^{x} \qquad (4)$$

So this distribution changes to negative binomial distribution with parameter $P = \frac{\lambda_i}{\lambda_i + \lambda_j}$.

In the second case when the demand of any item (j) is compound Poisson then the probability of demand x in T time distributed exponentially is derived using the same procedure, hence the demand probability is

$$P(x) = \sum_{y=0}^{\infty} \binom{y+\ Q_i - 1}{y} \left(\frac{\lambda_i}{\lambda_i+\lambda_j}\right)^{Q_i} \left(\frac{\lambda_j}{\lambda_i+\lambda_j}\right)^{y} f_x^y \quad (5)$$

III. Policy Evaluation

Let now $\emptyset(R_i,\ Q_i)$ denote the average cost of item i, and $\emptyset(k_j, T)$ denote the individual average costs of an item j, under the policy proposed. The average cost consists of average minor order, average holding cost plus average shortage cost per unit time. The total averages costs are the sum of the average major cost and the average individual cost for items i& j.

$$TC = \frac{A}{T} + \emptyset(R_i,\ Q_i) + \sum_{j=2}^{n} \emptyset\ (k_j\ , T) \quad (6)$$

Using an approximate technique, the average cost $\emptyset(R_i,\ Q_i)$ is evaluated by

$$\emptyset(R_i,\ Q_i) = \frac{a_i}{Q_i/\lambda_i} + h_i\left(R_i + \frac{Q_i}{2} - \lambda_i LT\right) + p_{u_i}\frac{\lambda_i}{Q_i}\sum_{x=R_i}^{\infty}(\mathrm{x} - R_i)\mathrm{f}_{\mathrm{d_{LT}}}(\mathrm{x}) \quad (7)$$

In order to evaluate an approximate value for the function $\emptyset(k_j, T)$, we suppose that at the beginning of each cycle, the inventory positions for items (j) are raised to the level S_j. Suppose now that the inventory position at the first review period $(k_j - 1)T$ equals to (u_j), then if the value of (u_j) is less than s_j, an early order will be incurred and hence the average cost will be evaluated. To evaluate the total cost, the probability inventory position equals (u_j) is calculated using

$$P\big(IP\big(t = (k_j-1)T\big) = u_j\big) = P\big(D(t) =\ S_j - u_j\big) = \binom{S_j - u_j + (k_j-1)Q_i - 1}{S_j - u_j}\left(\frac{\lambda_i}{\lambda_i+\lambda_j}\right)^{(k_j-1)\ Q_j} \quad (8)$$

If the inventory position at $(k_j - 1)T$ is higher than the control level s_j, then the order will be procured at k_jT time units with no postponing. Let the inventory position at k_jT equals to w, because of the control level s_j which is set at $(k_j - 1)$ period, the inventory positions probability at k_j period come only from transition inventory position between s_j& S_j

$$P\big(IP(t = k_jT) = w_j\big) = \sum_{IP_j=\max(s_j+1,w_j)}^{S_j} P\big(IP\big(t = (k_j-1)T\big) = IP_j\big)\, P\big(D(t = T) = IP_j - w_j\big) \quad (9)$$

$$P\big(IP(t = k_jT) = w_j\big) = \sum_{IP_j=\min(s_j+1,u)}^{S_j} \binom{IP_j - w_j + Q_i - 1}{IP_j - w_j}\left(\frac{\lambda_i}{\lambda_i+\lambda_j}\right)^{Q_i}\left(\frac{\lambda_j}{\lambda_i+\lambda_j}\right)^{IP_j - w_j} * \binom{S_j -\ IP_j + (k_j-1)Q_i - 1}{S_j -\ IP_j}\left(\frac{\lambda_i}{\lambda_i+\lambda_j}\right)^{(k_j-1)Q_i}\left(\frac{\lambda_j}{\lambda_i+\lambda_j}\right)^{S_j -\ IP_j} \quad (10)$$

For compound Poisson distribution the probabilities $P\big(IP\big(t = (k_j-1)T\big) = u_j\big)$ and $P\big(IP(t = k_jT) = w_j\big)$ are

$$P\big(IP\big(t = (k_j-1)T\big) = u_j\big) = P\big(D(t) =\ S_j - u_j\big) = \sum_{y=0}^{\infty}\binom{y + (k_j-1)Q_i - 1}{y}\left(\frac{\lambda_i}{\lambda_i+\lambda_j}\right)^{(k_j-1)Q_i}\left(\frac{\lambda_j}{\lambda_i+\lambda_j}\right)^{y} f_{S_j-u_j}^{y} \quad (11)$$

And

$$P\big(IP(t = k_jT) = w_j\big) = \sum_{IP_j=\min(s_j+1,u)}^{S_j} P\big(D(t) =\ IP_j - w_j\big).P(\ IP_j) = \sum_{IP=\min(s_j+1,u)}^{S_j}\left(\sum_{y=0}^{\infty}\binom{y + Q_i - 1}{y}\left(\frac{\lambda_i}{\lambda_i+\lambda_j}\right)^{Q_i}\left(\frac{\lambda_j}{\lambda_i+\lambda_j}\right)^{y} f_{IP_j - w_j}^{y} * \sum_{y=0}^{\infty}\binom{y + (k_j-1)Q_i - 1}{y}\left(\frac{\lambda_i}{\lambda_i+\lambda_j}\right)^{(k_j-1)Q_i}\left(\frac{\lambda_j}{\lambda_i+\lambda_j}\right)^{y} f_{S_j -\ IP_j}^{y}\right) \quad (12)$$

It is now possible to calculate the total cost equation by averaging it over all possible inventory positions; the total cost for an item (j) is given by the next equation

$$\emptyset(k_j, T) = \left(\frac{a_j}{k_j} + h_j\left(\frac{S_j + w_j}{2} - \lambda_j LT\right) + \left(\frac{h_j}{2} + \frac{p_{uj}}{k_jT}\right) * \sum_{y=w_j}^{\infty}(y - w_j) * P(y, \lambda_{j,LT})\right) * P\big(IP(t) = w_j\big) + \left(\frac{a_j}{(k_j-1)} + h_j\left(\frac{S_j + u_j}{2} - \lambda_j LT\right) + \left(\frac{h_j}{2} + \frac{p_{uj}}{(k_j-1)T}\right) * \sum_{y=u_j}^{\infty}(y - u_j) * P(y, \lambda_{j,LT})\right) * P\big(IP(t) = u_j\big) \quad (13)$$

In the case shortage cost is unknown, control level is determined according to a fill rate percentage (β)

$$\sum_{y=u_j}^{\infty}(y - u_j)\,.\,P(y, \lambda_{j,LT}).\,P\big(IP(t) = u_j\big) + \sum_{y=w_j}^{\infty}(y - w_j)\,.\,P(y, \lambda_{j,LT}).\,P\big(IP(t) = w_j\big) > (1 - \beta) \quad (14)$$

Solution procedure

The first step in the solution procedure is to find an initial solution using the (k_iT, S_i) policy $[i = 1,2, \ldots, n]$, we begin by finding a near optimum T where a search procedure is carried between $[T_{min}, T_{max}]$, the T_{min} is the lowest stochastic optimum order period T results for optimizing each item separately using (T, S) , while T_{max} is the common order interval (T) resulting from solving a JRP with deterministic data, done using any Goyal's heuristic.

A simple technique is used to find the (k_iT, S_i) policy parameters, the interval between $[T_{min}$& $T_{max}]$ is divided into small equally spaced intervals, the optimum k_i& S_i for each item in each interval are determined, then total cost is calculated, the T with minimum cost is selected. The technique is as follows, for each interval T, the k_i for each item is set initially equals to one, S_i& TC are calculated using the next two equations, for next iteration k_i for each item increases by one separately until TC increases.

$$(S_i) = F^{-1}_{d_{LT+T}}\left(\frac{p_{u_i} - h_i k_i T}{p_{u_i}}\right) \quad (15)$$

$$TC = \frac{A}{T} + \sum_{i=1}^{n} \frac{a_i}{k_i T} + h_i\left(S_i - \left(\frac{k_i T}{2} + \mathrm{LT_i}\right) * E(d_i)\right) + \frac{p_{u_i}}{k_i T}\sum_{x=S_i}^{\infty}(\mathrm{x} - S_i)\mathrm{f}_{\mathrm{d}_{\mathrm{LT}+k_i T}}(\mathrm{x}) \quad (16)$$

The common order interval $(T\&k_i)$ for each item $[i = 1,2,\dots,n]$ are introduced into the second step, a continuous review policy is carried on an item with $k_i = 1$, Q_i is set equals to $T\lambda_i$. The reorder level for item *i* is set using the next equation

$$F_{d_{LT}}(R_i) = 1 - \frac{h_i Q_i}{p_{u_i}\lambda_i} \quad (17)$$

The third step concerns with finding the optimum control level s_j for each item j with $k_j > 1$, this done through an iterative process. The process starts by setting $s_j = S_j$ then $P\big(IP(t = k_j T) = w_j\big)$ and $P\big(IP(t = (k_j - 1)T) = u_j\big)$ are evaluated using (7) & (8), then using (12) $\emptyset(k_j, T)$ is calculated at that level. Due to convexity, s_j is reduced by one each and same process is carried again until cost increases.

In the fourth step an outer loop where S_j value is reduced by one each time, and an inner loop to find the optimum control level $s_j$for each value, this procedure stops total cost increases.

IV. Numerical Results

In this section, the experimental results applied under the (R_i, s_j, S_j) policy are compared to $(k_i T, S_i)$ and (R_i, Q_i) policies. Model is programmed using Visual Basic Application embedded in Microsoft excel, which can take up to 100 different items, and processed on an Intel Core 2 duo 1.73GHZ and 2GB RAM. Several numerical studies are conducted to investigate the cost reduction percentage; this is conducted by considering different data input combinations. The major order cost (A) ranges between [0-100] and a common lead time for all items ranges between [0.1-0.2]. The results revels that $(R_i, s_j, S_j)$is superior to others in all cases, except when major order cost equals zero, the (R_i, Q_i) is better than (R_i, s_j, S_j) policy. The basic settings with respect to the parameters are summarized in Table 1, showing seven distinct items which are considered in the study.

Table I Solution for 7-item problem (A=50, LT=0.1)

Data					Results			
Item	a	h	λ	p_u	$(k_i T, S_i)$	(R_i, s_j, S_j)	s_j	S_j
1	20	3	2500	25	797.6	739.5		
2	50	2.5	300	30	342.2	340.7	71	158
3	15	2	400	20	214.8	216.8	95	159
4	20	5	225	20	307.5	308.1	51	90
5	35	2	80	30	137	130.7	20	63
6	30	1	150	15	117.7	113.2	37	114
7	40	1.5	100	18	137.3	130.5	24	89
Total average cost					2054.1	1979.5		

A continuous review is carried on item 1; a cost reduction of 58.1 occurs for this item (item *i*), when its reorder point $R_i = 286$ and order quantity $Q_i = 299$. While for the rest of items (*j*), a cost reduction of 16.5 happens when applying a control level s_j and an up to level S_j shown in table 1.

The algorithm uses a search procedure on the control levelss_j. An effect occurs on the total cost by varying the s_j value, the total cost shape for item 5 is illustrated in Figure 1 for the initial and optimumS_j values.

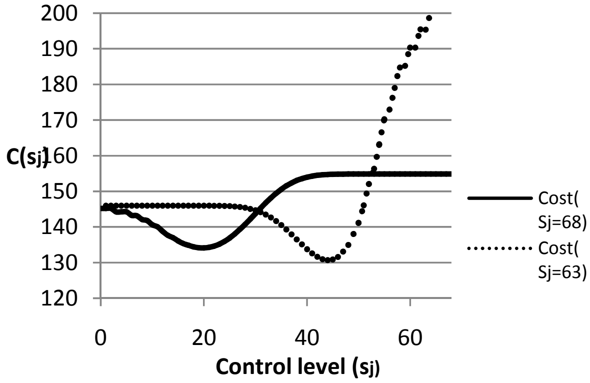

Fig. 1 The shape of the cost for item 5 at different control level.

Table II Total Average Cost Comparison

Problem	(R,Q)	$k_i T, S_i$	(R_i, s_j, S_j)
7-item problem(LT=0.1)			
A=0	1796.7	1964.5	1884.5
A=30	2517.8	2298.4	2226.5
A=60	3051.4	2537	2458.2
A=100	3632.2	2779.8	2699.4
7-item problem(LT=0.2)			
A=0	1908.7	2042.6	1976.8
A=30	2621.9	2369.9	2311.2
A=60	3151.6	2606.9	2543.1
A=100	3728.9	2846.2	2780.3

When considering the dataset in Table 2, we again see that the optimal policy (R_i, s_j, S_j) outperform the optimal$k_i T, S_i$ even by increasing the lead time, the (R,Q) policy outperform the rest of policies only when the major order cost equals zero. Furthermore, as lead time increases the cost percentage reduction decreases.

V. Conclusion And Future Work Recommendations

In this paper we have featured the proposed (R_i, s_j, S_j) and formulated an analytical solution procedure. The performance of the (R_i, s_j, S_j) is compared to the $k_i T, S_i$ and the (R, Q) policies at different values of lead time and major order cost. The cost reductions occurs by applying the (R_i, s_j, S_j) is due to reviewing the inventory for each item (j) once in its out of control period (T), so reducing the amount of safety stock needed.

An interesting extension for this work is to group the items to ABC classification, for items A, a contiguous review is carried on the sum of their demands, once a total of Q units are demanded then a new cycle begins and a joint replenishment is carried for all items, with each item has a lower control level. Moreover, a multiple product multiple location can be introduced to this policy.

APPENDIX A

Compound Poisson distribution:

This distribution characterizes with interaction of two distributions together, this means that customer arrives according to a Poisson process with rate $\boldsymbol{\lambda}$, and the order size of each arrival is also probabilistic variable with probability f_j and parameter β.

The number of customers in a time interval follows a Poisson distribution and so the probability of k customers $p\,(y) = \frac{\lambda t^y}{y!} e^{-\lambda t}$.

The mean demand $\mu = E\,(y) * E\,(J) = \lambda \sum_{j=1}^{\infty} j\, f_j$

The standard deviation σ= $E\,(y) * E\,(j^2) = \lambda \sum_{j=1}^{\infty} j^2 f_j$

In order to fit the geometric distribution on the model proposed the demand in a given time (t) should be observed, so it could be possible to determine $\lambda$and β and hence demand probability.

Let $\mu' = \lambda t\,(E(j)) = \frac{\lambda t}{(1-\beta)}$ and $\sigma' = \lambda t\,(E(J^2)) = \frac{\lambda t}{(1-\beta)^2}$

Therefore $\beta = 1 - \frac{2}{1+\sigma^2/\mu}$ and $\boldsymbol{\lambda} = (1-\beta)$

f_j^y is the probability that y customers give the total demand j. If D(t) be the demand generated under compound Poisson demand in the time interval (t). Then the distribution of D(t) is obtained recursively using complete convolution process $f_j^y = \sum_{i=y-1}^{j-1} f_i^{y-1} f_{j-i}$.

So, in the compound Poisson case, the probability for demand quantity j is determined using the next equation. $P(D\,(t) = j) = \sum_{y=0}^{\infty} \frac{\lambda t^y}{y!} e^{-\lambda t} f_j^y$.

APPENDIX B

The sum of two gamma functions are a gamma function

Proof: by using convolution theorem.

$$f_{X+Y}(a) = \frac{d}{da}\int_{-\infty}^{\infty} F_X(a-y) f_Y(y)dy$$

$$= \frac{1}{\Gamma(s)\Gamma(t)}\int_0^a \lambda e^{-\lambda(a-y)}\,[\lambda(a-y)]^{s-1}\lambda e^{-\lambda y}(\lambda y)^{t-1}$$

$$= Ke^{-\lambda a}\int_0^a (a-y)]^{s-1}\, y^{t-1}dy$$

$$= Ke^{-\lambda a} a^{s+t-1} \int_0^1 (1-x)^{s-1}\, x^{t-1}dx$$ by letting $x = \frac{y}{a}$

$$= Ce^{-\lambda a} a^{s+t-1}$$

Where C is a constant that doesn't depend on a. by integration the above function to 1 C is determined and

$$f_{X+Y}(a) = \frac{\lambda e^{-\lambda a}(\lambda a)^{s+t-1}}{\Gamma(s+t)}$$

Hence the proposition is proved.

REFERENCES

[1] Atkins, D., Iyogun, P., 1988. Periodic versus can order policies for coordinated multi-item inventory system. Management Science 34 (6), 791–796.
[2] Axsater, S., 2004.Inetory Control, Kluwer Academic Publisher Group.
[3] Eryan, A., Kropp, D.H., 2007. Effective and simple EOQ-like solutions for stochastic demand periodic review systems. Eur. J. Oper. Res. 180, 1135–1143.
[4] Eryan, A., Kropp, D.H., 1998. Periodic review and joint replenishment in stochastic demand environment.IIE Transactions 30, 1025-1033.
[5] Fung, R., MA, X., 2001.Anew method for joint replenishment problems. Journal of Operational Research Society52, 358-362.
[6] Forsberg, R., 1995. Optimization of order-up-to S policies for two-level inventory systems with compound Poisson demand. European Journal of Operational Research 81.
[7] Goyal, S.K., 1973. Determination of economic packaging frequency for items jointly replenished. Management Science 20 (2), 232.
[8] Goyal, S.K., 1974. Determination of optimum packaging frequency of items jointly replenished. Management Science 21 (4), 436–443.
[9] Hariga, M., 1994. Two new heuristic procedures for the joint replenishment problem, Journal of the Operational Research Society 45 463-471.
[10] Johansen, S.G., Melchior P., 2003. Can-order policy for the periodic-review joint replenishment problem, Journal of the Operational Research Society 54 283–290.
[11] Larsen,C., 2008.The Q(s,S) control policy for the joint replenishment problem extended to the case of correlation among item-demand. Journal of Production Economics,doi:10.1016/j.ijpe.2008.08.025
[12] Olsen, A.L., 2005. An evolutionary algorithm to solve the joint replenishment problem using direct grouping. Computers and Industrial Engineering 48 (2), 223–235.
[13] Pantumsinchai, P., 1992. A comparison of three joint ordering inventory policies. Decision Science 23, 111–127.
[14] Porras E. and Dekker R. 2005. An efficient optimal solution method for the joint replenishment problem with minimum order quantities. European Journal of Operational Research (forthcoming)
[15] Silver, E.A., 1976. A simple method of determining order quantities in joint replenishments under deterministic demand. Management Science 22 (12), 1351–1361.
[16] Van Eijs, M.J.G., 1993. A note on the joint replenishment problem under constant demand. Journal of Operational Research Society 44 (2), 185–191.
[17] Viswanathan, S., 1996. A new optimal algorithm for the joint replenishment problem. Journal of Operational Research Society 47 (7), 936–944.
[18] Viswanathan, S., 2002. On optimal algorithms for the joint replenishment problem. Journal of Operational Research Society 53 (11), 1286–1290.
[19] Wildeman, R.E, Frenk, J.B.G., Dekker, R., 1997. An efficient optimal solution for the joint replenishment problem. European Journal of Operational Research 99, 433–444.
[20] Zheng, Y.S., Federgruen, A., 1991. Finding optimal s̊; Sꝍpolicies is about as simple as evaluating a single policy. Operations Research 39 (4), 654–665.
[21] Zipkin, P., 2000. Foundations of Inventory Management. McGraw-Hill.