International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:12, No:6, 2018

Improving the Analytical Power of Dynamic DEA Models, by the Consideration of the Shape of the Distribution of Inputs/Outputs Data: A Linear Piecewise Decomposition Approach

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***Abstract*—**In Dynamic Data Envelopment Analysis (DDEA), which is a subfield of Data Envelopment Analysis (DEA), the productivity of Decision Making Units (DMUs) is considered in relation to time. In this case, as it is accepted by the most of the researchers, there are outputs, which are produced by a DMU to be used as inputs in a future time. Those outputs are known as intermediates. The common models, in DDEA, do not take into account the shape of the distribution of those inputs, outputs or intermediates data, assuming that the distribution of the virtual value of them does not deviate from linearity. This weakness causes the limitation of the accuracy of the analytical power of the traditional DDEA models. In this paper, the authors, using the concept of piecewise linear inputs and outputs, propose an extended DDEA model. The proposed model increases the flexibility of the traditional DDEA models and improves the measurement of the dynamic performance of DMUs.

***Keywords*—**Data envelopment analysis, Dynamic DEA, Piecewise linear inputs, Piecewise linear outputs.

I. Introduction

DEA is a multi-criteria methodology for the evaluation of the efficiency of DMUs. The methodology was, firstly, introduced by Charnes et al. [1]. Since then, DEA has become the leading methodology for the evaluation of the relative efficiency of peer DMUs in a multiple input/output setting [6]. The relative efficiency of a DMU is calculated by the ratio of a weighted sum of its outputs (or virtual output) to a weighted sum of its inputs (or virtual input) and it is measured on a bounded ratio scale [2]. The weights for inputs and outputs are estimated to the best advantage for each DMU so as to maximize its relative efficiency. Initially, the traditional DEA models had considered the evaluated DMUs as black boxes and they had not taken into account any differentiation in their weighting scheme with which the virtual inputs and virtual outputs were evaluated. The assumption of the DMU, as a black box, was released for the first time by Färe and Grosskopf [5].

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Additionally, the estimation of the efficiency of DMUs, and, the dependence of the production frontier by the time was firstly examined by Färe et al. [3]. A later work, by Färe and Grosskopf [4], entitled "Intertemporal Production Frontiers: With Dynamic DEA" had as ultimate goal to propose a dynamic model. This work could be considered as the beginning of the development of the Dynamic DEA (DDEA). Since then, several DDEA models have been proposed. The reconsideration of the assumptions which had been taken by the researchers, leaded to the creation of variants of the old models, in the direction of the accurate measurement of the efficiency of DMUs over time [11]. An important characteristic which should be taken into account in the development of a DDEA model has to do with the existence of intermediate inputs (or carry-overs) and their types. As intermediates are defined the outputs which are produced by a DMU at a time point and they used, by the same DMU, as inputs in a future time point.

According to Tone and Tsutsui [14], [15], those carry-overs may be categorized into the following four types:

- Desirable. The transition element of that kind is considered as preferable, i.e. as an output. So, its value should not be lower than the observed.
- Undesirable. In this case, the transition element is considered as undesirable. This carry-over performs as input and, its value should not be higher than the observed.
- Discretionary. This transition element is considered as free to change.
- Non-discretionary. This carry-over represents an intermediate that is not under the control of the DMU.

A major drawback of the most of DDEA models is that they do not provide a discrimination among the different types of intermediate inputs [11]. As Skevas et al. [13] asserted,

> "Most models overlook this detail and consider the intermediate interchangeably, without observing the peculiarities of the analytical condition".

Another issue, which has been emerged as a subject of discussion, is the way in which the DEA models calculate the virtual inputs and virtual outputs. The most of the models adopt the assumption that the virtual inputs and outputs follow a linear distribution. In consequence, the weights for the evaluation of the virtual inputs and outputs are not related with the level of value for each input or output. Recently, Despotis et al. [2] have relaxed the assumption of linearity for the inputs and outputs, suggesting a piecewise linear representation of the value function for inputs and outputs.

We believe that the shape of the distribution of the virtual intermediate inputs is of great importance in DDEA models and it should be taken into account for the determination of the weights, with which the total virtual intermediate should be calculated. We, also, believe that the carry-overs are, in many interesting cases, variables, which do not follow the assumption of linearity. For those cases, we suggest the utilization of a piecewise linear weighting scheme for those nonlinear intermediate inputs. In our opinion, this could provide the proper handling and the differentiation among the different types of carry-overs, improving the discrimination power of the DDEA models. We implement our approach transforming the DDEA model of Kao and Huang [8], Kao [7]. We choose this model because it utilizes all the carry-overs interchangeably. The rest of the paper is organized as follows. In Section II, we, briefly, present the model of Kao and Huang [8], Kao [7], we analyze the concept of the non-linear virtual carry-over, which is explained by an example, and we propose a variant model capable of taking into consideration intermediates deviating from linearity. The paper ends with some conclusions.

II. Methodology

A careful examination of the concepts of desirable, undesirable, discretional and non-discretional carry-overs leads the reader to the conclusion that the handling of each type of carry-over is differentiated according the existence of preference, dislike, indifference or weakness of control upon the value of the intermediate.

In many cases, the unit prices of products or the unit prices of services are uncertain, especially in cases of public services [9], [10], [12]. For those reasons, we believe that the above-mentioned treatment of the carry-over:

- Oversimplifies the nature and, in following, the handling of the carry-over in DDEA models.
- Deprives a researcher the opportunity to take into account a detailed opinion of the Decision Maker (briefly, DM) of the problem about the values of the carry-over, and
- Does not allow the model to incorporate DM preferences upon the desirable levels of the intermediates.

In this work, we improve the model of Kao and Huang [8], Kao [7], providing a variant, which could be capable of incorporating the different types of carry-overs in a more detailed manner. Our approach increases the ability of representation of the preferences of the decision makers, especially in case where the distributions of the carry-overs exhibit a deviation from linearity.

The proposed model by Kao and Huang [8], Kao [7], evaluates the dynamic production of DMUs as a sequence of technologies interconnected by intermediate inputs/outputs between the periods without taking into account the different types of intermediates.

The model considers the following variables and parameters: X_{ij}^t: input i of the jth DMU in period t, Y_{rj}^t: output r of the jth DMU in period t, Z_{fj}^t: intermediate input f of the jth DMU in period t. Taking

$$X_{ij} = \sum_{t=1}^{p} X_{ij}^t \tag{1}$$

and

$$Y_{rj} = \sum_{t=1}^{p} Y_{rj}^t \tag{2}$$

the model of Kao and Huang [8], Kao [7] is expressed, in the multiplier form, as follows:

$$1/_{E_k^R} = \min\left(\sum_{i=1}^{m} v_i X_{ik} + \sum_{f=1}^{g} w_f Z_{fk}^0\right) \tag{3}$$

s.t.

$$\left[\sum_{r=1}^{s} u_r Y_{rj} + \sum_{f=1}^{g} w_f Z_{fj}^t\right] = 1 \tag{4}$$

$$\left[\sum_{r=1}^{s} u_r Y_{rj} + \sum_{f=1}^{g} w_f Z_{fj}^0\right] - \left[\sum_{i=1}^{m} v_i X_{ij} + \sum_{f=1}^{g} w_f Z_{fj}^p\right] \le 0, \tag{5}$$

$$\left[\sum_{r=1}^{s} u_r Y_{rj}^t + \sum_{f=1}^{g} w_f Z_{fj}^t\right] - \left[\sum_{i=1}^{m} v_i X_{ij}^t + \sum_{f=1}^{g} w_f Z_{fj}^{t-1}\right] \le 0 \tag{6}$$

where j=1,…,n and t=1,…,p.

$$u_r, v_i, w_f \ge \varepsilon, \qquad r = 1, \dots, s;\ i = 1, \dots, m; f = 1, \dots, g$$

The previous inequalities show clearly that the implementation of this model considers the intermediates interchangeably. Additionally, the formation of each virtual input/output/intermediate does not take into account the shape of the distribution of the variables and, of course, the level of the exhibited value of each input/output/intermediate.

We believe that the DDEA models should be capable of handling the above-mentioned issues, and, we propose an approach which extends the thought of Despotis et al., [2], concerning the existence of non-linear virtual inputs and outputs, in DDEA.

In order to make clear our approach to the reader, let us suppose that there is an intermediate input Z_{kj}^t, where t=1,…,p; j=1,…,n. This carry-over, obviously, belongs in one of the above mentioned four types of intermediate inputs (i.e. desirable, undesirable, discretional or non-discretional). For this categorization, we believe that the opinion of the Decision Maker is valuable. As the analysts, in order to categorize the carry-over, we should ask the Decision maker, to provide us the intervals of his/her preference within which he/she believes that we should use different weigh values for the calculation of the virtual value of the carry-over, and the kind of the relation among those weights. The DM's estimations should be taken, by us, as a strong indication if the distribution of the virtual intermediate exhibits deviation from linearity. If the distribution of the virtual carry-over is expected not to follow the linearity, we propose the decomposition of the values of this carry-over for all the DMUs j, and, the time-periods t, into the sub-intervals using breakpoints. The Decision Maker could also, provide us those breakpoints. In this manner, a new augmented dataset for the carry-over is created.

This process, in details, is as follows:

Let us take the intermediate input Z_{fj}^{t}, which is supposed to deviate from linearity, and, a set of h breakpoints $b_{f,l}$ where:

- $b_{f,1}$=L_f is the minimum observed value (or the minimum value, according the way of measurement of the carry-over),
- $b_{f,L(f)}$=H_f is the maximum observed value within the dataset of $\widehat{Z_k}$ (or the maximum value, according the way of measurement of the carry-over), and
- $b_{f,l}$ are the breakpoints which are provided by the DM and satisfy the inequality $b_1 < b_{fl} < b_{f(l+1)} < b_{f,L(f)}$, $\forall l \in \{2,3,\ldots,L(f)-2\}$

In this case, the intermediate input Z_{fj}^{t}, for each DMU j, and time-period t, is decomposed into a new augmented set of values.

As an example, let us consider a case taken from the field of evaluation of school performance. Schools could be considered as DMUs, which operate using inputs (i.e. money for the teachers' wages, infrastructures etc.) and produce outputs (i.e. services which provide knowledge and education). The examination of the dynamic performance of a school over time leads to the conclusion that the performance is influenced by several carry-overs which could be:

- Desirable (like the added knowledge to the student by the staff of the school).
- Undesirable (like the percentage of penalties which were given to the students, for different causes, during last year).
- Discretional.
- Non-discretional.

Let us suppose that we evaluate the performance of some high schools. An intermediate input of the production technology has to do with the added value in knowledge, which is provided by the teachers of the school every year. This output is used as intermediate input at the next year(s). Usually, in Greece, this variable is measured using the scores of students' performance. The responsible Ministry has adopted a grading system for the students of the high schools which:

- Is extended from 0 to 20.
- Is divided in sub-intervals using the set of mid-breakpoints {5, 9.4, 13, 16, 18}.

The significance of the average performance of a student for the Ministry of Greece, and the Greek State is strongly influenced by the sub-interval to which it belongs.

Let us suppose that the following dataset contains the average annual performance of each one of the students who are going to attend the last year of their studies at a high school.

{8, 8, 9, 9.8, 11, 11, 11.5, 12, 12, 13, 13, 13.3, 13.3, 13.5, 13.5, 13.7, 13.9, 14, 15.1, 15.1, 15.4, 15.6, 15.6, 15.6, 16, 16.3, 16.4, 16.4, 16.9, 17.1, 17.3, 18, 18.3, 18.5}

The total average for this dataset is 14 and, using the common DEA approach, this value is weighted in order to provide the virtual intermediate input.

Following our approach, we use the above-mentioned set of breakpoints, in order to transform the initial dataset to the following distribution:

1. [0,5]:0 members, sub-interval average=2.5
2. [5.1,9.4]:3 members, sub-interval average=7,25
3. [9.5,13]: 8 members, sub-interval average=11,25
4. [13.1,16]: 14 members, sub-interval average=14,55
5. [16.1, 18]: 7 members, sub-interval average=17,05
6. [18,1.20]: 3 members, sub-interval average=19,05

It is obvious that the value function of the students 'performance, in this case, could be calculated with different weight values for each one of sub-interval, i.e. $U=U_1+U_2+U_3+U_4+U_5+U_6$.

All the students who belong to the same sub-interval, are weighted by the same weight-value. In this way, a piecewise linear virtual utility function is created for this intermediate,

In order to represent those ideas to the reader, we are going to evaluate the virtual carry-over, using the following four scenarios:

Scenario A. The considered carry-over deviates from the linearity. In this case, we decompose the total number of students into six sub-intervals and we create the virtual intermediate weighting with the vector of partial weights, $\widehat{W} = [0.05, 0.1, 0.15, 0.20, 0.25, 0.30]$, (column 3, Table I). The reader could notice that the values of the components of $\widehat{W}$ show increasing values within the sub-intervals of [0,20]. This is chosen because the selected carry-over is considered as preferable, i.e. it behaves as output. The total virtual intermediate is calculated in column 5 of Table I, and it is represented as the piecewise linear curve PLvi, in Fig. 1.

Scenario B. The carry-over follows the assumption of linearity, and, the total number of students is multiplied by the average of the components of $\widehat{W}$. That is equivalent with the weighting of the number of students in each sub-interval, with the constant weight W_B=Average($\widehat{W}_i$). The total virtual intermediate AvLvi is calculated in column 6 of Table I, and it is represented by the straight line AvLvi, in Fig. 1.

Scenario C. The carry-over follows the assumption of linearity, and, the total number of students is multiplied by the average of the components of $\widehat{W}$. That is equivalent with the weighting of the number of students in each sub-interval, with the constant weight W_C=Max($\widehat{W}_i$). The total virtual intermediate MaxLvi is calculated in column 7 of Table I, and it is represented by the straight line MaxLvi, in Fig. 1.

Scenario D. The carry-over follows the assumption of linearity and the total number of students is multiplied by the average of the components of $\widehat{W}$. That is equivalent with the weighting of the number of students in each sub-interval, with the constant weight W_D=Min(($\widehat{W}_i$). The total virtual intermediate MinLvi is calculated in column 8 of Table I, and it is represented by the straight line MinLvi, in Fig. 1.

Let us discuss the results of the example. The reader could notice that if we adopt the distribution of the virtual average score as linear, this will lead us to under-estimations or over-estimations of the real performance of the schools as it is described by the calculated virtual carry-over (Table I, Fig. 1).

TABLE I
EVALUATION OF VIRTUAL AVERAGE SCORE USING FOUR SCENARIOS

IAG	NoStPI	NoSt	weight	PLvi	AvLvi	MaxLvi	MinLvi
[0,5]	0	0	0.05	0	0	0	0
[5.1, 9.4]	3	3	0.1	0.3	0.5	0.9	0.15
[9.5, 13]	8	11	0.15	1.5	1.9	3.3	0.55
[13.1, 16]	14	25	0.2	4.3	4.3	7.5	1.25
[16.1, 18]	7	32	0.25	6.05	5.6	9.6	1.6
[18.1,20]	3	35	0.3	6.95	6.1	10.5	1.75

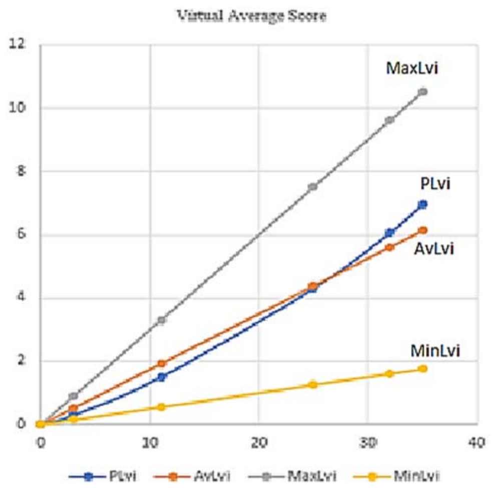

Fig. 1 Virtual carry-over of average score

The above described approach of the piecewise evaluation of the virtual carry-overs, of course, could be used in every input or output, with similar characteristics.

We have considered two main possibilities about the scale of measurement for the carry-overs, which are considered. Those are the carry-overs, which are:

1. Quantitative continuous variables. In this case, the utility function is

$$\Omega_{fj}^t = \sum_{l=1}^{\xi} w_{f,l} L_{k,l,j}^t + w_{f,\xi+1}(Z_{k,j}^t - b_{k\xi}^t) \qquad (7)$$

where: $Z_{k,l,j}^t - b_{k,\xi}^t$, stands for the difference of $Z_{k,j}^t$ from its immediate and lower breakpoint $b_{k,\xi}^t$, and the terms $L_{k,l,j}^t$ stand for the lengths of each one of the successive sub-intervals which are created with breakpoints $b_{k,l,j}^t \le b_{k,\xi}^t$.

2. Quantitative discrete variables. In this case, the utility function is

$$\Omega_{fj}^t = \sum_{l=1}^{h(k)} w_{f,l} N_{k,l,j}^t, \qquad (8)$$

where the term $N_{f,l,j}^t$ counts the number of cases of Z_{fj}^t which belong to sub-interval l, for all the sub-intervals which are created with breakpoints lower or equal to H_f^t.

Those utility functions are considered as convex and non-decreasing over the range of values of the carry-over.

Subsequently, we consider that the first c among the g carry-overs follows the assumption of linearity, while the rest of them do not. The values of those (g-c) carry-overs, should be decomposed into a new augmented dataset, using suitable breakpoints as described above.

Under these conditions, we propose the following variant of the model of Kao and Huang [8], Kao [7].

$$^1/_{E_k^R} = \min\left(\sum_{i=1}^m v_i X_{ik} + \sum_{f=1}^c w_f Z_{fk}^0 + \sum_{f=c+1}^g \Omega_{fj}^0\right), \qquad (9)$$

s.t.

$$\left[\sum_{r=1}^s u_r Y_{rj} + \sum_{f=1}^c w_f Z_{fj}^t + \sum_{f=c+1}^g \Omega_{fj}^t\right] = 1 \qquad (10)$$

$$\left[\sum_{r=1}^s u_r Y_{rj} + \sum_{f=1}^c w_f Z_{fj}^0 + \sum_{f=c+1}^g \Omega_{fj}^0\right] - \left[\sum_{i=1}^m v_i X_{ij} + \sum_{f=1}^c w_f Z_{fj}^p + \sum_{f=c+1}^g \Omega_{fj}^p\right] \le 0 \qquad (11)$$

$$\left[\sum_{r=1}^s u_r Y_{rj}^t + \sum_{f=1}^c w_f Z_{fj}^t + \sum_{f=c+1}^g \Omega_{fj}^t\right] - \left[\sum_{i=1}^m v_i X_{ij}^t + \sum_{f=1}^c w_f Z_{fj}^{t-1} + \sum_{f=c+1}^g \Omega_{fj}^{t-1}\right] \le 0 \qquad (12)$$

where j=1,…,n and t=1,…,p.

$$u_r, v_i, w_f \ge \varepsilon, \qquad r = 1, \dots, s;\ i = 1, \dots, m;\ f = 1, \dots, g$$

Our opinion is that the above described model should be strengthened by a set of weight restrictions appropriate to incorporate the type of the carry-overs in the analysis. More specifically, for the:

- Desirable carry-over(s). It is accepted that this type of carry-over exhibits characteristics of an output. For cases of that kind, we propose that the augmented dataset of the carry-over should be weighted using weights which follow the weight restriction

$$0 < a_f \le {}^{w_{f,l}}/_{w_{f,l+1}} \le \gamma_f \le 1, \qquad (13)$$

where f=(c+1),…,g and l=1, ..., ξ-1.

Special attention should be given in the determination of these weight values, in a way to avoid effects of convergence.

- Undesirable carry-over(s). It is accepted that this type of carry-over exhibits characteristics of an input. For cases of that kind, we propose that the augmented dataset of the carry-over should be weighted using weights which follow the weight restriction

$$0 < a_f \le {}^{w_{f,l+1}}/_{w_{f,l}} \le \gamma_f \le 1, \qquad (14)$$

where f=(c+1),…,g and l=1, ..., ξ-1.

- Discretional carry-over(s). This intermediate is, usually, considered as free to change. In those cases, we propose that the augmented dataset of each of the carry-over should be weighted using weights pointed-out by the DM.
- Non-discretional carry-over(s). This type of carry-over represents an intermediate that is not under the control of the DMU. In this case, the DM could provide low weight values, in order to get approximately zero values, thereby

limited contribution.

III. Conclusions

It is argued by many researchers that there are four basic types of intermediates in DDEA model implementations [11], [13]-[15]. Until now, most of the models lacked handling the different types of those carry-overs.

Our aim, in this paper, was to provide a DDEA model capable of incorporating, in the same setting, the traditional inputs/outputs/intermediates with those which deviate from the linearity (as described above).

We confronted this problem, sharing the same way of thought with Despotis et al. [2], and, we proposed the decomposition of the carry-overs' values, into an augmented new dataset. We, also, believe that the analyst(s) should take into account the Decision Maker's preferences. Those preferences could be translated into weight restrictions, which reflect the type of each of carry-over of the problem. In this way, a new distribution of the virtual carry-over(s) is created, that of a piecewise linear shape.

We used an example of a preferable carry-over from the field of school evaluation. We presented four different scenarios for the evaluation of the virtual average score, showing the pitfalls of the traditional way with which, the virtual intermediate is calculated. We also proved why our proposed approach could manage the issue of the correct calculation of the virtual carry-over.

We applied the above-mentioned approach, extending the model of Kao and Huang [8], Kao [7] into a more sophisticated variant, which improves the representation and handling of intermediates and refines the way of the dynamic evaluation of the DMUs over time.

We focused on the treatment of the different types of carry-overs. A similar treatment is possible in case of inputs and outputs, which deviate from linearity.

Concluding, we believe that, during the development of a DDEA model, a critical task of the analyst is to examine if the virtual inputs, outputs or intermediates which are under consideration, follow a linear distribution or not. Especially, the examination of the shape of the distribution of the virtual carry-overs of the problem could improve the model, as it highlights the inherent characteristics of the variables of the problem and could offer to the analyst a refined way to incorporate the preferences of the DM.

Acknowledgment

This work has been partly supported by the University of Piraeus Research Center.

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