

I-Vague Normal Groups

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Abstract—The notions of I-vague normal groups with membership and non-membership functions taking values in an involutory dually residuated lattice ordered semigroup are introduced which generalize the notions with truth values in a Boolean algebra as well as those usual vague sets whose membership and non-membership functions taking values in the unit interval $[0, 1]$. Various operations and properties are established.

Keywords—Involutory dually residuated lattice ordered semigroup, I-vague set, I-vague group and I-vague normal group.

I. INTRODUCTION

VAGUE groups are studied by M. Demirci[2]. R. Biswas[1] defined the notion of vague groups analogous to the idea of Rosenfeld [4]. He defined vague normal groups of a group and studied their properties. N. Ramakrishna[3] studied vague normal groups and introduced vague normalizer and vague centralizer.

In his paper, T. Zelalem [9] studied the concept of I-vague groups. In this paper using the definition of I-vague groups, we defined and studied I-vague normal groups where I is an involutory DRL-semigroup. To be self contained we shall recall some basic results in [5], [6], [7], [9] in this paper.

II. DUALY RESIDUATED LATTICE ORDERED SEMIGROUP

Definition 2.1: [5] A system $A = (A, +, \leq, -)$ is called a dually residuated lattice ordered semigroup (in short DRL-semigroup) if and only if

- $A = (A, +)$ is a commutative semigroup with zero "0";
- $A = (A, \leq)$ is a lattice such that

$$a + (b \cup c) = (a + b) \cup (a + c) \quad \text{and} \quad a + (b \cap c) = (a + b) \cap (a + c) \quad \text{for all } a, b, c \in A;$$
- Given $a, b \in A$, there exists a least x in A such that $b + x \geq a$, and we denote this x by $a - b$ (for a given a, b this x is uniquely determined);
- $(a - b) \cup 0 + b \leq a \cup b$ for all $a, b \in A$;
- $a - a \geq 0$ for all $a \in A$.

Theorem 2.2: [5] Any DRL-semigroup is a distributive lattice.

Definition 2.3: [10] A DRL-semigroup A is said to be involutory if there is an element $1 (\neq 0)$ (0 is the identity w.r.t. $+$) such that

- $a + (1 - a) = 1 + 1$;
- $1 - (1 - a) = a$ for all $a \in A$.

Theorem 2.4: [6] In a DRL-semigroup with 1 , 1 is unique.

Theorem 2.5: [6] If a DRL-semigroup contains a least element x , then $x = 0$. Dually, if a DRL-semigroup with 1 contains a largest element α , then $\alpha = 1$.

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Throughout this paper let $I = (I, +, -, \vee, \wedge, 0, 1)$ be a dually residuated lattice ordered semigroup satisfying $1 - (1 - a) = a$ for all $a \in I$.

Lemma 2.6: [10] Let 1 be the largest element of I . Then for $a, b \in I$

- $a + (1 - a) = 1$.
- $1 - a = 1 - b \iff a = b$.
- $1 - (a \cup b) = (1 - a) \cap (1 - b)$.

Lemma 2.7: [10] Let I be complete. If $a_\alpha \in I$ for every $\alpha \in \Delta$, then

- $1 - \bigvee_{\alpha \in \Delta} a_\alpha = \bigwedge_{\alpha \in \Delta} (1 - a_\alpha)$.
- $1 - \bigwedge_{\alpha \in \Delta} a_\alpha = \bigvee_{\alpha \in \Delta} (1 - a_\alpha)$.

III. I-VAGUE SETS

Definition 3.1: [10] An I-vague set A of a non-empty set G is a pair (t_A, f_A) where $t_A : G \rightarrow I$ and $f_A : G \rightarrow I$ with $t_A(x) \leq 1 - f_A(x)$ for all $x \in G$.

Definition 3.2: [10] The interval $[t_A(x), 1 - f_A(x)]$ is called the I-vague value of $x \in G$ and is denoted by $V_A(x)$.

Definition 3.3: [10] Let $B_1 = [a_1, b_1]$ and $B_2 = [a_2, b_2]$ be two I-vague values. We say $B_1 \geq B_2$ if and only if $a_1 \geq a_2$ and $b_1 \geq b_2$.

Definition 3.4: [10] An I-vague set $A = (t_A, f_A)$ of G is said to be contained in an I-vague set $B = (t_B, f_B)$ of G written as $A \subseteq B$ if and only if $t_A(x) \leq t_B(x)$ and $f_A(x) \geq f_B(x)$ for all $x \in G$. A is said to be equal to B written as $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Definition 3.5: [10] An I-vague set A of G with $V_A(x) = V_A(y)$ for all $x, y \in G$ is called a constant I-vague set of G .

Definition 3.6: [10] Let A be an I-vague set of a non empty set G . Let $A_{(\alpha, \beta)} = \{x \in G : V_A(x) \geq [\alpha, \beta]\}$ where $\alpha, \beta \in I$ and $\alpha \leq \beta$. Then $A_{(\alpha, \beta)}$ is called the (α, β) cut of the I-vague set A .

Definition 3.7: Let $S \subseteq G$. The characteristic function of S denoted as $\chi_S = (t_{\chi_S}, f_{\chi_S})$, which takes values in I is defined as follows:

$$t_{\chi_S}(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_{\chi_S}(x) = \begin{cases} 0 & \text{if } x \in S \\ 1 & \text{otherwise} \end{cases}$$

χ_S is called the I-vague characteristic set of S in I . Thus

$$V_{\chi_S}(x) = \begin{cases} [1, 1] & \text{if } x \in S; \\ [0, 0] & \text{otherwise} \end{cases}$$

Definition 3.8: [10] Let $A = (t_A, f_A)$ and $B = (t_B, f_B)$ be I-vague sets of a set G .

(i) Their union $A \cup B$ is defined as $A \cup B = (t_{A \cup B}, f_{A \cup B})$ where $t_{A \cup B}(x) = t_A(x) \vee t_B(x)$ and

$f_{A \cup B}(x) = f_A(x) \wedge f_B(x)$ for each $x \in G$.

(ii) Their intersection $A \cap B$ is defined as $A \cap B = (t_{A \cap B}, f_{A \cap B})$ where $t_{A \cap B}(x) = t_A(x) \wedge t_B(x)$ and $f_{A \cap B}(x) = f_A(x) \vee f_B(x)$ for each $x \in G$.

Definition 3.9: [10] Let $B_1 = [a_1, b_1]$ and $B_2 = [a_2, b_2]$ be I-vague values. Then

(i) $\text{isup}\{B_1, B_2\} = [\text{sup}\{a_1, a_2\}, \text{sup}\{b_1, b_2\}]$.

(ii) $\text{iinf}\{B_1, B_2\} = [\text{inf}\{a_1, a_2\}, \text{inf}\{b_1, b_2\}]$.

Lemma 3.10: [10] Let A and B be I-vague sets of a set G. Then $A \cup B$ and $A \cap B$ are also I-vague sets of G.

Let $x \in G$. From the definition of $A \cup B$ and $A \cap B$ we have

(i) $V_{A \cup B}(x) = \text{isup}\{V_A(x), V_B(x)\}$;

(ii) $V_{A \cap B}(x) = \text{iinf}\{V_A(x), V_B(x)\}$.

Definition 3.11: [10] Let I be complete and $\{A_i : i \in \Delta\}$ be a non empty family of I-vague sets of G where $A_i = (t_{A_i}, f_{A_i})$. Then

(i) $\bigcap_{i \in \Delta} A_i = (\bigwedge_{i \in \Delta} t_{A_i}, \bigvee_{i \in \Delta} f_{A_i})$

(ii) $\bigcup_{i \in \Delta} A_i = (\bigvee_{i \in \Delta} t_{A_i}, \bigwedge_{i \in \Delta} f_{A_i})$

Lemma 3.12: [10] Let I be complete. If $\{A_i : i \in \Delta\}$ is a non empty family of I-vague sets of G, then $\bigcap_{i \in \Delta} A_i$ and $\bigcup_{i \in \Delta} A_i$ are I-vague sets of G.

Definition 3.13: [10] Let I be complete and $\{A_i = (t_{A_i}, f_{A_i}) : i \in \Delta\}$ be a non empty family of I vague sets of G. Then for each $x \in G$,

(i) $\text{isup}\{V_{A_i}(x) : i \in \Delta\} = [\bigvee_{i \in \Delta} t_{A_i}(x), \bigvee_{i \in \Delta} (1 - f_{A_i})(x)]$.

(ii) $\text{iinf}\{V_{A_i}(x) : i \in \Delta\} = [\bigwedge_{i \in \Delta} t_{A_i}(x), \bigwedge_{i \in \Delta} (1 - f_{A_i})(x)]$.

IV. I-VAGUE GROUPS

Definition 4.1: [9] Let G be a group. An I-vague set A of a group G is called an I-vague group of G if

(i) $V_A(xy) \geq \text{iinf}\{V_A(x), V_A(y)\}$ for all $x, y \in G$ and

(ii) $V_A(x^{-1}) \geq V_A(x)$ for all $x \in G$.

Lemma 4.2: [9] If A is an I-vague group of a group G, then $V_A(x) = V_A(x^{-1})$ for all $x \in G$.

Lemma 4.3: [9] If A is an I-vague group of a group G, then $V_A(e) \geq V_A(x)$ for all $x \in G$.

Lemma 4.4: [9] A necessary and sufficient condition for an I-vague set A of a group G to be an I-vague group of G is that $V_A(xy^{-1}) \geq \text{iinf}\{V_A(x), V_A(y)\}$ for all $x, y \in G$.

Lemma 4.5: [9] Let H be a subgroup of G and $[\gamma, \delta] \leq [\alpha, \beta]$ with $\alpha, \beta, \gamma, \delta \in I$ where $\alpha \leq \beta$ and $\gamma \leq \delta$. Then the I-vague set A of G defined by

$$V_A(x) = \begin{cases} [\alpha, \beta] & \text{if } x \in H \\ [\gamma, \delta] & \text{otherwise} \end{cases}$$

is an I-vague group of G.

Lemma 4.6: [9] Let $H \neq \emptyset$ and $H \subseteq G$. The I-vague characteristic set of H, χ_H is an I-vague group of G iff H is a subgroup of G.

Lemma 4.7: [9] If A and B are I-vague groups of a group G, then $A \cap B$ is also an I-vague group of G.

Lemma 4.8: [9] Let I be complete. If $\{A_i : i \in \Delta\}$ is a non empty family of I-vague groups of G, then $\bigcap_{i \in \Delta} A_i$ is an I-vague group of G.

Lemma 4.9: [9] Let A be an I-vague group of G and B be a constant I-vague group of G. Then $A \cup B$ is an I-vague group of G.

Theorem 4.10: [9] An I-vague set A of a group G is an I-vague group of G if and only if for all $\alpha, \beta \in I$ with $\alpha \leq \beta$, the I-vague cut $A_{(\alpha, \beta)}$ is a subgroup of G whenever it is non empty.

Theorem 4.11: [9] Let A be an I-vague group of a group G. If $V_A(xy^{-1}) = V_A(e)$ for $x, y \in G$, then $V_A(x) = V_A(y)$.

Lemma 4.12: [9] Let A be an I-vague group of a group G. Then $GV_A = \{x \in G : V_A(x) = V_A(e)\}$ is a subgroup of G.

V. I-VAGUE NORMAL GROUPS

Definition 5.1: Let G be a group. An I-vague group A of a group G is called an I-vague normal group of G if for all $x, y \in G$, $V_A(xy) = V_A(yx)$.

If the group G is abelian, then every I-vague group of G is an I-vague normal group of G.

Lemma 5.2: Let A be an I-vague group of a group G. A is an I-vague normal group of G if and only if $V_A(x) = V_A(yxy^{-1})$ for all $x, y \in G$.

Proof: Let A be an I-vague group of a group G.

Suppose that A is an I-vague normal group of G.

Let $x, y \in G$. Then

$$V_A(x) = V_A(xy^{-1}y) = V_A(yxy^{-1}).$$

$$V_A(x) = V_A(yxy^{-1}).$$

Conversely, suppose that $V_A(x) = V_A(yxy^{-1})$ for all $x, y \in G$.

$$\text{Then } V_A(xy) = V_A(y(xy)y^{-1}) = V_A(yx).$$

We have $V_A(xy) = V_A(yx)$. Hence the lemma follows.

Lemma 5.3: Let H be a normal subgroup of G and $[\gamma, \delta] \leq [\alpha, \beta]$ for $\alpha, \beta, \gamma, \delta \in I$ with $\alpha \leq \beta$ and $\gamma \leq \delta$. Then the I-vague set A of G defined by

$$V_A(x) = \begin{cases} [\alpha, \beta] & \text{if } x \in H \\ [\gamma, \delta] & \text{otherwise} \end{cases}$$

is an I-vague normal group of G.

Proof: Let H be a normal subgroup of G. By lemma 4.5, A is an I-vague group of G.

We show that $V_A(x) = V_A(yxy^{-1})$ for every $x, y \in G$.

Let $x, y \in G$.

If $x \in H$, then $yxy^{-1} \in H$. Thus $V_A(x) = V_A(yxy^{-1})$.

If $x \notin H$, then $yxy^{-1} \notin H$. Thus $V_A(x) = V_A(yxy^{-1})$.

Hence $V_A(x) = V_A(yxy^{-1})$ for every $x, y \in G$.

Therefore A is an I-vague normal group of G.

Lemma 5.4: Let $H \neq \emptyset$. The I-vague characteristic set of H, χ_H is an I-vague normal group of a group G iff H is a normal subgroup of G.

Proof: Suppose that H is a normal subgroup of G. By Lemma 5.3, χ_H is an I-vague normal group of G since

$$V_{\chi_H}(x) = \begin{cases} [1, 1] & \text{if } x \in H \\ [0, 0] & \text{otherwise.} \end{cases}$$

Conversely, suppose that χ_H is an I-vague normal group of G. We show that H is a normal subgroup of G.

By lemma 4.6, H is a subgroup of G. Let $y \in H$ and $x \in G$.

Now we prove that $xyx^{-1} \in H$.

$V_{X_H}(xyx^{-1}) = V_{X_H}(y) = [1, 1]$. This implies $xyx^{-1} \in H$.

It follows that H is a normal subgroup of G .

Hence the lemma holds true.

Theorem 5.5: If A and B are I-vague normal groups of G , then $A \cap B$ is also an I-vague normal group of G .

Proof: If A and B are I-vague groups of a group G , then $A \cap B$ is also an I-vague group of G by lemma 4.7.

Now it remains to show that $V_{A \cap B}(xy) = V_{A \cap B}(yx)$ for every $x, y \in G$. Let $x, y \in G$. Then

$$\begin{aligned} V_{A \cap B}(xy) &= \text{iinf}\{V_A(xy), V_B(xy)\} \\ &= \text{iinf}\{V_A(yx), V_B(yx)\} \\ &= V_{A \cap B}(yx). \end{aligned}$$

Hence $V_{A \cap B}(xy) = V_{A \cap B}(yx)$ for each $x, y \in G$.

Therefore $A \cap B$ is an I-vague normal group of G .

Lemma 5.6: Let I be complete. If $\{A_i : i \in \Delta\}$ is a non empty family of I-vague normal groups of G , then $\bigcap_{i \in \Delta} A_i$ is an I-vague normal group of G .

Proof: Let $A = \bigcap_{i \in \Delta} A_i$. Then A is an I-vague group of G by lemma 4.8.

Now we prove that $V_A(xyx^{-1}) = V_A(y)$ for every $x, y \in G$. Let $x, y \in G$. Then

$$\begin{aligned} V_A(xyx^{-1}) &= \text{iinf}\{V_{A_i}(xyx^{-1}) : i \in \Delta\} \\ &= \text{iinf}\{V_{A_i}(y) : i \in \Delta\} \\ &= V_A(y) \end{aligned}$$

Therefore $\bigcap_{i \in \Delta} A_i$ is an I-vague normal group of G .

Lemma 5.7: Let A be an I-vague normal group of G and B be a constant I-vague group of G . Then $A \cup B$ is an I-vague normal group of G .

Proof: Let A be an I-vague normal group of G and B be a constant I-vague group of G . Hence $V_B(x) = V_B(y)$ for all $x, y \in G$. By lemma 4.9, $A \cup B$ is an I-vague group of G .

For each $x, y \in G$,

$$\begin{aligned} V_{A \cup B}(yxy^{-1}) &= \text{isup}\{V_A(yxy^{-1}), V_B(yxy^{-1})\} \\ &= \text{isup}\{V_A(x), V_B(x)\} \\ &= V_{A \cup B}(x) \end{aligned}$$

Hence $V_{A \cup B}(yxy^{-1}) = V_{A \cup B}(x)$ for every $x, y \in G$.

Therefore $A \cup B$ is an I-vague normal group of G .

Remark Even if $V_{A \cup B}(xyx^{-1}) = V_{A \cup B}(y)$ for I-vague normal groups A and B , $A \cup B$ is not be an I-vague group of G as we have seen in I-vague groups[9].

Theorem 5.8: An I-vague set A of a group G is an I-vague normal group of G if and only if for all $\alpha, \beta \in I$ with $\alpha \leq \beta$, the I-vague cut $A_{(\alpha, \beta)}$ is a normal subgroup of G whenever it is non-empty.

Proof: By theorem 4.10, an I-vague set A of a group G is an I-vague group of G if and only if for all $\alpha, \beta \in I$ with $\alpha \leq \beta$, the I-vague cut $A_{(\alpha, \beta)}$ is a subgroup of G whenever it is non-empty.

Suppose that A is an I-vague normal group of G .

Consider $A_{(\alpha, \beta)}$. Let $y \in A_{(\alpha, \beta)}$ and $x \in G$. We prove that $xyx^{-1} \in A_{(\alpha, \beta)}$.

$y \in A_{(\alpha, \beta)}$ implies $V_A(y) \geq [\alpha, \beta]$. Since $V_A(y) = V_A(xyx^{-1})$, $V_A(xyx^{-1}) \geq [\alpha, \beta]$. Hence $xyx^{-1} \in A_{(\alpha, \beta)}$, so $A_{(\alpha, \beta)}$ is a normal subgroup of G .

Conversely, suppose that for all $\alpha, \beta \in I$ with $\alpha \leq \beta$, the non

empty set $A_{(\alpha, \beta)}$ is a normal subgroup of G .

Now it remains to prove that $V_A(y) = V_A(xyx^{-1})$ for all $x, y \in G$. Suppose that $V_A(y) = [\alpha, \beta]$. Then $y \in A_{(\alpha, \beta)}$. Since $A_{(\alpha, \beta)}$ is a normal subgroup of G , $xyx^{-1} \in A_{(\alpha, \beta)}$. It follows that $V_A(xyx^{-1}) \geq [\alpha, \beta] = V_A(y)$ for all $x \in G$. Hence $V_A(xyx^{-1}) \geq V_A(y)$ for all $x \in G$. This implies $V_A(x^{-1}yx) \geq V_A(y)$ for all $x, y \in G$. Put xyx^{-1} instead of y . Hence $V_A(x^{-1}(xyx^{-1})x) \geq V_A(xyx^{-1})$, so $V_A(y) \geq V_A(xyx^{-1})$. Consequently, $V_A(xyx^{-1}) = V_A(y)$ for all $x, y \in G$. Thus A is an I-vague normal group of G . Hence the theorem follows.

Theorem 5.9: If A is an I-vague normal group of G , then GV_A is a normal subgroup of G .

Proof: We prove that GV_A is a normal subgroup of G .

By lemma 4.12, $GV_A = \{x \in G : V_A(x) = V_A(e)\}$ is a subgroup of G . Now we show that $xyx^{-1} \in GV_A$ for $x \in G$ and $y \in GV_A$. Since A is an I-vague normal group of G , $V_A(xyx^{-1}) = V_A(y)$. $y \in GV_A$ implies $V_A(y) = V_A(e)$.

Hence $V_A(xyx^{-1}) = V_A(e)$, so $xyx^{-1} \in GV_A$.

Thus GV_A is a normal subgroup of G .

Theorem 5.10: If A is an I-vague group of a group G and B is an I-vague normal group of G , then $A \cap B$ is an I-vague normal group of GV_A .

Proof: GV_A is a subgroup of G because A is an I-vague group of G . Since A and B are I-vague groups of G , it follows that $A \cap B$ is an I-vague group of G by lemma 4.7.

So $A \cap B$ is an I-vague group of GV_A . Now we prove that $V_{A \cap B}(xy) = V_{A \cap B}(yx)$ for all $x, y \in GV_A$.

Let $x, y \in GV_A$. Then $xy, yx \in GV_A$. Hence

$V_A(xy) = V_A(yx) = V_A(e)$. $V_B(xy) = V_B(yx)$ because B is an I-vague normal group of G .

$V_{A \cap B}(xy) = \text{iinf}\{V_A(xy), V_B(xy)\} = \text{iinf}\{V_A(yx), V_B(yx)\} = V_{A \cap B}(yx)$. It follows that $V_{A \cap B}(xy) = V_{A \cap B}(yx)$ for every $x, y \in GV_A$. Therefore $A \cap B$ is an I-vague normal group of GV_A .

Theorem 5.11: Let A be an I-vague group of G . Then A is an I-vague normal group of G iff $V_A([x, y]) \geq V_A(x)$ for all $x, y \in G$.

Proof: Let A be an I-vague group of G .

Suppose that A is an I-vague normal group of G .

We prove that $V_A([x, y]) \geq V_A(x)$ for $x, y \in G$.

Let $x, y \in G$. Then

$$\begin{aligned} V_A([x, y]) &= V_A(x^{-1}(y^{-1}xy)) \\ &\geq \text{iinf}\{V_A(x^{-1}), V_A(y^{-1}xy)\} \\ &= \text{iinf}\{V_A(x), V_A(x)\} \text{ since } A \text{ is an I-vague normal} \\ &= V_A(x) \text{ group of } G. \end{aligned}$$

Hence $V_A([x, y]) \geq V_A(x)$.

Conversely, suppose that $V_A([x, y]) \geq V_A(x)$ for all $x, y \in G$. We prove that A is an I-vague normal group of G .

Let $x, z \in G$. Then

$$\begin{aligned} V_A(x^{-1}zx) &= V_A(ex^{-1}zx) \\ &= V_A(zx^{-1}x^{-1}zx) \\ &= V_A(z[z, x]) \\ &\geq \text{iinf}\{V_A(z), V_A([z, x])\} \\ &= V_A(z) \text{ by our supposition.} \end{aligned}$$

Hence $V_A(x^{-1}zx) \geq V_A(z)$ for $x, z \in G$. It implies $V_A(xzx^{-1}) \geq V_A(z)$ for $x, z \in G$. Instead of z put $x^{-1}zx$. Then we get $V_A(z) \geq V_A(x^{-1}zx)$.

Thus $V_A(z) = V_A(x^{-1}zx)$ for every $x, z \in G$.

Therefore A is an I-vague normal group of G.

Hence the theorem follows.

Definition 5.12: : Let A be an I-vague group of a group G. Then the set

$N(A) = \{a \in G : V_A(axa^{-1}) = V_A(x) \text{ for all } x \in G\}$ is called an I-vague normalizer of A.

Theorem 5.13: Let A be an I-vague group of G. Then

- (i) A is an I-vague normal group of N(A).
- (ii) I-vague normalizer N(A) is a subgroup of G .
- (iii) A is an I-vague normal group of G iff N(A)= G.

Proof: Let A be an I-vague group of G.

(i) We prove that A is an I-vague normal group of N(A). Let $x, a \in N(A)$.

By definition, $V_A(axa^{-1}) = V_A(x)$ for all $x, a \in N(A)$.

Thus A is an I-vague normal group of N(A).

(ii) Let $a, b \in N(A)$. We show that $a^{-1} \in N(A)$ and $ab \in N(A)$.

Let $a \in N(A)$. Then $V_A(axa^{-1}) = V_A(x)$ for all $x \in G$.

$V_A(x) = V_A(a(a^{-1}xa)a^{-1}) = V_A(a^{-1}xa)$.

Hence $V_A(a^{-1}xa) = V_A(x)$, so $a^{-1} \in N(A)$.

Let $a, b \in N(A)$. Then

$V_A(axa^{-1}) = V_A(x)$ and $V_A(bxb^{-1}) = V_A(x)$ for all $x \in G$.

Then $V_A(abx(ab)^{-1}) = V_A(a(bxb^{-1})a^{-1}) = V_A(bxb^{-1}) = V_A(x)$.

Thus $ab \in N(A)$. Therefore N(A) is a subgroup of G.

(iii) Suppose that A is an I-vague normal group of G.

We prove that N(A) = G.

Let $a \in G$. Since A is an I-vague normal group of G,

$V_A(axa^{-1}) = V_A(x)$ for all $x \in G$. It follows that $a \in N(A)$.

Hence $G \subseteq N(A)$.

Since $N(A) \subseteq G, G = N(A)$.

Conversely, assume that N(A) = G. For all $a, x \in G$,

$V_A(axa^{-1}) = V_A(x)$.

By definition, A is an I-vague normal group of G.

Theorem 5.14: Let A be an I-vague group of a group G.

Then GV_A is a normal subgroup of N(A).

Proof: Let A be an I-vague group of G. We prove that GV_A is a normal subgroup of N(A).

First we prove that $GV_A \subseteq N(A)$.

Let $x \in GV_A$. Then $x \in GV_A, V_A(x) = V_A(e)$.

For $y \in G, V_A(xy x^{-1}) \geq \text{iinf}\{V_A(x), V_A(yx^{-1})\}$

$$\geq \text{iinf}\{V_A(x), V_A(y)\}$$

$$= \text{iinf}\{V_A(e), V_A(y)\}$$

$$= V_A(y).$$

Hence $V_A(xy x^{-1}) \geq V_A(y)$ for $y \in G$ and $x \in GV_A$.

$x \in GV_A$ implies $x^{-1} \in GV_A$. Thus $V_A(x^{-1}yx) \geq V_A(y)$

where $x \in GV_A$ and $y \in G$. Put xyx^{-1} instead of y .

We have $V_A(x^{-1}(xyx^{-1})x) \geq V_A(xy x^{-1})$ and hence

$V_A(y) \geq V_A(xy x^{-1})$.

Therefore $V_A(y) = V_A(xy x^{-1})$ for each $y \in G$.

Thus $x \in N(A)$. Therefore $GV_A \subseteq N(A)$.

Since GV_A is a subgroup of G and $GV_A \subseteq N(A), GV_A$ is a subgroup of N(A).

Now we show that $yay^{-1} \in GV_A$ for all $a \in GV_A$ and for all $y \in N(A)$.

Since $y \in N(A), V_A(yay^{-1}) = V_A(a)$. Since $a \in GV_A,$

$V_A(a) = V_A(e)$. Hence $V_A(yay^{-1}) = V_A(e)$, so $yay^{-1} \in GV_A$. Therefore GV_A is a normal subgroup of N(A).

Definition 5.15: Let A be an I-vague group of a group G. Then the set

$C(A) = \{a \in G : V_A([a, x]) = V_A(e) \text{ for all } x \in G\}$ is called an I-vague centralizer of A.

Theorem 5.16: Let A be an I-vague group of a group G. Then C(A) is a normal subgroup of G.

Proof: Let A be an I-vague group of G. We prove that $C(A) = \{a \in G : V_A([a, x]) = V_A(e) \text{ for all } x \in G\}$ is a normal subgroup of G.

Step(1) We show that $a \in C(A)$ implies $V_A(xa) = V_A(ax)$ for all $x \in G$.

Let $a \in C(A)$. Then $V_A([a, x]) = V_A(e)$ for all $x \in G$.

$V_A([a, x]) = V_A(e) \Rightarrow V_A(a^{-1}x^{-1}ax) = V_A(e)$

$$\Rightarrow V_A((xa)^{-1}ax) = V_A(e)$$

$$\Rightarrow V_A((xa)^{-1}((ax)^{-1})^{-1}) = V_A(e)$$

$$\Rightarrow V_A((xa)^{-1}) = V_A((ax)^{-1}) \text{ by thm 4.11}$$

$$\Rightarrow V_A(xa) = V_A(ax).$$

Therefore $V_A(xa) = V_A(ax)$ for all $x \in G$.

Step(2) We show that $a \in C(A)$ implies $V_A([x, a]) = V_A(e)$ for all $x \in G$.

$V_A([x, a]) = V_A(x^{-1}a^{-1}xa) = V_A((x^{-1}a^{-1}xa)^{-1}) =$

$V_A(a^{-1}x^{-1}ax) = V_A([a, x]) = V_A(e)$

Hence $V_A([x, a]) = V_A(e)$ for each $a \in C(A)$ and for all $x \in G$.

Step(3) We prove that C(A) is a subgroup of G.

We show that (i) $a \in C(A)$ implies $a^{-1} \in C(A)$.

(ii) $a, b \in C(A)$ implies $ab \in C(A)$.

Now proof of (i)

For all $x \in G, V_A([a^{-1}, x]) = V_A(ax^{-1}a^{-1}x)$

$$= V_A(x^{-1}a^{-1}xa) \text{ by step (1)}$$

$$= V_A((x^{-1}a^{-1}xa)^{-1})$$

$$= V_A(a^{-1}x^{-1}ax)$$

$$= V_A([a, x])$$

$$= V_A(e).$$

Thus $V_A([a^{-1}, x]) = V_A(e)$ for all $x \in G$.

Hence we have that $a^{-1} \in C(A)$.

Proof of (ii) Let $a, b \in C(A)$. Then $V_A([a, x]) = V_A([b, x]) = V_A(e)$ for all $x \in G$.

$V_A([ab, x]) = V_A((ab)^{-1}x^{-1}(ab)x)$

$$= V_A(b^{-1}(a^{-1}x^{-1}abx))$$

$$= V_A((a^{-1}x^{-1}abx)b^{-1}) \text{ by step(1)}$$

$$= V_A((a^{-1}x^{-1}ax)(x^{-1}bxb^{-1}))$$

$$= V_A([a, x][x, b^{-1}]).$$

$$\geq \text{iinf}\{V_A([a, x]), V_A([x, b^{-1}])\}$$

$$= \text{iinf}\{V_A(e), V_A(e)\} \text{ since } b^{-1} \in C(A)$$

$$= V_A(e).$$

This implies $V_A([ab, x]) \geq V_A(e)$ for all $x \in G$.

Since $V_A(e) \geq V_A([ab, x]), V_A([ab, x]) = V_A(e)$ for all $x \in G$. Hence $ab \in C(A)$.

From (i) and (ii) C(A) is a subgroup of G.

Step(4) Now we show that $g^{-1}ag \in C(A)$ for all $a \in C(A)$ and for all $g \in G$.

That is $V_A([g^{-1}ag, x]) = V_A(e)$ for all $g, x \in G$ and for all $a \in C(A)$.

$V_A([g^{-1}ag, x]) = V_A((g^{-1}ag)^{-1}x^{-1}g^{-1}agx)$

$$= V_A(g^{-1}a^{-1}g^{-1}x^{-1}g^{-1}agx)$$

$$\begin{aligned}
&=V_A(g^{-1}a^{-1}gaa^{-1}x^{-1}g^{-1}agx) \\
&=V_A([g, a]a^{-1}(gx)^{-1}agx) \\
&=V_A([g, a][a, gx]) \\
&\geq \text{iinf}\{V_A([g, a]), V_A([a, gx])\} \\
&= \text{iinf}\{V_A(e), V_A(e)\} \\
&= V_A(e).
\end{aligned}$$

Hence $V_A([g^{-1}ag, x]) \geq V_A(e)$.

Since $V_A(e) \geq V_A([g^{-1}ag, x])$, $V_A([g^{-1}ag, x]) = V_A(e)$.

This implies $g^{-1}ag \in C(A)$.

From step(3) and step(4), we have $C(A)$ is a normal subgroup of G .

Theorem 5.17: Let A be an I-vague normal group of a group G . Then GV_A is a subgroup of $C(A)$.

Proof: Let A be an I-vague normal group of a group G . We prove that GV_A is a subgroup of $C(A)$.

Let $x \in GV_A$. Then $V_A(x) = V_A(e)$. Consider $V_A([x, y])$ for each $y \in G$.

$$\begin{aligned}
V_A([x, y]) &= V_A(x^{-1}(y^{-1}xy)) \geq \text{iinf}\{V_A(x^{-1}), V_A(y^{-1}xy)\} \\
&= \text{iinf}\{V_A(x), V_A(x)\} \\
&= V_A(x) \\
&= V_A(e).
\end{aligned}$$

Hence $V_A([x, y]) \geq V_A(e)$.

Since $V_A(e) \geq V_A([x, y])$, $V_A([x, y]) = V_A(e)$.

By the definition of $C(A)$, $x \in C(A)$.

Thus $GV_A \subseteq C(A)$. Since GV_A is a subgroup of G , GV_A is a subgroup of $C(A)$.

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