

Intuitionistic Fuzzy Positive Implicative Ideals with Thresholds (λ, μ) of BCI-Algebras

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Abstract—The aim of this paper is to introduce the notion of intuitionistic fuzzy positive implicative ideals with thresholds (λ, μ) of BCI-algebras and to investigate its properties and characterizations.

Keywords—BCI-algebra, intuitionistic fuzzy set, intuitionistic fuzzy ideal with thresholds (λ, μ) , intuitionistic fuzzy positive implicative ideal with thresholds (λ, μ) .

I. INTRODUCTION

A BCI-algebra is an important class of logical algebra and was introduced by Iséki [1], [2]. K. Atanassov [3] introduced the concept of intuitionistic fuzzy sets. In 2003, K. Hur [4] applied the concept to the theory of rings, and introduced the concepts of intuitionistic fuzzy subgroups and subrings. M. Jiang and X.L. Xin [5] later introduced the concepts of (λ, μ) intuitionistic fuzzy subrings (ideals); some meaningful results are obtained. In [6], [7], we have given the concepts of intuitionistic fuzzy subalgebras (ideals) with thresholds (λ, μ) and intuitionistic fuzzy implicative ideals with thresholds (λ, μ) of BCI-algebras, in this paper, we introduce the notion of intuitionistic fuzzy positive implicative ideals with thresholds (λ, μ) of BCI-algebras and give several properties and characterizations of it.

II. PRELIMINARIES

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a BCI-algebra if it satisfies the following axioms:

- (BCI-1) $((x * y) * (x * z)) * (z * y) = 0$,
- (BCI-2) $(x * (x * y)) * y = 0$,
- (BCI-3) $x * x = 0$,
- (BCI-4) $x * y = 0$ and $y * x = 0$ imply $x = y$,

for all $x, y, z \in X$. In a BCI-algebra X , we can define a partial ordering \leq by putting $x \leq y$ if and only if $x * y = 0$.

In any BCI-algebra X , the following hold:

1. $(x * y) * z = (x * z) * y$,
2. $x * 0 = x$,
3. $0 * (x * y) = (0 * x) * (0 * y)$,

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$$4. (x * z) * (y * z) \leq x * y,$$

$$5. x * (x * (x * y)) = x * y,$$

for all $x, y, z \in X$.

In this paper, X always means a BCI-algebra unless otherwise specified.

A nonempty subset K of X is called an ideal of X if $(I_1): 0 \in K, (I_2): x * y \in K$ and $y \in K$ imply $x \in K$. A nonempty subset K of X is called a positive implicative ideal of X if it satisfies (I_1) and $(I_3): ((x * z) * z) * (y * z) \in K$ and $y \in K$ imply $x * z \in K$.

Definition 1. [3] Let S be any set. An intuitionistic fuzzy subset A of S is an object of the following form

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle : x \in S \} \text{ where } \mu_A : S \rightarrow [0, 1]$$

and $\nu_A : S \rightarrow [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in S$ respectively and for every $x \in S, 0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Denote $\langle I \rangle = \{ \langle a, b \rangle : a, b \in [0, 1] \}$.

Definition 2. Let $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle : x \in S \}$ be an intuitionistic fuzzy set in a set S . For $\langle \alpha, \beta \rangle \in \langle I \rangle$, the set $A_{\langle \alpha, \beta \rangle} = \{ x \in S : \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta \}$ is called a cut set of A .

Definition 3. [6] Let $\lambda, \mu \in (0, 1]$ and $\lambda < \mu$.

An intuitionistic fuzzy set A in X is said to be an intuitionistic fuzzy ideal with thresholds (λ, μ) of X if the following are satisfied:

$$\begin{aligned} (IF_1) \mu_A(0) \vee \lambda &\geq \mu_A(x) \wedge \mu, \\ (IF_2) \nu_A(0) \wedge \mu &\leq \nu_A(x) \vee \lambda, \\ (IF_3) \mu_A(x) \vee \lambda &\geq \mu_A(x * y) \wedge \mu_A(y) \wedge \mu, \\ (IF_4) \nu_A(x) \wedge \mu &\leq \nu_A(x * y) \vee \nu_A(y) \vee \lambda, \end{aligned}$$

for all $x, y \in X$.

Proposition 1. [6] Let A be an intuitionistic fuzzy ideal with thresholds (λ, μ) of X . If $x \leq y$ holds in X , then

$$\mu_A(x) \vee \lambda \geq \mu_A(y) \wedge \mu, \nu_A(x) \wedge \mu \leq \nu_A(y) \vee \lambda.$$

Proposition 2. [6] Let A be an intuitionistic fuzzy ideal with thresholds (λ, μ) of X . If the inequality $x * y \leq z$ holds in X , then for all $x, y, z \in X$,

$$\begin{aligned} \mu_A(x) \vee \lambda &\geq \mu_A(y) \wedge \mu_A(z) \wedge \mu, \\ \nu_A(x) \wedge \mu &\leq \nu_A(y) \vee \nu_A(z) \vee \lambda. \end{aligned}$$

III. INTUITIONISTIC FUZZY POSITIVE IMPLICATIVE IDEALS WITH THRESHOLDS (λ, μ) OF BCI- ALGEBRAS

Definition 4. Let $\lambda, \mu \in (0,1]$ and $\lambda < \mu$. An intuitionistic fuzzy set A in X is called an intuitionistic fuzzy positive implicative ideal with thresholds (λ, μ) of X if it satisfies $(IF_1), (IF_2)$ and

$$\begin{aligned} (IF_5) \mu_A(x * z) \vee \lambda &\geq \mu_A(((x * z) * z) * (y * z)) \wedge \mu_A(y) \wedge \mu, \\ (IF_6) \nu_A(x * z) \wedge \mu &\leq \nu_A(((x * z) * z) * (y * z)) \vee \nu_A(y) \vee \lambda, \end{aligned}$$

for all $x, y, z \in X$.

Example 1. Let $X = \{0,1,2\}$ with Cayley table given by

TABLE I
RESULT OF CALCULATION

*	0	1	2
0	0	0	0
1	1	0	0
2	2	2	0

Define $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle : x \in S \}$ where $\mu_A : X \rightarrow [0,1]$ and $\nu_A : X \rightarrow [0,1]$ by $\mu_A(0) = 2/3, \mu_A(1) = \mu_A(2) = 1/3, \nu_A(0) = 1/4, \nu_A(1) = \nu_A(2) = 1/2$. Let $\lambda = 1/8$ and $\mu = 3/4$. By routine calculations give that A is an intuitionistic fuzzy positive implicative ideal with thresholds (λ, μ) of X .

The following proposition gives a relation between intuitionistic fuzzy positive implicative ideals with thresholds (λ, μ) and intuitionistic fuzzy ideals with thresholds (λ, μ) of X .

Proposition 3. Any intuitionistic fuzzy positive implicative ideal with thresholds (λ, μ) of X is an intuitionistic fuzzy ideal with thresholds (λ, μ) of X , but the converse does not hold.

Proof. Assume that A is an intuitionistic fuzzy positive implicative ideal with thresholds (λ, μ) of X and put $z = 0$ in (IF_5) and (IF_6) , we get

$$\begin{aligned} \mu_A(x) \vee \lambda &\geq \mu_A(x * y) \wedge \mu_A(y) \wedge \mu, \\ \nu_A(x) \wedge \mu &\leq \nu_A(x * y) \vee \nu_A(y) \vee \lambda. \end{aligned}$$

This means that A satisfies (IF_3) and (IF_4) . Combining (IF_1) and (IF_2) , A is an intuitionistic fuzzy ideal with thresholds (λ, μ) of X .

To show the last half part, we see the following example.

Example 2. Let $X = \{0,1,2\}$ with Cayley table given by

TABLE II
RESULT OF COMPUTATION

*	0	1	2
0	0	0	0
1	1	0	0
2	2	1	0

Define $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle : x \in S \}$ where $\mu_A : X \rightarrow [0,1]$ and $\nu_A : X \rightarrow [0,1]$ by $\mu_A(0) = 2/3, \mu_A(1) = \mu_A(2) = 1/3, \nu_A(0) = 1/4, \nu_A(1) = \nu_A(2) = 1/2$. Let $\lambda = 1/8$ and $\mu = 3/4$. It is easy to verify that A is an intuitionistic fuzzy ideal with thresholds (λ, μ) of X . But it is not an intuitionistic fuzzy positive implicative ideal with thresholds (λ, μ) of X since:

$$\mu_A(2 * 1) \vee \lambda < \mu_A(((2 * 1) * 1) * (0 * 1)) \wedge \mu_A(0) \wedge \mu.$$

Next, we give characterizations of intuitionistic fuzzy positive implicative ideals with thresholds (λ, μ) of X .

Proposition 4. Let A be an intuitionistic fuzzy ideal with thresholds (λ, μ) of X . Then the following are equivalent:

- (i). A is an intuitionistic fuzzy positive implicative ideal with thresholds (λ, μ) of X ,
- (ii). $\mu_A((x * y) * z) \vee \lambda \geq \mu_A(((x * z) * z) * (y * z)) \wedge \mu, \nu_A((x * y) * z) \wedge \mu \leq \nu_A(((x * z) * z) * (y * z)) \vee \lambda,$ for all $x, y, z \in X$,
- (iii). $\mu_A(x * y) \vee \lambda \geq \mu_A(((x * y) * y) * (0 * y)) \wedge \mu, \nu_A(x * y) \wedge \mu \leq \nu_A(((x * y) * y) * (0 * y)) \vee \lambda,$ for all $x, y \in X$.

Proof. (i) \Rightarrow (ii) Suppose that A is an intuitionistic fuzzy positive implicative ideal with thresholds (λ, μ) of X . Since

$$\begin{aligned} (((x * y) * z) * z) * (0 * z) &= (((x * y) * z) * z) * ((y * y) * z) \\ &= (((x * z) * z) * y) * ((y * z) * y) \leq ((x * z) * z) * (y * z), \end{aligned}$$

by $(IF_5), (IF_6), (IF_1), (IF_2)$ and Proposition 1, we have

$$\begin{aligned} \mu_A((x * y) * z) \vee \lambda &= (\mu_A((x * y) * z) \vee \lambda) \vee \lambda \\ &\geq (\mu_A(((x * y) * z) * z) * (0 * z)) \wedge \mu_A(0) \wedge \mu \vee \lambda \\ &= (\mu_A(((x * y) * z) * z) * (0 * z)) \vee \lambda \wedge (\mu_A(0) \vee \lambda) \wedge (\mu \vee \lambda) \\ &\geq (\mu_A(((x * z) * z) * (y * z)) \wedge \mu) \\ &\wedge (\mu_A(((x * z) * z) * (y * z)) \wedge \mu) \wedge \mu = \mu_A(((x * z) * z) * (y * z)) \wedge \mu, \\ \nu_A((x * y) * z) \wedge \mu &= (\nu_A((x * y) * z) \wedge \mu) \wedge \mu \end{aligned}$$

$$\begin{aligned} &\leq (v_A(((x*y)*z)*z)*(0*y)) \vee v_A(0) \vee \lambda \wedge \mu \\ &= (v_A(((x*y)*z)*z)*(0*y) \wedge \mu) \vee (v_A(0) \wedge \mu) \vee (\lambda \wedge \mu) \\ &\leq (v_A(((x*z)*z)*(y*z)) \vee \lambda) \\ &\vee (v_A(((x*z)*z)*(y*z)) \vee \lambda) \vee \lambda = v_A(((x*z)*z)*(y*z)) \vee \lambda. \end{aligned}$$

Hence

$$\begin{aligned} \mu_A((x*y)*z) \vee \lambda &\geq \mu_A(((x*z)*z)*(y*z)) \wedge \mu, \\ v_A((x*y)*z) \wedge \mu &\leq v_A(((x*z)*z)*(y*z)) \vee \lambda \end{aligned}$$

and (ii) holds.

- (ii) \Rightarrow (iii) Substituting 0 for y and y for z in (ii), respectively, we have (iii).
- (iii) \Rightarrow (i) Since

$$(((x*y)*y)*(0*y)) * (((x*y)*y)*(z*y)) \leq (z*y) * (0*y) \leq z,$$

by Proposition 2, we obtain

$$\begin{aligned} \mu_A(((x*y)*y)*(0*y)) \vee \lambda &\geq \mu_A(((x*y)*y)*(z*y)) \wedge \mu_A(z) \wedge \mu, \\ v_A(((x*y)*y)*(0*y)) \wedge \mu &\leq v_A(((x*y)*y)*(z*y)) \vee v_A(z) \vee \lambda. \end{aligned}$$

From (iii), we have

$$\begin{aligned} \mu_A(x*y) \vee \lambda &= (\mu_A(x*y) \vee \lambda) \vee \lambda \\ &\geq (\mu_A(((x*y)*y)*(0*y)) \wedge \mu) \vee \lambda \\ &= (\mu_A(((x*y)*y)*(0*y)) \vee \lambda) \wedge (\mu \vee \lambda) \\ &\geq \mu_A(((x*y)*y)*(z*y)) \wedge \mu_A(z) \wedge \mu, \\ &\quad v_A(x*y) \wedge \mu \\ (v_A(x*y) \wedge \mu) \wedge \mu &\leq (v_A(((x*y)*y)*(0*y)) \vee \lambda) \wedge \mu \\ &= (v_A(((x*y)*y)*(0*y)) \wedge \mu) \vee (\lambda \wedge \mu) \\ &\leq v_A(((x*y)*y)*(z*y)) \vee v_A(z) \vee \lambda. \end{aligned}$$

Hence, A is an intuitionistic fuzzy positive implicative ideal with thresholds (λ, μ) of X.

Proposition 5. An intuitionistic fuzzy set A of X is an intuitionistic fuzzy positive implicative ideal with thresholds (λ, μ) of X if and only if, for all $\alpha, \beta \in (\lambda, \mu]$, $A_{\langle \alpha, \beta \rangle}$ is either empty or a positive implicative ideal of X.

Proof. Let A be an intuitionistic fuzzy positive implicative ideal with thresholds (λ, μ) of X and $A_{\langle \alpha, \beta \rangle} \neq \emptyset$ for some $\alpha, \beta \in (\lambda, \mu]$. It is clear that $0 \in A_{\langle \alpha, \beta \rangle}$. Let $((x*z)*z)*(y*z) \in A_{\langle \alpha, \beta \rangle}$ and $y \in A_{\langle \alpha, \beta \rangle}$, then

$$\begin{aligned} \mu_A(((x*z)*z)*(y*z)) &\geq \alpha, \mu_A(y) \geq \alpha, \\ v_A(((x*z)*z)*(y*z)) &\leq \beta, v_A(y) \leq \beta. \end{aligned}$$

It follows from (IF_5) and (IF_6) ,

$$\begin{aligned} \mu_A(x*z) \vee \lambda &\geq \mu_A(((x*z)*z)*(y*z)) \wedge \mu_A(y) \wedge \mu \geq \alpha, \\ v_A(x*z) \wedge \mu &\leq v_A(((x*z)*z)*(y*z)) \vee v_A(y) \vee \lambda \leq \beta. \end{aligned}$$

Namely, $\mu_A(x*z) \geq \alpha$, $v_A(x*z) \leq \beta$ and $x*z \in A_{\langle \alpha, \beta \rangle}$. This shows that $A_{\langle \alpha, \beta \rangle}$ is a positive implicative ideal of X. Conversely, suppose that for each $\alpha, \beta \in (\lambda, \mu]$, $A_{\langle \alpha, \beta \rangle}$ is either empty or a positive implicative ideal of X. For any $x \in X$, let $\alpha = \mu_A(x) \wedge \mu, \beta = v_A(x) \vee \lambda$. Then $\mu_A(x) \geq \alpha, v_A(x) \leq \beta$, hence $x \in A_{\langle \alpha, \beta \rangle}$ and $A_{\langle \alpha, \beta \rangle}$ is a positive implicative ideal of X, therefore $0 \in A_{\langle \alpha, \beta \rangle}$, i.e., $\mu_A(0) \geq \alpha$ and $v_A(0) \leq \beta$. We get

$$\begin{aligned} \mu_A(0) \vee \lambda &\geq \mu_A(0) \geq \alpha = \mu_A(x) \wedge \mu, \\ v_A(0) \wedge \mu &\leq v_A(0) \leq \beta = v_A(x) \vee \lambda, \end{aligned}$$

i.e., $\mu_A(0) \vee \lambda \geq \mu_A(x) \wedge \mu$ and $v_A(0) \wedge \mu \leq v_A(x) \vee \lambda$, for all $x \in X$. Now we only need to show that A satisfies (IF_5) and (IF_6) . Let

$$\begin{aligned} \alpha &= \mu_A(((x*z)*z)*(y*z)) \wedge \mu_A(y) \wedge \mu, \\ \beta &= v_A(((x*z)*z)*(y*z)) \vee v_A(y) \vee \lambda. \end{aligned}$$

Then

$$\begin{aligned} \mu_A(((x*z)*z)*(y*z)) &\geq \alpha, \mu_A(y) \geq \alpha, \\ v_A(((x*z)*z)*(y*z)) &\leq \beta, v_A(y) \leq \beta. \end{aligned}$$

Hence $((x*z)*z)*(y*z) \in A_{\langle \alpha, \beta \rangle}$ and $y \in A_{\langle \alpha, \beta \rangle}$. Since $A_{\langle \alpha, \beta \rangle}$ is a positive implicative ideal of X, thus $x*z \in A_{\langle \alpha, \beta \rangle}$, i.e., $\mu_A(x*z) \geq \alpha, v_A(x*z) \leq \beta$. We get

$$\begin{aligned} \mu_A(x*z) \vee \lambda &\geq \mu_A(x*z) \geq \alpha = \mu_A(((x*z)*z)*(y*z)) \wedge \mu_A(y) \wedge \mu, \\ v_A(x*z) \wedge \mu &\leq v_A(x*z) \leq \beta = v_A(((x*z)*z)*(y*z)) \vee v_A(y) \vee \lambda. \end{aligned}$$

Namely,

$$\begin{aligned} \mu_A(x*z) \vee \lambda &\geq \mu_A(((x*z)*z)*(y*z)) \wedge \mu_A(y) \wedge \mu, \\ v_A(x*z) \wedge \mu &\leq v_A(((x*z)*z)*(y*z)) \vee v_A(y) \vee \lambda. \end{aligned}$$

Summarizing the above arguments, A is an intuitionistic fuzzy positive implicative ideal with thresholds (λ, μ) of X.

Proposition 6 Let J be a positive implicative ideal of X . Then there exists an intuitionistic fuzzy positive implicative ideal A with thresholds (λ, μ) of X such that $A_{(\alpha, \beta)} = J$ for some $\alpha, \beta \in (\lambda, \mu]$.

Proof. Define $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle : x \in S \}$ by

$$\mu_A(x) = \begin{cases} \alpha & \text{if } x \in J, \\ \lambda & \text{if } x \notin J, \end{cases}$$

$$\nu_A(x) = \begin{cases} \beta & \text{if } x \in J, \\ \mu & \text{if } x \notin J, \end{cases}$$

where α, β are two fixed numbers in $(\lambda, \mu]$. Since J is a positive implicative ideal of X , if $((x*z)*z)*(y*z) \in J$ and $y \in J$ then $x*z \in J$. Hence

$$\mu_A(((x*z)*z)*(y*z)) = \mu_A(y) = \mu_A(x*z) = \alpha,$$

$$\nu_A(((x*z)*z)*(y*z)) = \nu_A(y) = \nu_A(x*z) = \beta,$$

Thus,

$$\mu_A(x*z) \vee \lambda \geq \mu_A(((x*z)*z)*(y*z)) \wedge \mu_A(y) \wedge \mu,$$

$$\nu_A(x*z) \wedge \mu \leq \nu_A(((x*z)*z)*(y*z)) \vee \nu_A(y) \vee \lambda.$$

If at least one of $((x*z)*z)*(y*z)$ and y is not in J , then at least one of $\mu_A(((x*z)*z)*(y*z))$ and $\mu_A(y)$ is λ , and at least one of $\nu_A(((x*z)*z)*(y*z))$ and $\nu_A(y)$ is μ . Therefore,

$$\mu_A(x*z) \vee \lambda \geq \mu_A(((x*z)*z)*(y*z)) \wedge \mu_A(y) \wedge \mu,$$

$$\nu_A(x*z) \wedge \mu \leq \nu_A(((x*z)*z)*(y*z)) \vee \nu_A(y) \vee \lambda.$$

This means that A satisfies (IF_5) and (IF_6) . Since $0 \in J$, $\mu_A(0) \vee \lambda = \alpha \geq \mu_A(x) \wedge \mu$, $\nu_A(0) \wedge \mu = \beta \leq \nu_A(x) \vee \lambda$, for all $x \in X$ and so A satisfies (IF_1) and (IF_2) . Thus, A is an intuitionistic fuzzy positive implicative ideal with thresholds (λ, μ) of X . It is clear that $A_{(\alpha, \beta)} = J$.

Definition 5. Let S be any set. If

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle : x \in S \}, B = \{ \langle x, \mu_B(x), \nu_B(x) \rangle : x \in S \}$$

be any two intuitionistic fuzzy subsets of S , then

$$A \cap B = \{ \langle x, (\mu_A \cap \mu_B)(x), (\nu_A \cup \nu_B)(x) \rangle : x \in S \}$$

$$= \{ \langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x) \rangle : x \in S \}$$

Proposition 7 Let A and B be two intuitionistic fuzzy positive implicative ideals with thresholds (λ, μ) of X . Then

$A \cap B$ is also an intuitionistic fuzzy positive implicative ideal with thresholds (λ, μ) of X .

Proof. For all $x, y, z \in X$, by Definition 4, we have

$$\mu_{A \cap B}(0) \vee \lambda = (\mu_A(0) \wedge \mu_B(0)) \vee \lambda = (\mu_A(0) \vee \lambda) \wedge (\mu_B(0) \vee \lambda)$$

$$\geq (\mu_A(x) \wedge \mu) \wedge (\mu_B(x) \wedge \mu) = (\mu_A(x) \wedge \mu_B(x)) \wedge \mu = \mu_{A \cap B}(x) \wedge \mu,$$

$$\nu_{A \cap B}(0) \wedge \mu = (\nu_A(0) \vee \nu_B(0)) \wedge \mu = (\nu_A(0) \wedge \mu) \vee (\nu_B(0) \wedge \mu)$$

$$\leq (\nu_A(x) \vee \lambda) \vee (\nu_B(x) \vee \lambda) = (\nu_A(x) \vee \nu_B(x)) \vee \lambda = \nu_{A \cap B}(x) \vee \lambda,$$

$$\mu_{A \cap B}(x*z) \vee \lambda = (\mu_A(x*z) \wedge \mu_B(x*z)) \vee \lambda$$

$$= (\mu_A(x*z) \vee \lambda) \wedge (\mu_B(x*z) \vee \lambda)$$

$$\geq (\mu_A(((x*z)*z)*(y*z)) \wedge \mu_A(y) \wedge \mu)$$

$$\wedge (\mu_B(((x*z)*z)*(y*z)) \wedge \mu_B(y) \wedge \mu)$$

$$= (\mu_A(((x*z)*z)*(y*z)) \wedge \mu_B(((x*z)*z)*(y*z)))$$

$$\wedge (\mu_A(y) \wedge \mu_B(y)) \wedge \mu = \mu_{A \cap B}(((x*z)*z)*(y*z)) \wedge \mu_{A \cap B}(y) \wedge \mu.$$

$$\nu_{A \cap B}(x*z) \wedge \mu = (\nu_A(x*z) \vee \nu_B(x*z)) \wedge \mu$$

$$= (\nu_A(x*z) \wedge \mu) \vee (\nu_B(x*z) \wedge \mu)$$

$$\leq (\nu_A(((x*z)*z)*(y*z)) \vee \nu_A(y) \vee \lambda)$$

$$\vee (\nu_B(((x*z)*z)*(y*z)) \vee \nu_B(y) \vee \lambda)$$

$$= (\nu_A(((x*z)*z)*(y*z)) \vee \nu_B(((x*z)*z)*(y*z)))$$

$$\vee (\nu_A(y) \vee \nu_B(y)) \vee \lambda = \nu_{A \cap B}(((x*z)*z)*(y*z)) \vee \nu_{A \cap B}(y) \vee \lambda.$$

Hence $A \cap B$ is an intuitionistic fuzzy positive implicative ideal with thresholds (λ, μ) of X .

Definition 6. Let A and B be two intuitionistic fuzzy sets of a set X . The Cartesian product of A and B is defined by

$$A \times B = \{ \langle \mu_{A \times B}(x, y), \nu_{A \times B}(x, y) \rangle : x, y \in X \}$$

where

$$\mu_{A \times B}(x, y) = \mu_A(x) \wedge \mu_B(y), \nu_{A \times B}(x, y) = \nu_A(x) \vee \nu_B(y).$$

Proposition 8. Let A and B be two intuitionistic fuzzy positive implicative ideals with thresholds (λ, μ) of X . Then $A \times B$ is also an intuitionistic fuzzy positive implicative ideal with thresholds (λ, μ) of $X \times X$.

Proof. For all $(x, y) \in X \times X$, by Definition 4, we get

$$\mu_{A \times B}(0, 0) \vee \lambda = (\mu_A(0) \wedge \mu_B(0)) \vee \lambda = (\mu_A(0) \vee \lambda) \wedge (\mu_B(0) \vee \lambda)$$

$$\geq (\mu_A(x) \wedge \mu) \wedge (\mu_B(y) \wedge \mu) = \mu_{A \times B}(x, y) \wedge \mu,$$

$$\nu_{A \times B}(0, 0) \wedge \mu = (\nu_A(0) \vee \nu_B(0)) \wedge \mu = (\nu_A(0) \wedge \mu) \vee (\nu_B(0) \wedge \mu)$$

$$\leq (\nu_A(x) \vee \lambda) \vee (\nu_B(y) \vee \lambda) = \nu_{A \times B}(x, y) \vee \lambda,$$

For all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$, we have

$$\begin{aligned}
 & \mu_{A \times B}(x_1 * z_1, x_2 * z_2) \vee \lambda = (\mu_A(x_1 * z_1) \wedge \mu_B(x_2 * z_2)) \vee \lambda \\
 & = (\mu_A(x_1 * z_1) \vee \lambda) \wedge (\mu_B(x_2 * z_2) \vee \lambda) \\
 & \geq (\mu_A(((x_1 * z_1) * z_1) * (y_1 * z_1)) \wedge \mu_A(y_1) \wedge \mu) \\
 & \quad \wedge (\mu_B(((x_2 * z_2) * z_2) * (y_2 * z_2)) \wedge \mu_A(y_2) \wedge \mu) \\
 & = \mu_A(((x_1 * z_1) * z_1) * (y_1 * z_1)) \wedge \mu_B(((x_2 * z_2) * z_2) * (y_2 * z_2)) \\
 & \quad \wedge \mu_A(y_1) \wedge \mu_B(y_2) \wedge \mu \\
 & = \mu_{A \times B}(((x_1 * z_1) * z_1) * (y_1 * z_1), ((x_2 * z_2) * z_2) * (y_2 * z_2)) \\
 & \quad \wedge \mu_{A \times B}(y_1, y_2) \wedge \mu, \\
 & \nu_{A \times B}(x_1 * z_1, x_2 * z_2) \wedge \mu = (\nu_A(x_1 * z_1) \vee \nu_B(x_2 * z_2)) \wedge \mu \\
 & = (\nu_A(x_1 * z_1) \wedge \mu) \vee (\nu_B(x_2 * z_2) \wedge \mu) \\
 & \leq (\nu_A(((x_1 * z_1) * z_1) * (y_1 * z_1)) \vee \nu_A(y_1) \vee \lambda) \\
 & \quad \vee (\nu_B(((x_2 * z_2) * z_2) * (y_2 * z_2)) \vee \nu_A(y_2) \vee \lambda) \\
 & = \nu_A(((x_1 * z_1) * z_1) * (y_1 * z_1)) \vee \nu_B(((x_2 * z_2) * z_2) * (y_2 * z_2)) \\
 & \quad \vee \nu_A(y_1) \vee \nu_B(y_2) \vee \lambda \\
 & = \nu_{A \times B}(((x_1 * z_1) * z_1) * (y_1 * z_1), ((x_2 * z_2) * z_2) * (y_2 * z_2)) \\
 & \quad \vee \nu_{A \times B}(y_1, y_2) \vee \lambda,
 \end{aligned}$$

Hence $A \times B$ is an intuitionistic fuzzy positive implicative ideal with thresholds (λ, μ) of $X \times X$.

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