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Intuitionistic Fuzzy Dual Positive Implicative Hyper *K*- Ideals

M.M. Zahedi and L. Torkzadeh

Abstract- In this note first we define the notions of intuitionistic fuzzy dual positive implicative hyper K-ideals of types 1,2,3,4 and intuitionistic fuzzy dual hyper K-ideals. Then we give some classifications about these notions according to the level subsets. Also by given some examples we show that these notions are not equivalent, however we prove some theorems which show that there are some relationships between these notions. Finally we define the notions of product and anti-product of two fuzzy subsets and then give some theorems about the relationships between the intuitionistic fuzzy dual positive implicative hyper K-ideal of types 1,2,3,4 and their (anti-)products, in particular we give a main decomposition theorem.

Keywords- hyper K-algebra, intuitionistic fuzzy dual positive implicative hyper K-ideal.

I. Introduction

The hyperalgebraic structure theory was introduced by F. Marty [6] in 1934. Imai and Iseki [4] in 1966 introduced the notion of a BCK-algebra. Borzooei, Jun and Zahedi et.al. [3] applied the hyperstructure to BCK-algebras and introduced the concept of hyper K-algebra which is a generalization of BCK-algebra. The idea of "intuitionistic fuzzy set" was first published by Atanassov [1], as a generalization of the notion of fuzzy set. Now in this note by intuitionistic fuzzifications of the notions of dual positive implicative hyper K-ideals of types 1,2,3 and 4 we obtain some results.

II. Preliminaries

2.1 Definition [3] Let H be a nonempty set and " $\circ$ " be a *hyperoperation* on H, that is " $\circ$ " is a function from $H \times H$ to $\mathcal{P}^*(H) = \mathcal{P}(H)\backslash\{\emptyset\}$. Then H is called a hyper K-algebra if it contains a constant "0" and satisfies the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) < x \circ y$

(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$

(HK3) $x < x$

(HK4) $x < y,\ y < x \Rightarrow x = y$

(HK5) $0 < x$,

for all $x, y, z \in H$, where $x < y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists a \in A$, $\exists b \in B$ such that $a < b$.

Note that if $A, B \subseteq H$, then by $A \circ B$ we mean the subset $\bigcup_{a\in A, b\in B} a \circ b$ of H.

2.2 Theorem [3] Let $(H, \circ, 0)$ be a hyper K-algebra. Then for all $x, y, z \in H$ and for all non-empty subsets A, B and C of H the following hold:

(i) $A \circ B < A$, (ii) $x \circ y < x$,
(iii) $x \circ (x \circ y) < y$, (iv) $x \in x \circ 0$,
(v) $x \circ y < z \Leftrightarrow x \circ z < y$.

2.3 Definition [3] Let $(H, \circ, 0)$ be a hyper K-algebra. If there exists an element $1 \in H$ such that $1 < x$ for all $x \in H$, then H is called a bounded hyper K-algebra and 1 is said to be the unit of H. In a bounded hyper K-algebra, we denote $1 \circ x$ by Nx.

2.4 Definition [8,9] Let D be a nonempty subset of a bounded hyper K-algebra H with unit 1, such that $1 \in D$. Then D is said to be a dual positive implicative hyper K-ideal of

(i) type 1, if for all $x, y, z \in H$, $N((Nx \circ Ny) \circ Nz) \subseteq D$ and $N(Ny \circ Nz) \subseteq D$ imply that $N(Nx \circ Nz) \subseteq D$,
(ii) type 2, if for all $x, y, z \in H$, $N(Ny \circ Nz) \subseteq D$ implies that $N(Nx \circ Nz) \subseteq D$,
(iii) type 3, if for all $x, y, z \in H$, $N(Nx \circ Nz) \subseteq D$,
(iv) type 4, if for all $x, y, z \in H$, $N((Nx \circ Ny) \circ Nz) \subseteq D$ implies that $N(Nx \circ Nz) \subseteq D$.

Note that for simplicity of notation we write $DPIHKI - T1, 2, 3$ and 4 instead of dual positive implicative hyper K-ideals of types 1,2,3 and 4.

2.5 Proposition If $\{A_i | i \in \Lambda\}$ is a family of dual positive implicative hyper K-ideal of type $1(2, 3, 4)$, then $\bigcap_{i\in\Lambda} A_i$ is a $DPIHKI - T1(2, 3, 4)$.

2.6 Definition [3] Let H be a hyper$K$-algebra. An element $a \in H$ is called a *left(resp. right) scalar* if $|a \circ x| = 1$(resp. $|x \circ a| = 1$) for all $x \in H$. If $a \in H$ is both left and right scalar,we say that a is an *scalar element*.

M. M. Zahedi is with Department of Mathematics, Shahid Bahonar University of Kerman. Kerman, Iran (email: zahedi_mm@mail.uk.ac.ir)
L. Torkzadeh is with Department of Mathematics, Azad University of Kerman, Kerman, Iran (email: ltorkzadeh@yahoo.com)

2.7 Theorem Let H be a bounded hyper K-algebra and $NNx = x$, for all $x \in H$. Then:
(i) 1 is a left scalar,
(ii) 0 is a right scalar,
(iii) If $x < y$, then $Ny < Nx$.

2.8 Definition [3] Let H_1 and H_2 be two hyper K-algebras. Then a function $f : H_1 \longrightarrow H_2$ is called homomorphism if $\forall x, y \in H$, $f(x \circ y) = f(x) \circ f(y)$ and $f(0) = 0$.

2.9 Definition [10] Let μ be a fuzzy subset of a nonempty set H and $t \in [0, 1]$. Then the set:

$$U(\mu; t) = \{x \in H | \mu(x) \geq t\}$$

$$(resp. \quad L(\mu; t) = \{x \in H | \mu(x) \leq t\})$$

is called an upper (resp. lower) level subset of μ.

2.10 Definition [10] A fuzzy subset A of H satisfies the inf property condition, if for any subset T of H there exists $x_0 \in T$ such that

$$A(x_0) = \inf_{t \in T} A(t)$$

2.11 Definition [10] Let μ and ν be fuzzy subsets of X and Y, respectively. Then the fuzzy subsets $\mu \times \nu$ and $\mu \otimes \nu$, which are called the product and anti-product of μ and ν, resp. are defined by

$$(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\}$$

$$(\mu \otimes \nu)(x, y) = \max\{\mu(x), \nu(y)\}$$

2.12 Definition [1] An intuitionistic fuzzy set (briefly, IFS) A on a nonempty set X is an object having the form

$$A = \{(x, \mu_A(x), \gamma_A(x)) | x \in X\}$$

where the function $\mu_A : X \rightarrow [0, 1]$ and $\gamma_A : X \rightarrow [0, 1]$ denote the degree of membership and the degree of nonmembership, respectively, and

$$0 \leq \mu_A(x) + \gamma_A(x) \leq 1$$

for all $x \in X$. An intuitionistic fuzzy set $A = \{(x, \mu_A(x), \gamma_A(x)) | x \in X\}$ on X can be identified with an ordered pair (μ_A, γ_A) in $I^X \times I^X$. For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for IFS $A = \{(x, \mu_A(x), \gamma_A(x)) | x \in X\}$.

2.13 Definition [10] Let $f : X \longrightarrow Y$ be a function. An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ of X is said to be f-invariant, if $f(x) = f(y)$ implies that $\mu_A(x) = \mu_A(y)$ and $\gamma_A(x) = \gamma_A(y)$, for all $x, y \in H$.

III. Intuitionistic Fuzzy Dual Positive Implicative Hyper K-ideals

Henceforth H is a bounded hyper K-algebra with unit 1.

3.1 Definition Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy subset of H and let $\mu_A(1) \geq \mu_A(x)$ and $\gamma_A(1) \leq \gamma_A(y)$ for all $x, y \in H$. Then A is said to be an intuitionistic fuzzy dual positive implicative hyper K-ideal of
(i) type 1, if for all $t \in N(Nx \circ Nz)$,

$$\mu_A(t) \geq \min(\inf_{a \in N((Nx \circ Ny) \circ Nz)} \mu_A(a), \inf_{b \in N(Ny \circ Nz)} \mu_A(b))$$

and

$$\gamma_A(t) \leq \max(\sup_{a \in N((Nx \circ Ny) \circ Nz)} \gamma_A(a), \sup_{b \in N(Ny \circ Nz)} \gamma_A(b))$$

(ii) type 2, if for all $t \in N(Nx \circ Nz)$,

$$\mu_A(t) \geq \inf_{a \in N(Ny \circ Nz)} \mu_A(a)$$

and

$$\gamma_A(t) \leq \sup_{a \in N(Ny \circ Nz)} \gamma_A(a)$$

(iii) type 3, if for all $t \in N(Nx \circ Nz)$,

$$\mu_A(t) = \mu_A(1) \qquad \text{and} \qquad \gamma_A(t) = \gamma_A(1)$$

(iv) type 4, if for all $t \in N(Nx \circ Nz)$,

$$\mu_A(t) \geq \inf_{a \in N((Nx \circ Ny) \circ Nz)} \mu_A(a)$$

and

$$\gamma_A(t) \leq \sup_{a \in N((Nx \circ Ny) \circ Nz)} \gamma_A(a)$$

for all $x, y, z \in H$. For simplicity of notation we write $IFDPIHKI - T1(T2, T3, T4)$ instead of intuitionistic fuzzy dual positive implicative hyper K-ideal of type 1 (type 2, type 3, type 4).

3.2 Example The following table shows a hyper structure on $H = \{0, 1, 2\}$.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0, 1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0\}$

Define the IFS $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ on H as follows:

$$\mu_A(2) = 1/2, \ \mu_A(0) = \mu_A(1) = 1/3,$$

$$\gamma_A(2) = 0.2, \ \gamma_A(0) = \gamma_A(1) = 0.1,$$

$$\mu_B(1) = 1/3, \quad \mu_B(2) = 1/4, \quad \mu_B(0) = 1/5,$$

$$\gamma_B(1) = 0.1, \quad \gamma_B(2) = 0.3, \quad \gamma_B(0) = 0.4$$

Then A and B are $IFDPIHKI - T1$,$T2$ and $T4$, also A is an $IFDPIHKI - T3$, while B is not.

3.3 Theorem Let $A = (\mu_A, \gamma_A)$ be an IFS of H. Then A is an $IFDPIHKI - T1(T2, T3, T4)$ if and only if for all $s, t \in [0, 1]$, the nonempty level subsets $U(\mu_A; t)$ and $L(\gamma_A; s)$ of A is a $DPIHKI - T1(T2, T3, T4)$.

3.4 Theorem (i) Let $A = (\mu_A, \gamma_A)$ be an $IFDPIHKI - T3$. Then it is an $IFDPIHKI - T1(T2, T4)$.
(ii) Let $A = (\mu_A, \gamma_A)$ be an $IFDPIHKI - T2(T4)$. Then it is an $IFDPIHKI - T1$.

3.5 Example The following tables show two hyper K-algebra structures on $\{0, 1, 2\}$.

H_1	0	1	2
0	$\{0,2\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,2\}$	$\{2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

H_2	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,2\}$
1	$\{1\}$	$\{0,2\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

Define the intuitionistic fuzzy subsets $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ on H_1 and H_2, respectively as follows:

$$\mu_A(2) = 0, \quad \mu_A(1) = \mu_A(0) = 1/2,$$
$$\gamma_A(2) = 0.7, \quad \gamma_A(1) = \gamma_A(0) = 0.3,$$
$$\mu_B(0) = 0, \quad \mu_B(2) = \mu_B(1) = 1/3,$$
$$\gamma_B(0) = 0.5, \quad \gamma_B(1) = \gamma_B(2) = 0.2.$$

Then A is an $IFDPIHKI - T1$,$T2$ and $T4$, while it is not of type 3. B is an $IFDPIHKI - T1$, while it is not of type 2, 3 or 4.

3.6 Theorem Let H be a bonded hyper K-algebra and $NNx = x$, $\forall x \in H$. If $A = (\mu_A, \gamma_A)$ is an $IFDPIHKI - T2(3)$, then it is constant.

3.7 Definition Let A be a fuzzy subset of H and $A(1) \geq A(x)$ for all $x \in H$. Then A is said to be a fuzzy dual positive implicative hyper K-ideal of
(i) type 1, if for all $t \in N(Nx \circ Nz)$, $A(t) \geq \min(\inf_{a \in N((Nx \circ Ny) \circ Nz)} A(a), \inf_{b \in N(Ny \circ Nz)} A(b))$
(ii) type 2, if for all $t \in N(Nx \circ Nz)$,

$$A(t) \geq \inf_{a \in N(Ny \circ Nz)} A(a)$$

(iii) type 3, if for all $t \in N(Nx \circ Nz)$,

$$A(t) = A(1)$$

(iv) type 4, if for all $t \in N(Nx \circ Nz)$,

$$A(t) \geq \inf_{a \in N((Nx \circ Ny) \circ Nz)} A(a).$$

for all $x, y, z \in H$. For simplicity of notation we write $FDPIHKI - T1(T2, T3, T4)$ instead of fuzzy dual positive implicative hyper K-ideal of type 1 (type 2, type 3 and type 4)

3.8 Theorem An IFS $A = (\mu_A, \gamma_A)$ is an $IFDPIHKI - T1(T2, T3, T4)$ if and only if the fuzzy sets μ_A and $\bar{\gamma}_A$ are $FDPIHKI - T1(T2, T3, T4)$, where $\bar{\gamma}_A(x) = 1 - \gamma_A(x)$, for all $x \in H$.

3.9 Theorem Let H_1 and H_2 be bounded hyper K-algebras, and let $1_{H_1} \circ 1_{H_1} = 0_1$, $1_{H_2} \circ 1_{H_2} = 0_2$, μ and ν be fuzzy subsets of H_1 and H_2, respectively. If $\mu(1_{H_1}) = \nu(1_{H_2})$, $\mu(1) \geq \mu(x)$ and $\nu(1) \geq \nu(y)$, $\forall (x, y) \in H_1 \times H_2$, then $\mu \times \nu$ is an $FDPIHKI - T1(T2, T3, T4)$ of $H_1 \times H_2$ if and only if μ and ν are $FDPIHKI - T1(T2, T3, T4)$ on H_1 and H_2, respectively.

3.10 Theorem Let H_1 and H_2 be bounded hyper K-algebras, $1_{H_1} \circ 1_{H_1} = 0_1$, $1_{H_2} \circ 1_{H_2} = 0_2$ and let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy subsets of H_1 and H_2, respectively. If $\mu_A(1_{H_1}) = \mu_B(1_{H_2})$, $\gamma_A(1_{H_1}) = \gamma_B(1_{H_2})$, $\mu_A(1_{H_1}) \geq \mu_A(x)$, $\mu_B(1_{H_2}) \geq \mu_B(y)$, $\gamma_A(1_{H_1}) \leq \gamma_A(x)$ and $\gamma_B(1_{H_2}) \leq \gamma_B(y)$, for all $(x, y) \in H_1 \times H_2$, then $A \times B = (\mu_A \times \mu_B, \gamma_A \otimes \gamma_B)$ is an $IFDPIHKI - T1(T2, T3, T4)$ if and only if A and B are $IFDPIHKI - T1(T2, T3, T4)$.

3.11 Theorem Let H_1 and H_2 be two bounded hyper K-algebras, $1_{H_1} \circ 1_{H_1} = 0_1$ and $1_{H_2} \circ 1_{H_2} = 0_2$. If μ is an $FDPIHKI - T1(T2, T3, T4)$ on $H_1 \times H_2$, then there are μ_1 and μ_2 which are $FDPIHKI - T1(T2, T3, T4)$ on H_1 and H_2, respectively.

3.12 Theorem Let H_1 and H_2 be two bounded hyper K-algebras, $N_1 N_1 x = x$ and $N_2 N_2 y = y$, for all $(x, y) \in H_1 \times H_2$. If μ satisfies the anti-additive condition on $H_1 \times H_2$ and it is an $FDPIHKI - T4(T1, T2, T3)$, then there exist fuzzy subsets μ_1 and μ_2 on H_1 and H_2 which are $FDPIHKI - T4(T1, T2, T3)$ and $\mu = \mu_1 \times \mu_2$.

Proof. We prove theorem for type 4, the proofs of the other types are similar to type 4. Define $\mu_1(x) = \mu(x, 1)$ and $\mu_2(y) = \mu(1, y)$, $\forall (x, y) \in H_1 \times H_2$. By the proof of Theorem 3.11 μ_1 and μ_2 are $FDPIHKI - T4$. Now we show that $\mu = \mu_1 \times \mu_2$. Since μ satisfies the fuzzy anti-additive condition, then $(x, y) < (x, 1)$ and $(x, y) < (1, y)$ imply that $\mu(x, y) \leq \mu(x, 1) = \mu_1(x)$ and $\mu(x, y) \leq \mu(1, y) = \mu_2(y)$. Thus $\mu(x, y) \leq \min\{\mu_1(x), \mu_2(y)\} = \mu_1 \times \mu_2(x, y)$, for all $(x, y) \in H_1 \times H_2$. Let $(x, y) \in H_1 \times H_2$ and $(a, b) \in N(N(x, y) \circ N(1, 1))$. Then since μ is an $FDPIHKI - T4$ we have $\mu(a, b) \geq \inf_{(t_1, t_2) \in N((N(x,y) \circ N(1,y)) \circ N(1,1))} \mu(t_1, t_2)$, so by hypothesis we get that $\mu(x, y) \geq \mu(x, 1) = \mu_1(x)$, similarly $\mu(x, y) \geq \mu(1, y) = \mu_2(y)$, hence $\mu(x, y) \geq \min\{\mu_1(x), \mu_2(y)\} = \mu_1 \times \mu_2(x, y)$. Therefore $\mu = \mu_1 \times \mu_2$.

3.13 Theorem (Decomposition Theorem) Let H_1 and H_2 be two bounded hyper K-algebras, $N_1N_1x = x$ and $N_2N_2y = y$, for all $(x, y) \in H_1 \times H_2$. If $A = (\mu_A, \gamma_A)$ satisfies the anti-additive condition on $H_1 \times H_2$ and it is an $IFDPIHKI-T4(T1, T2, T3)$, then there exist intuitionistic fuzzy subsets $A_1 = (\mu_{A_1}, \gamma_{A_1})$ and $A_2 = (\mu_{A_2}, \gamma_{A_2})$ on H_1 and H_2 which are $IFDPIHKI - T4(T1, T2, T3)$ and $A \times B = (\mu_{A_1} \times \mu_{A_2}, \gamma_{A_1} \otimes \gamma_{A_2})$.

3.14 Definition Let $f : X \longrightarrow Y$ be a function and B be a fuzzy set of Y. Then the fuzzy set $f^{-1}(B)$ of X is defined by:

$$f^{-1}(B)(x) = B(f(x)), \forall x \in H$$

3.15 Theorem Let H_1 and H_2 be two bounded hyper K-algebras and $f : H_1 \longrightarrow H_2$ be a homomorphism such that $f(1) = 1$. If $A = (\mu_A, \gamma_A)$ is an $IFDPIHKI - T1(T2, T3, T4)$ of H_2, then $f^{-1}(A) = (f^{-1}(\mu_A), f^{-1}(\gamma_A))$ is an $IFDPIHKI - T1(T2, T3, T4)$ of H_1.

3.16 Definition Let $f : X \longrightarrow Y$ be a function and A be an intuitionistic fuzzy set of X. Then the intuitionistic fuzzy set $f(A)$ of Y is defined by the pair $(f(\mu_A), f(\gamma_A))$:

$$f(\mu_A)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_A(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & otherwise \end{cases}$$

$$f(\gamma_A)(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \gamma_A(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & otherwise \end{cases}$$

3.17 Theorem Let H_1 and H_2 be two bounded hyper K-algebras and $f : H_1 \longrightarrow H_2$ be an onto homomorphism and $f(1) = 1$. Then :
(i) if $A = (\mu_A, \gamma_A)$ is an $IFDPIHKI - T3$, then so is $f(A) = (f(\mu_A), f(\gamma_A))$.
(ii) if $A = (\mu_A, \gamma_A)$ is an f-invariant and an $IFDPIHKI - T1(2, 4)$, then so is $f(A) = (f(\mu_A), f(\gamma_A))$.

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