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Hyers-Ulam Stability of Functional Equation $f(3x) = 4f(3x-3) + f(3x-6)$

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Abstract—The functional equation $f(3x) = 4f(3x-3) + f(3x-6)$ will be solved and its Hyers-Ulam stability will be also investigated in the class of functions $f : \mathbf{R} \to X$, where X is a real Banach space.

Keywords—Functional equation, Lucas sequence of the first kind, Hyers-Ulam stability.

I. Introduction

IN 1940, Ulam gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems (ref. [20]). Among those was the question concerning the stability of homomorphisms:

> Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given any $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In the following year, Hyers affirmatively answered in his paper [8] the question of Ulam for the case where G_1 and G_2 are Banach spaces. Later, the result of Hyers has been generalized by Rassias (ref. [16]).

Let $(G_1, \cdot)$ be a groupoid and let $(G_2, +)$ be a groupoid with the metric d. The equation of homomorphism

$$f(x \cdot y) = f(x) + f(y)$$

is stable in the Hyers-Ulam sense (or has the Hyers-Ulam stability) if for every $\delta > 0$ there exists an $\varepsilon > 0$ such that for every function $h : G_1 \to G_2$ satisfying

$$d[h(x \cdot y), h(x) + h(y)] \leq \varepsilon$$

for all $x, y \in G_1$ there exists a solution $g : G_1 \to G_2$ of the equation of homomorphism with

$$d[h(x), g(x)] \leq \delta$$

for any $x \in G_1$ (see [15, Definition 1]).

This terminology is also applied to the case of other functional equations. It should be remarked that a lot of references concerning the stability of functional equations can be found in the books [3], [9], [12] (see also [1], [4], [5], [6], [7], [10], [11], [17], [18], [19]).

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The nth Fibonacci number will be denoted by F_n for $n \in \mathbf{N}$. It is well known that the Fibonacci numbers satisfy the equality $F_{3n} = 4F_{3n-3} + F_{3n-6}$ for all $n \geq 2$ (see [14, p. 89]). From this famous formula, the following functional equation

$$f(3x) = 4f(3x-3) + f(3x-6) \tag{1}$$

may be derived.

In this paper, using the idea from [13], the functional equation (1) will be solved and its Hyers-Ulam stability will be investigated in the class of functions $f : \mathbf{R} \to X$, where X is a real Banach space.

Throughout this paper, the positive and the negative root of the equation $x^2 - 4x - 1 = 0$ will be denoted by a and b, respectively, i.e.,

$$a = 2 + \sqrt{5} \quad \text{and} \quad b = 2 - \sqrt{5}.$$

Moreover, the Lucas sequence of the first kind will be denoted by $\{U_n(4,-1)\}$ and an abbreviation U_n will be used instead of $U_n(4,-1)$, i.e., U_n is defined by

$$U_n = U_n(4,-1) = \frac{a^n - b^n}{a - b}$$

for all integers n. It is not difficult to see that

$$U_{n+2} = 4U_{n+1} + U_n \tag{2}$$

for any integer n. For any $x \in \mathbf{R}$, $[x]$ stands for the largest integer that does not exceed x.

II. General Solution to Eq. (1)

Throughout this section, let X be a real vector space. The general solution of the functional equation (1) will be investigated.

Theorem 2.1. *Let X be a real vector space. A function $f : \mathbf{R} \to X$ is a solution of the functional equation* (1) *if and only if there exists a function $h : [-3, 3) \to X$ such that*

$$f(x) = U_{[x/3]+1} h(x - 3[x/3]) + U_{[x/3]} h(x - 3[x/3] - 3). \tag{3}$$

Proof. Since $a + b = 4$ and $ab = -1$, it follows from (1) that

$$\begin{cases} f(3x) - af(3x-3) = b\,[f(3x-3) - af(3x-6)]\,, \\ f(3x) - bf(3x-3) = a\,[f(3x-3) - bf(3x-6)]\,. \end{cases}$$

If a function $g : \mathbf{R} \to X$ is defined by $g(x) = f(3x)$ for each $x \in \mathbf{R}$, then it follows from the above equalities that

$$\begin{cases} g(x) - ag(x-1) = b\left[g(x-1) - ag(x-2)\right], \\ g(x) - bg(x-1) = a\left[g(x-1) - bg(x-2)\right]. \end{cases} \tag{4}$$

By the mathematical induction, it can be proved that

$$\begin{cases} g(x) - ag(x-1) \\ \quad = \; b^n\left[g(x-n) - ag(x-n-1)\right], \\ g(x) - bg(x-1) \\ \quad = \; a^n\left[g(x-n) - bg(x-n-1)\right] \end{cases} \tag{5}$$

for all $x \in \mathbf{R}$ and $n \in \{0, 1, 2, \ldots\}$. Substitute $x + n$ $(n \geq 0)$ for x in (5) and divide the resulting equations by b^n resp. a^n, and then substitute $-m$ for n in the resulting equations to obtain the equations in (5) with m in place of n, where $m \in \{0, -1, -2, \ldots\}$. Therefore, the equations in (5) are true for all $x \in \mathbf{R}$ and $n \in \mathbf{Z}$.

Multiply the first and the second equation of (5) by b and a, respectively. And subtract the first resulting equation from the second one to obtain

$$g(x) = U_{n+1}g(x-n) + U_n g(x-n-1) \tag{6}$$

for any $x \in \mathbf{R}$ and $n \in \mathbf{Z}$.

Putting $n = [x]$ in (6) yields

$$g(x) = U_{[x]+1}g(x-[x]) + U_{[x]}g(x-[x]-1),$$

i.e., by the definition of g, it holds that

$$f(x) = U_{[x/3]+1}f(x-3[x/3]) + U_{[x/3]}f(x-3[x/3]-3)$$

for all $x \in \mathbf{R}$.

Since $0 \leq x - 3[x/3] < 3$ and $-3 \leq x - 3[x/3] - 3 < 0$, if a function $h : [-3, 3) \to X$ is defined by $h := f|_{[-3,3)}$, then f is a function of the form (3).

Now, assume that f is a function of the form (3), where $h : [-3, 3) \to X$ is an arbitrary function. Then, it follows from (3) that

$$f(3x) = U_{[x]+1}h(3x-3[x]) + U_{[x]}h(3x-3[x]-3),$$

$$f(3x-3) = U_{[x]}h(3x-3[x]) + U_{[x]-1}h(3x-3[x]-3),$$

$$f(3x-6) = U_{[x]-1}h(3x-3[x]) + U_{[x]-2}h(3x-3[x]-3)$$

for any $x \in \mathbf{R}$. Thus, by (2), it holds that

$$\begin{aligned} & f(3x) - 4f(3x-3) - f(3x-6) \\ & \quad = \; (U_{[x]+1} - 4U_{[x]} - U_{[x]-1})h(3x-3[x]) \\ & \quad\quad\; +(U_{[x]} - 4U_{[x]-1} - U_{[x]-2})h(3x-3[x]-3) \\ & \quad = \; 0, \end{aligned}$$

which completes the proof.

III. Hyers-Ulam Stability of Eq. (1)

In this section, a denotes the positive root of the equation $x^2 - 4x - 1 = 0$ and b is its negative root. The Hyers-Ulam stability of the functional equation (1) will be proved in the following theorem.

Theorem 3.1. *Let $(X, \|\cdot\|)$ be a real Banach space. If a function $f : \mathbf{R} \to X$ satisfies the inequality*

$$\|f(3x) - 4f(3x-3) - f(3x-6)\| \leq \varepsilon \tag{7}$$

for all $x \in \mathbf{R}$ and for some $\varepsilon \geq 0$, then there exists a unique solution function $F : \mathbf{R} \to X$ of (1) such that

$$\|f(x) - F(x)\| \leq \frac{5+\sqrt{5}}{20}\varepsilon \tag{8}$$

for all $x \in \mathbf{R}$.

Proof. First, define a function $g : \mathbf{R} \to X$ by $g(x) = f(3x)$ for all $x \in \mathbf{R}$. Analogously to the first equation of (4), it follows from (7) that

$$\|g(x) - ag(x-1) - b[g(x-1) - ag(x-2)]\| \leq \varepsilon$$

for each $x \in \mathbf{R}$. Replacing x with $x - k$ in the last inequality yields

$$\begin{aligned} & \|g(x-k) - ag(x-k-1) \\ & \qquad - b[g(x-k-1) - ag(x-k-2)]\| \\ & \leq \; \varepsilon \end{aligned}$$

and further

$$\begin{aligned} & \left\|b^k[g(x-k) - ag(x-k-1)]\right. \\ & \qquad \left. - b^{k+1}[g(x-k-1) - ag(x-k-2)]\right\| \\ & \leq \; |b|^k\varepsilon \end{aligned} \tag{9}$$

for all $x \in \mathbf{R}$ and $k \in \mathbf{Z}$. By (9), it obviously holds that

$$\begin{aligned} & \|g(x) - ag(x-1) - b^n[g(x-n) - ag(x-n-1)]\| \\ & \leq \; \sum_{k=0}^{n-1}\|b^k[g(x-k) - ag(x-k-1)] \\ & \qquad\quad - b^{k+1}[g(x-k-1) - ag(x-k-2)]\| \\ & \leq \; \sum_{k=0}^{n-1}|b|^k\varepsilon \end{aligned} \tag{10}$$

for $x \in \mathbf{R}$ and $n \in \mathbf{N}$.

For any $x \in \mathbf{R}$, (9) implies that the sequence $\{b^n[g(x-n) - ag(x-n-1)]\}$ is a Cauchy sequence. (Note that $|b| < 1$). Therefore, a function $G_1 : \mathbf{R} \to X$ can be defined by

$$G_1(x) = \lim_{n\to\infty} b^n[g(x-n) - ag(x-n-1)],$$

since X is complete. It follows from the definition of G_1 that

$$\begin{aligned} & 4G_1(x-1)+G_1(x-2) \\ &= 4b^{-1}\lim_{n\to\infty} b^{n+1}[g(x-(n+1)) \\ &\qquad - ag(x-(n+1)-1)] \\ &\quad + b^{-2}\lim_{n\to\infty} b^{n+2}[g(x-(n+2)) \\ &\qquad - ag(x-(n+2)-1)] \\ &= 4b^{-1}G_1(x)+b^{-2}G_1(x) \\ &= G_1(x) \end{aligned} \tag{11}$$

for all $x \in \mathbf{R}$, since $b^2 = 4b + 1$. If n goes to infinity, then (10) yields that

$$\|g(x) - ag(x-1) - G_1(x)\| \le \frac{3+\sqrt{5}}{4}\varepsilon \tag{12}$$

for every $x \in \mathbf{R}$.

On the other hand, it also follows from (7) that

$$\|g(x) - bg(x-1) - a[g(x-1) - bg(x-2)]\| \le \varepsilon$$

(see the second equation in (4)). Analogously to (9), replacing x by $x + k$ in the above inequality and then dividing by a^k both sides of the resulting inequality yield

$$\begin{aligned} & \big\|a^{-k}[g(x+k) - bg(x+k-1)] \\ &\qquad - a^{-k+1}[g(x+k-1) - bg(x+k-2)]\big\| \\ &\le a^{-k}\varepsilon \end{aligned} \tag{13}$$

for all $x \in \mathbf{R}$ and $k \in \mathbf{Z}$. It further follows from (13) that

$$\begin{aligned} & \|a^{-n}[g(x+n) - bg(x+n-1)] - [g(x) - bg(x-1)]\| \\ &\le \sum_{k=1}^{n} \|a^{-k}[g(x+k) - bg(x+k-1)] \\ &\qquad - a^{-k+1}[g(x+k-1) - bg(x+k-2)]\| \\ &\le \sum_{k=1}^{n} a^{-k}\varepsilon \end{aligned} \tag{14}$$

for $x \in \mathbf{R}$ and $n \in \mathbf{N}$.

On account of (13), the sequence $\{a^{-n}[g(x+n) - bg(x+n-1)]\}$ is a Cauchy sequence for any fixed $x \in \mathbf{R}$. Hence, a function $G_2 : \mathbf{R} \to X$ can be defined by

$$G_2(x) = \lim_{n\to\infty} a^{-n}[g(x+n) - bg(x+n-1)].$$

It follows from the definition of G_2 that

$$\begin{aligned} & 4G_2(x-1)+G_2(x-2) \\ &= 4a^{-1}\lim_{n\to\infty} a^{-(n-1)}[g(x+n-1) \\ &\qquad - bg(x+(n-1)-1)] \\ &\quad + a^{-2}\lim_{n\to\infty} a^{-(n-2)}[g(x+n-2) \\ &\qquad - bg(x+(n-2)-1)] \\ &= 4a^{-1}G_2(x)+a^{-2}G_2(x) \\ &= G_2(x) \end{aligned} \tag{15}$$

for any $x \in \mathbf{R}$. By letting n go to infinity, (14) yields

$$\|G_2(x) - g(x) + bg(x-1)\| \le \frac{\sqrt{5}-1}{4}\varepsilon \tag{16}$$

for $x \in \mathbf{R}$.

From (12) and (16), it follows that

$$\begin{aligned} & \left\|g(x) - \left[\frac{b}{b-a}G_1(x) - \frac{a}{b-a}G_2(x)\right]\right\| \\ &= \frac{1}{|b-a|}\|(b-a)g(x) - [bG_1(x) - aG_2(x)]\| \\ &\le \frac{1}{a-b}\|bg(x) - abg(x-1) - bG_1(x)\| \\ &\quad + \frac{1}{a-b}\|aG_2(x) - ag(x) + abg(x-1)\| \\ &\le \frac{5+\sqrt{5}}{20}\varepsilon \end{aligned} \tag{17}$$

for all $x \in \mathbf{R}$. Now define a function $F : \mathbf{R} \to X$ by

$$F(x) = \frac{b}{b-a}G_1\Big(\frac{x}{3}\Big) - \frac{a}{b-a}G_2\Big(\frac{x}{3}\Big)$$

for all $x \in \mathbf{R}$. Then, it follows from (11) and (15) that

$$\begin{aligned} & 4F(3x-3)+F(3x-6) \\ &= \frac{4b}{b-a}G_1(x-1) - \frac{4a}{b-a}G_2(x-1) \\ &\quad + \frac{b}{b-a}G_1(x-2) - \frac{a}{b-a}G_2(x-2) \\ &= \frac{b}{b-a}[4G_1(x-1)+G_1(x-2)] \\ &\quad - \frac{a}{b-a}[4G_2(x-1)+G_2(x-2)] \\ &= \frac{b}{b-a}G_1(x) - \frac{a}{b-a}G_2(x) \\ &= F(3x) \end{aligned}$$

for each $x \in \mathbf{R}$, i.e., F is a solution of (1). Moreover, the inequality (8) follows from (17).

The uniqueness of F will be proved. Assume that $F_1, F_2 : \mathbf{R} \to X$ are solutions of (1) and that there exist positive constants C_1 and C_2 with

$$\|f(x) - F_1(x)\| \le C_1 \quad \text{and} \quad \|f(x) - F_2(x)\| \le C_2 \tag{18}$$

for all $x \in \mathbf{R}$. According to Theorem 2.1, there exist functions $h_1, h_2 : [-3, 3) \to X$ such that

$$\begin{aligned} F_1(x) &= U_{[x/3]+1}h_1(x - 3[x/3]) \\ &\quad + U_{[x/3]}h_1(x - 3[x/3] - 3), \\ F_2(x) &= U_{[x/3]+1}h_2(x - 3[x/3]) \\ &\quad + U_{[x/3]}h_2(x - 3[x/3] - 3) \end{aligned} \tag{19}$$

for any $x \in \mathbf{R}$, since F_1 and F_2 are solutions of (1).

Fix a t with $0 \le t < 3$. It then follows from (18) and (19) that

$$\begin{aligned} & \|U_{n+1}[h_1(t) - h_2(t)] + U_n[h_1(t-3) - h_2(t-3)]\| \\ &= \|[U_{n+1}h_1(t) + U_nh_1(t-3)] \\ &\qquad - [U_{n+1}h_2(t) + U_nh_2(t-3)]\| \\ &= \|F_1(3n+t) - F_2(3n+t)\| \\ &\le \|F_1(3n+t) - f(3n+t)\| + \|f(3n+t) - F_2(3n+t)\| \\ &\le C_1 + C_2 \end{aligned}$$

for each $n \in \mathbf{Z}$, i.e.,

$$\begin{aligned} &\left\| \frac{a^{n+1} - b^{n+1}}{a-b} [h_1(t) - h_2(t)] \right. \\ &\qquad \left. + \ \frac{a^n - b^n}{a-b} [h_1(t-3) - h_2(t-3)] \right\| \\ &\leq \ C_1 + C_2 \end{aligned} \tag{20}$$

for every $n \in \mathbf{Z}$. Dividing both sides by a^n yields that

$$\begin{aligned} &\left\| \frac{a - (b/a)^n b}{a-b} [h_1(t) - h_2(t)] \right. \\ &\qquad \left. + \ \frac{1 - (b/a)^n}{a-b} [h_1(t-3) - h_2(t-3)] \right\| \\ &\leq \ \frac{C_1 + C_2}{a^n}. \end{aligned}$$

Let $n \to \infty$ to get

$$a[h_1(t) - h_2(t)] + [h_1(t-3) - h_2(t-3)] = 0. \tag{21}$$

Analogously, divide both sides of (20) by $|b|^n$ and let $n \to -\infty$ to get

$$b[h_1(t) - h_2(t)] + [h_1(t-3) - h_2(t-3)] = 0. \tag{22}$$

From (21) and (22), it follows that

$$\begin{pmatrix} a & 1 \\ b & 1 \end{pmatrix} \begin{pmatrix} h_1(t) - h_2(t) \\ h_1(t-3) - h_2(t-3) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Because $a - b \neq 0$, it should hold that

$$h_1(t) - h_2(t) = h_1(t-3) - h_2(t-3) = 0$$

for any $t \in [0, 3)$, i.e., $h_1(t) = h_2(t)$ for all $t \in [-3, 3)$. Therefore, it is true that $F_1(x) = F_2(x)$ for any $x \in \mathbf{R}$.

Remark 1. The presented proof of uniqueness of F is due to an idea of Professor Changsun Choi. It should be remarked that the uniqueness of F can be obtained directly from [2, Proposition 1].

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