Image Segmentation by Mathematical Morphology: An Approach through Linear, Bilinear and Conformal Transformation

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Abstract—Image segmentation process based on mathematical morphology has been studied in the paper. It has been established from the first principles of the morphological process, the entire segmentation is although a nonlinear signal processing task, the constituent wise, the intermediate steps are linear, bilinear and conformal transformation and they give rise to a non linear affect in a cumulative manner.

Keywords—Image segmentation, linear transform, bilinear transform, conformal transform, mathematical morphology.

I. INTRODUCTION

IMAGE segmentation [1]-[3] has been found to be an essential process for most subsequent processing like image analysis and understanding tasks such as image representation, description and object recognition, image visualization by the machines and object based image compression [4]-[8]. In general, segmentation related problems arise during partitioning of an image into a number of homogeneous segments (i.e. spatially connected groups of pixels) such that the union of any two neighboring segments can give a homogeneous segments. A large number of techniques and algorithms has been proposed and applied for image segmentation and out of them; a major portion belongs to hybrid algorithm [1]-[8]. Apart from this, noise reduction is an inherent problem for all types of image processing.

Mathematical morphology is a non linear area of the signal processing and related to the application of set theory concept to image analysis. Morphology deals with the study of shapes and structures from a general techno-scientific point of view [16]-[27]. Various image processing operation can be implemented in spatial domain, few of them can be implemented in frequency domain. (mainly various filters) and most of them can be realize through morphological operation. Morphological filters or operators are non linear transformations [9]-[13] which either modify or tend to modify geometric features of images. All these operators transform the original image into another image through various iterations with other image of a certain shape and size which is called structuring elements [14], [15]. In present

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study, attempt has been made to establish that all mathematical morphological operation may be thought of as a linear, bilinear transform or conformal transform.

II. BASIC BINARY MORPHOLOGICAL OPERATIONS

The basic operations involved in the morphological process are erosion and dilation. All other operation may be derived from these two basic operations which are associated with translation, reflection etc.

A. Erosion

When A is the image under test and B is the structuring element and both are sets in Z^2, the erosion of A by B denoted by $A \ominus B$ is defined as

$$A \ominus B = \{z | (B)_z \subseteq A\} \tag{1}$$

Equation says that the erosion of A by B translated is contained in A.

B. Dilation

For binary images with A and B as sets in Z^2, the dilation of A and B denoted $A \oplus B$ is defined as

$$A \oplus B = \{z | (\hat{B})_z \cap A \neq \varnothing\} \tag{2}$$

The equation is based on reflecting B about in origin and shifting this reflection by Z. The dilation of image A by structuring element B is the set of all displacements. Based on the interpretation, equation can also be written by

$$A \oplus B = \{z | [(\hat{B})_z \cap A] \subseteq A\} \tag{3}$$

C. Opening and Closing

The opening of set A by structuring element B, denoted $A \circ B$ is defined as

$$A\ B = (A \ominus B) \oplus B \tag{4}$$

The closing of set A by structuring elements B denoted by $A \bullet B$ is defined as

$$A \bullet B = ((A \oplus B) \ominus B) \tag{5}$$

III. MORPHOLOGY

The binary morphology can be easily extended to grayscale morphology. The main differences come from the definition of

erosion and dilation as other operations basically depend on them.

A. Grayscale Erosion and Dilation

A gray scale image can be considered as 3-D set when the first two elements are the x and y co-ordinates of a pixel and the third element is gray scale value. The same concept can be applied to the grayscale structuring elements.

Thus gray scale erosion, denoted by $f \ominus b$,is denoted as

$$(f \ominus b)(s,t) = \max \{ f(s-x, t+y) - \frac{b(x,y)}{s+x}, (t+y)\epsilon d_f(s,y)\epsilon D_b \} \quad (6)$$

When D_f and D_b are the domains of each image or function e.g, f and b respectively. Grayscale dilation of f by $f \oplus b$ denoted by is defined as

$$(f \oplus b)(s,t) = \max \{ f(s-x, t-y) - \frac{b(x,y)}{s-x}, (t-y)\epsilon d_f(s,y)\epsilon D_b \} \quad (7)$$

Like binary dilation and erosion, gray scale dilation and erosion are dulas w.r.t. function completion and reflection. The relation is given by

$$(f \ominus b)^c(s,t) = (f^c \oplus b)(s,t) \quad (8)$$

B. Gray Scale Opening and Closing

The opening of a gray image f by a structuring element b, denoted by $f \circ b$ is defined as

$$f\ b = (f \ominus b) \oplus b \quad (9)$$

and closing is denoted by $f \bullet b$ as

$$f \bullet b = (f \oplus b) \ominus b \quad (10)$$

C. Morphological Gradient

Morphological gradient can be generated using dilation and erosion. Dilation gives the original set plus an extra boundary, the size and shape of the boundary depends on the shape and size of the structuring element. Erosion gives the points for which the structuring element is contained in the original set. The outer boundary of the original shape is removed by erosion. The morphological gradient is generated by subtracting an eroded image from its dilated version. The morphological gradient highlights sharp gray-level transitions in the input image.

Where we have denoted erosion as $f \ominus b$ and dilation as $f \oplus b$. With erosion and dilation morphological gradient can be denoted as

$$MG = (f \oplus b) - (f \ominus b) \quad (11)$$

D. Morphological Smoothing

One way to achieve smoothing is to perform a morphological opening followed by a closing. Opening smoothes the contour by removing thin bridges and eliminating thin protrusions. Closing also smoothes the contour, but by enforcing bridges and closing small holes. The boundary of opening with a circular structuring element

corresponds to rolling a ball on the inside of the set. The boundary of closing corresponds to rolling a ball on the outside of the set.

E. Multiscale Edge Detector

To achieve more robustness to noise, a multiscale gradient algorithm can be applied. Multiscale means image analysis with structuring elements of different or multiple sizes. The combination of morphological gradients in different scales is insensitive to noise as well as to extraction of various finenesses of the edges. The acceptable multiscale edge detector was proposed to obtain the gradient of the image

$$MG(f) = \frac{1}{n}\sum_{i=1}^{n}[((f \oplus b_i - (f \ominus b_i)) \oplus b_{i-1}] \quad (12)$$

where, n is scale and b_i denotes the assembly of square structural elements where sizes are (2i+1)*(2i-1) pixels.

IV. LINEAR, BILINEAR AND CONFORMAL TRANSFORMATIONS

A. Linear Transformation

In mathematics, a linear map (also called a linear mapping, linear transformation or, in some contexts, linear function) is a mapping V → W between two modules (including vector spaces) that preserves (in the sense defined below) the operations of addition and scalar multiplication. An important special case is when V = W, in which case the map is called a linear operator, or an endomorphism of V. Sometimes the definition of a linear function coincides with that of a linear map, while in analytic geometry it does not.

A linear map always maps linear subspaces to linear subspaces (possibly of a lower dimension); for instance it maps a plane through the origin to a plane, straight line or point.

In the language of abstract algebra, a linear map is a homomorphism of modules. In the language of category theory it is morphism in the category of modules over a given ring.

Let V and W be vector spaces over the same field K. A function f: $V \rightarrow W$ is said to be a linear map if for any two vectors x and y in V and any scalar α in K, the following two conditions are satisfied:

$$f(x+y) = f(x) + f(y) \quad (13)$$

$$f(\sigma x) = \sigma f(x) \quad (14)$$

This is equivalent to requiring the same for any linear combination of vectors, i.e. that for any vectors $x_1, ..., x_m \in V$ and scalars $a_1, ..., a_m \in K$, the following equality holds:

$$f(a_1 x_1 + \cdots + a_m x_m) = a_1 f(x_1) + \cdots + a_m f(x_m) \quad (15)$$

Denoting the zero elements of the vector spaces V and W by 0_V and 0_W respectively, it follows that $f(0_V) = 0_W$ because letting α = 0 in the equation for homogeneity of degree 1,

$$f(0_v) = f(0.0_v) = 0.f(0_v) = 0_w \qquad (16)$$

Occasionally, *V* and *W* can be considered to be vector spaces over different fields. It is then necessary to specify which of these ground fields is being used in the definition of "linear". If *V* and *W* are considered as spaces over the field *K* as above, we talk about *K*-linear maps. For example, the conjugation of complex numbers is an R-linear map C → C, but it is not C-linear.

B. Bi-linear Transformation

The bilinear transformation is used in digital signal processing and discrete-time control theory to transform continuous-time system representations to discrete-time and vice versa.

In mathematics, the transformation

$$w = \frac{az+b}{cz+d}, ad - bc \neq 0 \qquad (17)$$

where a, b, c, d are real or complex constants, is called a bilinear transformation

The bilinear transform is a special case of a conformal mapping, often used to convert a transfer function H_aS of a linear, time-invariant (LTI) filter in the continuous-time domain (often called an analog filter) to a transfer function $H_d(z)$ of a linear, shift-invariant filter in the discrete-time domain (often called a digital filter although there are analog filters constructed with switched capacitors that are discrete-time filters). It maps positions on the *jw* axis, *Re(s)=0*, in the s-plane to the unit circle, *|z|=1*, in the z-plane. Other bilinear transforms can be used to warp the frequency response of any discrete-time linear system (for example to approximate the non-linear frequency resolution of the human auditory system) and are implementable in the discrete domain by replacing a system's unit delays z^{-1} with first order all-pass filters.

The transform preserves stability and maps every point of the frequency response of the continuous-time filter, $H_a(jw_a)$ to a corresponding point in the frequency response of the discrete-time filter, $H_d(e^{jwdt})$ although to a somewhat different frequency, as shown in the Frequency warping section below. This means that for every feature that one sees in the frequency response of the analog filter, there is a corresponding feature, with identical gain and phase shift, in the frequency response of the digital filter but, perhaps, at a somewhat different frequency. This is barely noticeable at low frequencies but is quite evident at frequencies close to the Nyquist frequency.

C. Conformal Transformation

In mathematics, a conformal map is a function which preserves angles locally. In the most common cause the function has a domain and range in the complex plane.

More formally, a map,

$$f: U \to V \text{ with } U, V \subset C^n \qquad (18)$$

is called conformal (or angle-preserving) at a point u_0 if it preserves oriented angles between curves through u_0 with respect to their orientation (i.e., not just the magnitude of the angle). Conformal maps preserve both angles and the shapes of infinitesimally small figures, but not necessarily their size or curvature.

The conformal property may be described in terms of the Jacobian derivative matrix of a coordinate transformation. If the Jacobian matrix of the transformation is everywhere a scalar times a rotation matrix, then the transformation is conformal.

Conformal maps can be defined between domains in higher-dimensional Euclidean spaces, and more generally on a Riemannian or semi-Riemannian manifold.

An important family of examples of conformal maps comes from complex analysis. If *U* is an open subset of the complex plane, C, then a function *f:U→ C* is conformal if and only if it is holomorphic and its derivative is everywhere non-zero on *U*. If *f* is anti-holomorphic it still preserves angles, but it reverses their orientation.

The Riemann mapping theorem, one of the profound results of complex analysis, states that any non-empty open simply connected proper subset of C admits a bijective conformal map to the open unit disk in C.

A map of the extended complex plane (which is conformally equivalent to a sphere) onto itself is conformal if and only if it is a Möbius transformation. Again, for the conjugate, angles are preserved, but orientation is reversed.

V. DISCUSSIONS

Mathematical morphology mainly deals with set theory to analyze the images and subsequent understanding and visualization by the machines. Various image processing techniques like filtering, enhancement, smoothing, sharpening etc. which are normally accomplished either in spatial or frequency domain may also carried out by morphological operations. Erosion and dilation are the two basic operations that are associated with translation, reflection and rotation process of the test image and structuring element and these are linear or bilinear in nature. Although the overall effect of morphological operations are non linear signal processing. Phenomena, the constituent sub-processes associated with or related to the linear, bilinear or conformal transformation. As it is evident from conformal transformation, the one to one mapping is maintained. It may be considered that after the mathematical morphological processes are carried out, the ultimate changes that occur between input test image and output image (i.e. after morphological operation) are simple one to one mapping prevalent in conformal transformation. In the latter, the image may require to be transformed between real plane to complex plane but the main image components will remain the same. The angles between two curve lines are also preserved during conformal transformation. The topographic space may be altered from Euclidean to other spaces by conformal transformation.

VI. CONCLUSION

Mathematical morphological processes are dealt as a part of nonlinear signal processing area and this concept is widely in vogue. The present study has meticulously analyzed the every sub processes that are involved to materialize the image segmentation and established that the constituent steps are mainly linear, bilinear and conformal transformation. The inverse transformations may be applied for implementing the segmentation of various images. The given explanation also holds good for all types of images.

DEDICATION

One of the others (Dibeyendu Ghoshal) dedicates the entire study to the loveliest and loving memory of his only one and younger sister Kumari Sumita Ghoshal who herself was a gem of the scholars, a symbol of wisdom and art, peerless beauty and simplicity, unfathomable knowledge and generosity.

REFERENCES

[1] Milan Sonka et. al. Image Processing, Analysis and Machine Vision. PWS Publishing, second edition, 1999.
[2] Pierre Soille, Morphological Image Analysis. Springer-Verlag, 2003.
[3] Rafael C. Gonzalez, Richard E Woods. Digital Image Processing. Prentice Hall, second edition, 2002.
[4] P. Suetnes, P.Fua and A. J. Hanson, "Computational strategies for object recognition," ACM Computing Surveys, Vol. 24, pp. 05-61, 1992.
[5] R. Besl, R. Jain, "Three dimensional object recognition," ACM Computing Surveys, Vol. 17, pp. 75-145, 1985.
[6] K. Hohne, H. Fuchs, S. Pizer, "3D imaging in medicine: Algorithms, systems, Applications", Berlin, Germany, Springer –Verlag, 1990.
[7] M. Kunt, M. Bernard and R. Leonardi, "Recent results in high compression image coding," IEEE Trans. on Circuits and Systems, Vol.34, pp.1306-1336, 1987.
[8] M. Bomans, K. Hohne, U. Tiede and M. Riemer, "3D segmentation of MR images of the head for 3D display," IEEE Transactions on Medical imaging, Vol.9, pp. 253-277, 1990.
[9] Phani krishna kishore M and Venugopal IVS, "A Cryptographic Algorithm based on Bilinear Transformation", International Journal of Computer Science Issues, Vol. 8, Issue 5, No 2, pp. 298-302, September 2011.
[10] R K jain and S R K lyengar, Advanced engineering mathematics, Narosa publication, pp. 14.1-14.28.
[11] E.P. Dolzhenko (2001), "Conformal mapping", in Hazewinkel, Michiel, Encyclopedia of Mathematics, Springer, ISBN 978-1-55608-010-4.
[12] Dai Okano, Hidenori Ogata, Kaname Amano, Masaaki Sugihara, "Numerical conformal mappings of bounded multiply connected domains by the charge simulation method", Journal of Computational and Applied Mathematics, pp. 109117, 2003.
[13] M.E.Klonowska and W.J.Prosnak," On an effective method forconfor malmapping of multiply connected domains", ActaMechanica, pp. 3552, 1996.
[14] A.P. Ilayaraja, M. P. Gopinath, "An Analysis of Morphological Operation Using Arbitrary Structuring Element", International Journal of Modern Engineering Research, Vol.2, Issue.2, pp-354-357, 2012.
[15] S. Thilagamani, N.Shanthi, "A Novel Recursive Clustering Algorithm for Image Oversegmentation", European Journal of Scientific Research, Vol.52, No.3, pp.430-436, 2011.
[16] S. Beucher, "Watershed, hierarchical segmentation and water fall algorithm," in Mathematical Morphology and Its Applications to Image Processing, Dordrecht, The Netherlands: Kluwer, 1994, pp. 69–76.
[17] Beucher, S., and Meyer, F. The morphological approach to segmentation: the watershed transformation. In Mathematical Morphology in Image Processing, E. R. Dougherty, Ed. Marcel Dekker, New York, ch. 12, pp. 433-481, 1993.
[18] F. Meyer, S. Beucher, "Morphological Segmentation," Journal of Visual Communication and Image Representation,vol. 1, pp. 21-46, 1990.
[19] C. R. Giardina and E. R. Dougherty, Morphological Methods in Image and Signal Processing, Prentice-Hall, Upper Saddle River, NJ, USA, 1988.
[20] Santiago Velasco-Forero, Student Member, IEEE, and Jesus Angulo, "Random Projection Depth for Multivariate Mathematical Morphology", IEEE Journal Of Selected Topics In Signal Processing, Vol. 6, No. 7, November 2012.
[21] S. Velasco-Forero and J. Angulo, "Mathematical morphology for vector images using statistical depth," in Mathematical Morphology and Its Applications to Image and Signal Processing, ser. Lecture Notes in Computer Science, P. Soille, M. Pesaresi, and G. Ouzounis, Eds. Berlin/Heidelberg, Germany: Springer, 2011, vol. 6671, pp.355–366.
[22] Rafael C. Gonzalez, Richard E. Woods, Steven L. Eddins, "Digital Image Processing Using MATLAB," Second Edition, Gatesmark Publishing, 2009.
[23] P. Jackway, "Gradient watersheds in morphological scalespace," IEEE Trans. Image Processing vol. l5, pp. 913–921, June, 1996.
[24] P. Soille, Morphological Image Analysis. New York: Springer Verlag, 1999.
[25] Image Analysis and Mathematical Morphology by Jean Serra, ISBN 0-12-637240-3, 1982.
[26] Image Analysis and Mathematical Morphology, Volume 2: Theoretical Advances by Jean Serra, ISBN 0-12-637241-1, 1988.
[27] An Introduction to Morphological Image Processing by Edward R. Dougherty, ISBN 0-8194-0845-X, 1992.