

IFS on the Multi-Fuzzy Fractal Space

Nadia M. G. AL-Sa'idi, Muhammad Rushdan Md. Sd., and Adil M. Ahmed

Abstract—The IFS is a scheme for describing and manipulating complex fractal attractors using simple mathematical models. More precisely, the most popular “fractal-based” algorithms for both representation and compression of computer images have involved some implementation of the method of Iterated Function Systems (IFS) on complete metric spaces. In this paper a new generalized space called Multi-Fuzzy Fractal Space was constructed. On these spaces a distance function is defined, and its completeness is proved. The completeness property of this space ensures the existence of a fixed-point theorem for the family of continuous mappings. This theorem is the fundamental result on which the IFS methods are based and the fractals are built. The defined mappings are proved to satisfy some generalizations of the contraction condition.

Keywords—Fuzzy metric space, Fuzzy fractal space, Multi fuzzy fractal space.

I. INTRODUCTION

REPRESENTATION as well as compression of a computer images involves implementation to the methods of Iterated Function Systems (IFS) on complete metric space [1]. In 1981, Hutchinson [17] introduced the formal definition of iterated function systems (IFS). These methods are useful tools to build fractals and other similar sets. In many scientific and engineering applications the fuzzy set concept plays an important role [15]. The fuzziness appears when we need to perform, calculations with imprecision variable on a manifold. The concept of fuzzy sets was introduced initially by Zadeh [16] in 1965 as an extension to the classical notion of set. Since then, various concepts of fuzzy metric spaces were considered. George and Veeramani [14] and Kramosil and Michalek [12] have introduced the concept of fuzzy metric space which has important application in quantum particle physics. Fadel S.F. [2], in his thesis generalized the fixed-point theorem to IFS. Fixed-point theory of contractive type mappings in fuzzy metric spaces for different is closely related to that in probabilistic metric spaces [13].

Nadia M. G. AL-Sa'idi, is with Institute for Mathematical Research (INSPEM) ,University Putra Malaysia(UPM), 43400, Serdang, Selangor, Malaysia (phone: 03-8946 6841, fax: 03-8942 3789, H/P: 0060162144183, e-mail: nadia_alsaidi@hotmail.com, m_nadia@putra.upm.edu.my).

Muhammad Rushdan Md. Sd., is with Institute for Mathematical Research (INSPEM) ,University Putra Malaysia(UPM), 43400, Serdang, Selangor, Malaysia (phone: 03-8946 6841, fax: 03-8942 3789, H/P: 006012-216 5335, e-mail: mrushdan@putra.upm.edu.my).

Adil M. Ahmed is with Mathematical Department, IBB University, Yemen, H/P:00967711770370 (e-mail: adilm_ahmed@yahoo.com).

In this paper, new fuzzy metric space are introduced and used to construct fuzzy fractal space, some of their properties are studied in terms of their completeness to ensure the existence and uniqueness of a fixed point for a family of continuous mappings. A generalization of the theory of fuzzy metric space to construct a new space called Multi-Fuzzy Fractal Space is presented.

The layout of this paper is organized as follows. In section II, some background material is included to assist readers less familiar with the detailed exposition of the fundamentals of metric space and Hausdorff space, and IFS concepts, to consider. Section III, is devoted to the fundamental result on fuzzy fractal space, the uniform structure of this space and the theory of a fuzzy contractive mapping. In section IV, the structure of the multi-fractal space is drawn, and then the concept of IFS is applied. In section V, new space called the multi-fuzzy fractal space is constructed, proving its completeness and an extension of the contraction property (which represents the fundamental idea of this paper) is proved. Conclusions are drawn in section VI.

II. PRELIMINARIES

In this section, some definitions and theories of metric space and Hausdorff space are given to fill in some background for the reader. A more detailed review of the topics in this section was given in [6]

Definition 1. Let (X,d) be a complete metric space. The Iterated Function System (IFS) is a set of n contraction maps $W=\{w_1,w_2,\dots,w_n\}$, $w_i : X \rightarrow X$.

The space of fractal denoted by $H(X)$ is the place where fractals live. The elements of this space are the non-empty compact subsets of the complete metric space (X,d) . There is a relation between the spaces X and $H(X)$, since a point A in $H(X)$ corresponds to a compact subset in X [3]. In order to show that $H(X)$ is a metric space we must define a distance function between any two compact sets called Hausdorff distance defined as follows.

Definition 2. $D(A,B)=\max \{d(A,B),d(B,A)\}$, $(A,B \in H(X)) = \max\{\max_x \min_y d(x,y), \max_y \min_x d(x,y) : x \in A, y \in B\}$

where $d(A,B)=\max\{d(x,B):x \in A\}$ and $d(x,B)=\min\{d(x,y):y \in B\}$.

D defined in this form is a metric function, and $(H(X),D)$ is a metric space. The fixed point theorem is proved in [6], after proving the completeness of $(H(X), D)$.

Definition 3. Let (X,d) be a complete metric space. A mapping $f: X \rightarrow X$ is called a contraction mapping (or

contraction on X) if there exists a real number r with $0 \leq r < 1$ such that $d(f(x), f(y)) \leq rd(x, y)$ for every $x, y \in X$.

Let (X, d) be a complete metric space. Then $(H(X), D)$ is a complete metric space. Moreover if $\{A_n \in H(X)\}_{n=1}^\infty$ is a Cauchy sequence, then $A = \lim_{n \rightarrow \infty} A_n \in H(X)$, where $A = \{x \in X : \text{there is a Cauchy sequence } \{x_n \in A_n\} \text{ that converges to } x\}$.

Definition 4 [4]. The transformation $W: H(X) \rightarrow H(X)$, is defined, where $W(B) = \bigcup_{n=1}^N w_n(B)$ for any $B \in H(X)$, it is easily shown that W is a strict contraction, with contractivity factor $s = \max_{1 \leq n \leq N} \{s_n\}$. According to the contraction mapping theorem, if (X, d) is complete, then W has a unique fixed point A in $H(X)$, satisfying the remarkable self-covering condition, $A = W(A) = \bigcup_{n=1}^N w_n(A)$.

III. FUZZY FRACTAL SPACE

A fuzzy singleton as introduced by Zadeh [16], is an ordered pair (x, t) , where $x \in X$ and $t \in I$. These singleton represent the element of the set $\tilde{X} = X \times I$. A new fuzzy space is constructed depending on the elements of this set, which is defined as follows. Most of the definitions and concepts in this section can be found in [11].

Definition 1. Let (X, d) be a metric space. Let \tilde{X} be a set consisting of all fuzzy singleton (x, t) defined as a mapping $x_t: X \rightarrow [0, 1]$ where $x \in X$ and $t \in [0, 1]$, associating with each element x_t its grade of membership μ_{x_t} , defined as a nonempty subset. $x_t = \{(x, \mu_{x_t}) : x \in X, t \in I\}$, these elements are defined also as:

$$x_t(u) = \begin{cases} 0 & x \neq u \\ t & x = u \end{cases}$$

Definition 2. A distance function d^* is defined as $d^*(x_t, y_s) = \max\{d(x, y), |t - s|\}$, $x_t, y_s \in \tilde{X}$. It's easy to show that d^* satisfies the following axioms:-

- $d^*(x_t, y_s) \geq 0 \quad \forall x_t, y_s \in \tilde{X}$
- $d^*(x_t, y_s) = 0$ iff $x_t = y_s$
- $d^*(x_t, y_s) = d^*(y_s, x_t) \quad \forall x_t, y_s \in \tilde{X}$
- $d^*(x_t, y_s) \leq d^*(x_t, z_p) + d^*(z_p, y_s), \quad \forall x_t, y_s, z_p \in \tilde{X}$

Definition 3. The space, $F(X) \subseteq P(\tilde{X})$ where $F(X) = \{A \in P(\tilde{X}) \mid A: X \rightarrow I, A(x) = t\}$, and $P(\tilde{X})$ is the power set of $X \times I$, is called fuzzy space. The space $(F(X), d^*)$ is a complete metric space.

Definition 4. $w^*: \tilde{X} \rightarrow \tilde{X}$, is defined as $w^*(x, t) = (w(x), rt)$.

Theorem 5. w^* is a contraction mapping on the space $(F(X), d^*)$.

Proof. $d^*(w^*(x, t), w^*(y, s)) = d^*((w(x), rt), (w(y), rs))$
 $= \max\{d(w(x), w(y)), |rt - rs|\}$
 $\leq \max\{rd(x, y), r|t - s|\}$
 $\leq r \max\{d(x, y), |t - s|\}$
 $\leq r d^*(x, t), (y, s) \quad \square$

Definition 6. The space $H(F(X)) = \{A \in F(X) \mid A \neq \emptyset, A \text{ compact}\} \subseteq F(X)$, is called fuzzy Hausdorff space or fuzzy fractal space.

Definition 7. The distance function D^* on $H(F(X))$, is defined as follows.

$$D^* = \max \left\{ \begin{aligned} & \sup_{x \in X} \inf_{y \in Y} \max \{d(x, y), |A(x) - B(y)|\}, \\ & \sup_{y \in Y} \inf_{x \in X} \max \{d(x, y), |A(x) - B(y)|\} \end{aligned} \right\}$$

Definition 8. Let $w^*: H(F(X)) \rightarrow H(F(X))$ be defined as $w^*(A) = \{w^*(x, t) \mid (x, t) \in A\}$, $A = \{(x, A(x)) \mid x \in X\}$, and $w^*(A) = \{w^*(x, rt) \mid (x, t) \in A\} = \{(w(x), rA(x)) \mid x \in X\} \in F(X)$.

Theorem 9. $w^*: H(F(X)) \rightarrow H(F(X))$, is a contraction mapping on the space $(H(F(X)), D^*)$.

Proof. $D^*(w^*(A), w^*(B)) = D^*((w(x), rA(x)), (w(y), rB(y)))$

$$= \max \left\{ \begin{aligned} & \sup_x \inf_y \max \{d(w(x), w(y)), |rA(x) - rB(y)|\}, \\ & \sup_y \inf_x \max \{d(w(x), w(y)), |rA(x) - rB(y)|\} \end{aligned} \right\}$$

$$\leq \max \left\{ \begin{aligned} & \sup_x \inf_y \max \{rd(x, y), r|A(x) - B(y)|\}, \\ & \sup_y \inf_x \max \{rd(x, y), r|A(x) - B(y)|\} \end{aligned} \right\}$$

$$= r \max \left\{ \begin{aligned} & \sup_x \inf_y \max \{d(x, y), |A(x) - B(y)|\}, \\ & \sup_y \inf_x \max \{d(x, y), |A(x) - B(y)|\} \end{aligned} \right\}$$

$$= r D^*(A, B)$$

Therefore w^* is a contraction mapping on the space $(H(F(X)), D^*)$. \square

Let $(F(X), w_1^*, w_2^*, \dots, w_n^*)$ be a hyperbolic fuzzy dynamical system, where $(F(X), d^*)$ is a complete metric space, and each $w_r^*: F(X) \rightarrow F(X)$, is a contractive function with corresponding contractivity factor $s_r, r = 1, 2, \dots, N$.

Definition 10. Let $W^*: H(F(X)) \rightarrow H(F(X))$, be defined as:

$$W^*(B) = \bigcup_{i=1}^n w_i^*(B) \text{ for any } B \in H(F(X), B = \{(x, B(x)) \mid x \in X\}.$$

Theorem 11. W^* is a contraction mapping provided w^* is a contraction.

Proof. Let $A, B \in H(F(X))$, $q^* > D^*(A, B)$ be given, if x_t is any element of $W^*(A)$, then

$$(\forall x_t)(x_t \in w_i^*(A)), (\exists i)(\exists x'_t), (x'_t \in A) \text{ such that } w_i^*(x'_t) = x_t$$

Since $D^*(A, B) < q \Rightarrow (\exists y'_s)(y'_s \in B), \exists d^*(x'_t, y'_s) < q^*$.

$$\Rightarrow (\exists y_s), y_s \in w_i^*(B), \exists w_i^*(y'_s) = y_s, \text{ then}$$

$$\begin{aligned} d^*(x_t, y_s) &= d^*(w_i^*(x'_t), w_i^*(y'_s)) \\ &= \max\{d(w(x'), w(y')), |rt - rs|\} \\ &\leq \max\{rd(x', y'), r/t - s\} \\ &\leq r \max\{d(x', y'), |t - s|\} \\ &= rd^*(x'_t, y'_s) \\ &\leq rq^* \end{aligned}$$

this is true for all $x_t \in W^*(A)$, so $W^*(A)$ is in the rq -neighborhood $W^*(B)$, also if y_s is any element of $W^*(B)$, then $(\forall y_s)(y_s \in w_i^*(B)), (\exists i)(\exists y'_s), (y'_s \in B)$ such that

$$w_i^*(y'_s) = y_s,$$

since $D^*(A, B) < q \Rightarrow (\exists x'_t)(x'_t \in A), \exists d^*(x'_t, y'_s) < q^*$.

$$\Rightarrow (\exists x_t), x_t \in w_i^*(A), \exists w_i^*(x'_t) = x_t, \text{ then}$$

$$\begin{aligned} d^*(x_t, y_s) &= d^*(w_i^*(x'_t), w_i^*(y'_s)) \\ &= \max\{d(w(x'), w(y')), |rt - rs|\} \\ &\leq \max\{rd(x', y'), r/t - s\} \\ &\leq r \max\{d(x', y'), |t - s|\} \\ &= rd^*(x'_t, y'_s) \\ &\leq rq^* \end{aligned}$$

this is true for all $y_s \in W^*(B)$, so $W^*(B)$ is in the rq -neighborhood of $W^*(A)$, therefore $D^*(W^*(A), W^*(B)) \leq rq^*$, and this is true for every $q^* > D^*(A, B)$, which implies $D^*(W^*(A), W^*(B)) \leq D^*(A, B)$, and W^* is a contraction in the complete metric space $H(F(X))$. \square

IV. MULTI FRACTAL SPACE

Let J be a set, assume that for each $j \in J$ we have a metric space (X_j, d_j) . The product space $\mathcal{X} = \prod_{j \in J} X_j$ is the space that

consists of all J -tuples $(X_j)_{j \in J}$ with $x_j \in X_j$, define $d: X \times X \rightarrow R$ as follows: If $X = (x_1, x_2, \dots, x_N)$ and $Y = (y_1, y_2, \dots, y_N)$ are elements of X , then $d(X, Y) = \max_{i=1, \dots, N} d_i(x_i, y_i)$. It is easy to verify that d

satisfies the requirements to be a distance function, if d represents the product metric on X then (X, d) is a metric space. This result follows also from the following theorem.

Theorem 1. [10] The product of a finite number of metric spaces is a metric spaces.

Let (X_i, d_i) , be complete metric spaces, for $i=1, 2, \dots, N$. Let

$$\mathcal{X} = \prod_{i=1}^N X_i = X_1 \times X_2 \times \dots \times X_N, (\mathcal{X}, d) \text{ be a complete metric}$$

space. For each complete metric space $((X_i, d_i), i=1, \dots, N)$ there is an associated Hausdorff space $((H(X_i), D_i), i=1, \dots, N)$, which is also complete metric spaces.

Definition 2. Let $\mathcal{X}(X) = H(X_1) \times H(X_2) \times \dots \times H(X_N)$ then $\mathcal{X}(X)$ is a Hausdorff space (since the product of Hausdorff spaces is a Hausdorff space) [4]. To prove that $\mathcal{X}(X)$ is a metric space the following steps are important, let $A = (A_1, A_2, \dots, A_N)$ and $B = (B_1, B_2, \dots, B_N)$ are element of $H(X)$.

Definition 3. Let [9] $\mathcal{D}(A, B) = \max_{1 \leq i \leq N} \{D_i(A_i, B_i)\}$, where $A, B \in H(X)$ (each of A & B are an n -tuple of elements of $H(X_i)$), \mathcal{D} satisfies the requirements to be a distance function, so $(\mathcal{X}(X), \mathcal{D})$ is a metric space, and moreover it is a complete metric space.

Let $(X; w_1, w_2, \dots, w_N)$ be a hyperbolic IFS on a complete metric space X . then there exists unique $A \subseteq X$ such that

$$A = \bigcup_{i=1}^N w_i(A).$$

Let $(X_i, d_i), i=1, 2, \dots, N$, be complete metric space. Let $w_{ij}^k: X_j \rightarrow X_i, k=1, 2, \dots, m(i, j), i, j=1, 2, \dots, N$ be contraction mapping. Then there exists a unique $A = (A_1, \dots, A_N)$ such that

$$A_i = \bigcup_{j=1}^N \bigcup_{k=1}^{m(i, j)} w_{ij}^k(A_j), \quad i=1, 2, \dots, N \quad \text{and} \quad \text{hence}$$

$$\left(\prod_{i=1}^N A_i = \prod_{i=1}^N \bigcup_{j=1}^N \bigcup_{k=1}^{m(i, j)} w_{ij}^k(A_j) \right).$$

Definition 4. [12]. Let $\mathcal{W}^\circ: \prod_{i=1}^N H(X_i) \rightarrow \prod_{i=1}^N H(X_i)$ be defined by:

$$\mathcal{W}^\circ(B) = \prod_{i=1}^N \left(\bigcup_{j=1}^N \bigcup_{k=1}^{m(i, j)} w_{ij}^k(B_j) \right)$$

Theorem 5. [12] \mathcal{W} is a contraction mapping on the complete metric space $(\prod_{i=1}^N H(X_i), \mathcal{D})$, where

$$\mathcal{W}^\circ \left(\prod_{i=1}^N B_i \right) = \prod_{i=1}^N \left(\bigcup_{j=1}^N \bigcup_{k=1}^{m(i, j)} w_{ij}^k(B_j) \right).$$

V. MULTI-FUZZY FRACTAL SPACE

In this section we define a new space called "multi-fuzzy fractal space" as a generalization of the fuzzy fractal space, and then we give some definitions and prove some results about this space. Let $(F(X_i), D_i), i=1, 2, \dots, N$ be a complete metric space. Let $w_{ij}^{*k}: F(X_j) \rightarrow F(X_i) (k=1, 2, \dots, m(i, j))$,

$i, j=1, 2, \dots, N$ be a contraction mapping such that

$$A_i = \bigcup_{j=1}^N \bigcup_{k=1}^m w_{ij}^{*k} (A_j) \text{ for each } i=1, 2, \dots, N$$

Definition 1. A multi-fuzzy fractal space is defined as $H(F(X)) = \prod_{i=1}^N H(F(X_i)) = H(F(X_1)) \times H(F(X_2)) \times \dots \times H(F(X_N))$

Definition 2. A distance function on this space defined as.

$$\mathcal{D}^*(A, B) = \max \{D_i^*(A, B) : i = 1, \dots, N\}, (A, B \in H(F(X)) = \prod_{i=1}^N H(F(X_i)))$$

then $\mathcal{H}(F(X))$ with \mathcal{D}^* is a complete metric space.

Definition 3. Define $W^* : \prod_{i=1}^N H(F(X_i)) \rightarrow \prod_{i=1}^N H(F(X_i))$ such

$$\text{that } W^*(B) = \prod_{i=1}^N \bigcup_{j=1}^N \bigcup_{k=1}^m w_{ij}^{*k} (B_j)$$

Theorem 4. W^* is a contraction mapping on the complete metric space $(\mathcal{H}(F(X)), \mathcal{D}^*)$ if it satisfies, $\mathcal{D}^*(W^*(A), W^*(B)) \leq r \mathcal{D}^*(A, B)$ where $\mathcal{D}^* = \max\{D_i^*\}$ and D_i^* is a distance function defined on the space $H(F(X_i))$

Proof. $W^*(A) = W^*(\prod_{i=1}^N A_i) = W^*(A_1, \dots, A_N) =$

$$\left(W_1^*(A_1, \dots, A_N), W_2^*(A_1, \dots, A_N), \dots, W_N^*(A_1, \dots, A_N) \right)$$

where $W_i^*(\prod_{j=1}^N B_j) = \prod_{j=1}^N \bigcup_{k=1}^m w_{ij}^{*k} (B_j)$,

Let $A, B \in H(F(X)) = \prod_{i=1}^N H(F(X_i))$

where $A = \prod_{i=1}^N A_i$, and $B = \prod_{i=1}^N B_i$

Let $q^* > \mathcal{D}^*(A, B)$ be given, then

$$(\forall X_t)(X_t = (x_{t_1}^1, x_{t_2}^2, \dots, x_{t_N}^N) = \prod_{i=1}^N x_{t_i}^i = \prod_{i=1}^N (x_i, t_i))$$

where $x_{t_i}^i = (x_i, t_i) \in H(F(X_i))$,

$$X_t \in W^*(A) = \prod_{i=1}^N \bigcup_{j=1}^N \bigcup_{k=1}^m w_{ij}^{*k} (A_j)$$

$$(\forall_i)(x_{t_i}^i \in \bigcup_{j=1}^N \bigcup_{k=1}^m w_{ij}^{*k} (A_j)), (\exists j)(\exists k), \text{ such that}$$

$$(\exists x_{t_j}^{i'} \in A_j), \exists (w_{ij}^{*k}(x_{t_j}^{i'}) = x_{t_i}^i) \text{ since } \mathcal{D}^*(A, B) < q^*, \text{ then}$$

$$(\forall_i)(\exists y_{s_i}^{i'})(y_{s_i}^{i'} \in B_i), \exists (d^*(x_{t_i}^{i'}, y_{s_i}^{i'}) < q_i^*) \Rightarrow$$

$$\exists y_{s_i}^{i'} \in \bigcup_{j,k} w_{ij}^{*k} (B_j),$$

$$X_t = \prod_{i=1}^N x_{t_i}^i$$

$$\begin{aligned} d^*(X_t, Y_s) &= \max_{1 \leq i \leq N} d^*(x_{t_i}^i, y_{s_i}^i) \\ &= \max (d^*(w_{ij}^{*k}(x_{t_j}^{i'}), w_{ij}^{*k}(y_{s_i}^{i'}))) \\ &\leq \max_{1 \leq i \leq N} \{d(w_{ij}^k(x_{t_j}^{i'}), w_{ij}^k(y_{s_i}^{i'})) / r_{ij} t_j - r_{ij} s_j\} \\ &\leq \max_{1 \leq i \leq N} \{r_{ij} d(x_{t_j}^{i'}, y_{s_i}^{i'}) / t_j - s_j\} \\ &\leq r_{ij} \max_{1 \leq i \leq N} \{d(x_{t_j}^{i'}, y_{s_i}^{i'}) / t_j - s_j\} \\ &\leq r_{ij} d^*(x_{t_j}^{i'}, y_{s_i}^{i'}) \\ &\leq r_i q_i^* \text{ where } r_i = \max_j \{r_{ij}\} \\ &\leq r q^* \text{ for all } r = \max_i \{r_i\} \end{aligned}$$

which implies that $W^*(A)$ is in the $r q^*$ -neighborhood of $W^*(B)$. By the same steps we have,

$$(\forall Y_t)(Y_t = (y_{t_1}^1, y_{t_2}^2, \dots, y_{t_N}^N) = \prod_{i=1}^N y_{t_i}^i = \prod_{i=1}^N (y_i, s_i))$$

$$y_{t_i}^i = (y_i, s_i) \in H(F(X_i)), Y_t \in W^*(A) = \prod_{i=1}^N \bigcup_{j=1}^N \bigcup_{k=1}^m w_{ij}^{*k} (A_j)$$

$$(\forall_i)(y_{t_i}^i \in \bigcup_{j=1}^N \bigcup_{k=1}^m w_{ij}^{*k} (A_j)), (\exists j)(\exists k), \text{ such that}$$

$$(\exists y_{t_j}^{i'} \in B_j) \exists (w_{ij}^{*k}(y_{t_j}^{i'}) = y_{t_i}^i), \text{ since } \mathcal{D}^*(A, B) < q^*, \text{ then}$$

$$(\forall_i)(\exists x_{t_j}^{i'})(x_{t_j}^{i'} \in A_j), \exists (d^*(x_{t_j}^{i'}, y_{s_i}^{i'}) < q_i) \Rightarrow \exists x_{t_i}^i \in \bigcup_{j,k} w_{ij}^{*k} (A_j)$$

$$Y_s = \prod_{i=1}^N y_{s_i}^i$$

$$\begin{aligned} d^*(X_t, Y_s) &= \max_{1 \leq i \leq N} d^*(x_{t_i}^i, y_{s_i}^i) \\ &= \max (d^*(w_{ij}^{*k}(x_{t_j}^{i'}), w_{ij}^{*k}(y_{s_i}^{i'}))) \\ &\leq \max_{1 \leq i \leq N} \{d(w_{ij}^k(x_{t_j}^{i'}), w_{ij}^k(y_{s_i}^{i'})) / r_{ij} t_j - r_{ij} s_j\} \\ &\leq \max_{1 \leq i \leq N} \{r_{ij} d(x_{t_j}^{i'}, y_{s_i}^{i'}) / t_j - s_j\} \\ &\leq r_{ij} \max_{1 \leq i \leq N} \{d(x_{t_j}^{i'}, y_{s_i}^{i'}) / t_j - s_j\} \\ &\leq r_{ij} d^*(x_{t_j}^{i'}, y_{s_i}^{i'}) \\ &\leq r_i q_i^* \text{ where } r_i = \max_j \{r_{ij}\} \\ &\leq r q^* \text{ for all } r = \max_i \{r_i\} \end{aligned}$$

which implies that $\mathcal{W}^*(B)$ is in the $r q^*$ -neighborhood of $\mathcal{W}^*(A)$ and hence, $\mathcal{S}^*(\mathcal{W}^*(A), \mathcal{W}^*(B)) \leq r q$, which implies $\mathcal{S}^*(\mathcal{W}^*(A), \mathcal{W}^*(B)) < r \mathcal{S}^*(A, B)$. Therefore, \mathcal{W}^* is a contraction with contractivity factor, $r = \max\{r_1, r_2, \dots, r_N\}$. \square

By the contraction mapping theorem on the space $(\mathcal{A}(F(X)), \mathcal{S}^*)$, has a unique fixed point A , i.e. $\mathcal{W}^*(A) = A$.

VI. CONCLUSION

New fuzzy metric spaces are introduced and used to construct fuzzy fractal space; some of their properties are studied in term of their completeness to ensure the existence and uniqueness of a fixed point for a family of continuous mappings. A generalization of the theory of fuzzy metric space to construct a new space called Multi-Fuzzy Fractal Space is provided.

ACKNOWLEDGMENT

The researchers would like to acknowledge the Institute for Mathematical Research (INSPERM), University Putra Malaysia (UPM) for its continuous support.

REFERENCES

- [1] B. Forte, E.R. Vrscay. Theory of generalized fractal transforms. *Fractal Image Encoding and Analysis*, July, 1995.
- [2] F.S. Fadhel. About fuzzy fixed point theorem. Ph.D. thesis, Al-Nahreen University, Baghdad, Iraq, 1998.
- [3] G.A. Edger, J. Golds. A fractal dimension estimate for a graph-directed iterated function system of non-similarities. *Indiana University Mathematics Journal*, vol.48, no.2, pp. 429-447, 1999.
- [4] G.A. Edger. *Measure, topology, and fractal geometry*. New York: Springer-Verlag, 1990.
- [5] L. Zoushu, Z. Quan. Fuzzy metric spaces and fixed point theorems. *J. Fuzzy Math.*, vol.1, no.2, 1993.
- [6] M.F. Barnsley. *Fractals everywhere*. 2nd ed. Academic Press Professional, Inc., San Diego, CA, USA, 1993.
- [7] P.E. Kloeden. Fuzzy dynamical systems. *Fuzzy Sets and Systems*, vol. 7, no. 3, pp. 275-296, 1982.
- [8] V. Gregori, A. Sapena. On fixed point theorem in fuzzy metric spaces. *Fuzzy Sets and Systems*, vol. 125, no. 2, pp. 245-252, 1998.
- [9] W. F. Al-Shameri. On nonlinear iterated function systems. Ph.D. thesis, Al-Mustansiriyah University, Baghdad, Iraq, 2001.
- [10] W. Rudin. *Principles of mathematical analysis*. 3rd ed., McGraw-Hill Book Co., 1976.
- [11] N. Al-Saidi. On multi fuzzy metric space. Ph.D. thesis, Al-Nahreen University, Baghdad, Iraq, 2002.
- [12] O. Kramsil, J. Michalek. Fuzzy metric and statistical metric spaces, *Kybernetika*, vol. 11, pp. 326-334, pp.1975.
- [13] M. Dorel. On fuzzy contractive mappings in fuzzy metric spaces, *Fuzzy Set and Systems*, vol. 158, no. 8, pp. 915-921, 2007.
- [14] A. George. P. Veeramani. On some result in fuzzy metric spaces. *Fuzzy Sets and Systems*, vol.64, pp. 395-399, 1994.
- [15] H.M. Abu-Donia. Common fixed point theorems for fuzzy mappings under ϕ -contraction condition. *Chaos, Solitons and Fractals*. Vol.34, pp. 538-543, Oct. 2007.
- [16] L.A. Zadah. Fuzzy sets. *Inform Control*, vol. 8, pp. 338-353, 1965.
- [17] J. Hutchinson. Fractals and self-similarity. *Indiana Univ. J.Math.* vol. 30, no. 5, pp. 713-747, 1981.



Nadia M. G. Al-Saidi was born in Iraq-Meysan 1967. She completed her Bachelor of Science and Master of Science degrees in applied mathematics, from applied sciences department, university of technology, Baghdad, Iraq, in 1989, and 1995, respectively. She received her Ph.D. degree in mathematics and computer application sciences from Al-Nahreen University, Baghdad, Iraq 2003.

In 1989 she joined the applied sciences department, university of technology, as a staff member, then from 1995-1999 as a lecturer, where she is now Associate Professor, Recently she joined the Institute for Mathematical Research (INSPERM), University Putra Malaysia (UPM) as a post doctorate researcher. Her research interests are on fractals and Cryptography.

Dr. Nadia is the author of numerous technical papers. Since 1994 she has been a member of the Editorial board of International Journal of Advancement in Computing Technology (IGACT).