International Journal of Mechanical, Industrial and Aerospace Sciences
ISSN: 2517-9950
Vol:2, No:7, 2008

Group invariant solutions for radial jet having finite fluid velocity at orifice

I. Naeem, R. Naz

***Abstract*—The group invariant solution for Prandtl's boundary layer equations for an incompressible fluid governing the flow in radial free, wall and liquid jets having finite fluid velocity at the orifice are investigated. For each jet a symmetry is associated with the conserved vector that was used to derive the conserved quantity for the jet elsewhere. This symmetry is then used to construct the group invariant solution for the third-order partial differential equation for the stream function. The general form of the group invariant solution for radial jet flows is derived. The general form of group invariant solution and the general form of the similarity solution which was obtained elsewhere are the same.

***Keywords*—Two-dimensional jets, radial jets, group invariant solution

I. Introduction

The flow in radial jets is governed either by the system of two partial differential equations for the velocity components or by a single third-order partial differential equation for the stream function. The similarity solution for the third-order partial differential equation for the stream function for the radial free jet was discussed by Schwarz [1]. Schlichting [2] and Bickley [3] derived the similarity solution for the third-order partial differential equation for the stream function for two-dimensional free jet. Mason in [4] constructed the group invariant solution for the same equation. In [5] authors found the group invariant solution for system of equations for the velocity components for both radial and two-dimensional free jets. Glauret [6] obtained the similarity solution for radial and two-dimensional wall jets. Riley in [7] established the similarity solution for the radial free, wall and liquid jets. In all these problems the fluid velocity at the orifice was infinite.

Watson suggested the general form of similarity solution for the flows having finite velocity at the orifice. Riley [8] considered the problem of radial and two-dimensional wall jets for which the velocity remains finite at orifice and so our solution has some significance even near axis. The similarity solution was constructed for both radial and two-dimensional wall jets. The similarity solution for the radial free, wall and liquid jets with finite velocity at the orifice was studied by Riley [9]. Schwarz in [1] derived the similarity solution for the stream function equation for radial free jet having finite as well as infinite velocity at orifice. Watson in [10] derived the similarity solution for system of equations for velocity components for the radial and two-dimensional liquid jets having finite velocity at the orifice. To the best of our knowledge, the group invariant solution for the third-order partial differential equation for stream function for radial free, wall and liquid jets, having finite fluid velocity at the orifice is still not attempted in the literature. It is considered in this paper.

inqau@yahoo.com

The similarity solution transforms the third-order partial differential equation to a third-order ordinary differential equation. By using the certain transformations same third-order ordinary differential equation can be deduced for the radial jets whether the velocity at the orifice is finite or infinite. The third-order ordinary differential equation for radial free jet was first solved numerically by Schlichting [2] and later, Bickley [3] found the analytical solution. For the wall jet, Glauert [6] solved the third-order ordinary differential equation. Riley in [7], [9] found the solution for ordinary differential equation which appeared for radial liquid jet. The authors in [11] solved the third-order ordinary differential equations for radial free, wall and liquid jets by symmetry methods.

In this paper we will derive the group invariant solution for the radial free, wall, liquid jets having finite velocity at the orifice. In [12] the conserved quantities for radial free, wall and liquid jets have been derived using the conservation laws. The symmetry associated with the conserved vector which is used to establish the conserved quantity for each jet generates the group invariant solution for the third-order partial differential equation for the stream function. This symmetry is obtained by using the approach introduced by Kara and Mahomed [13]. We give explicitly the general form of group invariant solution for radial jet flows. We concluded that the group invariant solution and similarity solution are equivalent. In similarity solution method the form of stream function was assumed whereas in the group invariant method the form of stream function is derived.

II. Radial jets

The coordinate axis are chosen such that the x-axis is along the jet and the y-axis is perpendicular to the jet. The jet is symmetrical about the x-axis and the origin is at the orifice. Prandtl's boundary layer equations for an incompressible fluid governing the flow in radial jets, in absence of a pressure gradient, are

$$uu_x + vu_y = \nu u_{yy}, \tag{1}$$

$$(xu)_x + (xv)_y = 0, \tag{2}$$

where $u(x, y)$ and $v(x, y)$ are velocity components in the x and y directions respectively, and ν is the kinematic viscosity of the fluid. Introduce a stream function

$$u = \frac{1}{x}\psi_y,\ v = -\frac{1}{x}\psi_x, \tag{3}$$

then system (1)-(2) is transformed to the third-order partial differential equation

$$\frac{1}{x}\psi_y\psi_{xy} - \frac{1}{x^2}\psi_y^2 - \frac{1}{x}\psi_x\psi_{yy} - \nu\psi_{yyy} = 0. \quad (4)$$

A. Lie point symmetries for third-order partial differential equation for the stream function

The Lie point symmetry generator

$$X = \xi^1(x,y,\psi)\frac{\partial}{\partial x} + \xi^2(x,y,\psi)\frac{\partial}{\partial y} + \eta(x,y,\psi)\frac{\partial}{\partial \psi}, \quad (5)$$

of the third-order partial differential equation (4) are derived by solving

$$X^{[3]}G \mid_{G=0} = 0, \quad (6)$$

where

$$G = \frac{1}{x}\psi_y\psi_{xy} - \frac{1}{x^2}\psi_y^2 - \frac{1}{x}\psi_x\psi_{yy} - \nu\psi_{yyy}. \quad (7)$$

In equation (6), $X^{[3]}$ is the third prolongation of the operator X and is defined by

$$X^{[3]} = \xi^1\frac{\partial}{\partial x} + \xi^2\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial \psi} + \zeta_x\frac{\partial}{\partial \psi_x} + \zeta_y\frac{\partial}{\partial \psi_y}$$

$$+\zeta_{xx}\frac{\partial}{\partial \psi_{xx}} + \zeta_{xy}\frac{\partial}{\partial \psi_{xy}} + \zeta_{yy}\frac{\partial}{\partial \psi_{yy}}, \quad (8)$$

with

$$\zeta_i = D_i(\eta) - \psi_s D_i(\xi^s), \quad (9)$$

$$\zeta_{ij} = D_j(\zeta_i) - \psi_{is}D_j(\xi^s), \quad (10)$$

$$\zeta_{ijk} = D_k(\zeta_{ij}) - \psi_{ijs}D_k(\xi^s), \quad (11)$$

where D_i are the total derivative operators defined as

$$D_1 = D_x = \frac{\partial}{\partial x} + \psi_x\frac{\partial}{\partial \psi} + \psi_{xx}\frac{\partial}{\partial \psi_x} + \psi_{xy}\frac{\partial}{\partial \psi_y} + \cdots, \quad (12)$$

$$D_2 = D_y = \frac{\partial}{\partial y} + \psi_y\frac{\partial}{\partial \psi} + \psi_{yy}\frac{\partial}{\partial \psi_y} + \psi_{yx}\frac{\partial}{\partial \psi_x} + \cdots. \quad (13)$$

Equation (6) is separated according to the different combinations of derivatives of ψ and resulting system is solved for unknown coefficients ξ^1, ξ^2 and η. The Lie point symmetry generator of equation (4) is

$$X = [c_1x + \frac{c_2}{x^2}]\frac{\partial}{\partial x} + [(2c_1 - \frac{c_2}{x^3} - c_3)y + k(x)]\frac{\partial}{\partial y}$$

$$+[c_3\psi + c_4]\frac{\partial}{\partial \psi}, \quad (14)$$

where c_1, c_2, c_3 and c_4 are constants and $k(x)$ is an arbitrary function.

III. Group invariant solution for radial free jet

The governing equations for radial free jet are (1) and (2). The boundary conditions and the conserved quantity for radial free jet are

$$y = 0: \quad v = 0,\ u_y = 0, \quad (15)$$

$$y = \pm\infty: \quad u = 0, \quad (16)$$

and

$$J = 2\rho\int_0^\infty xu^2dy. \quad (17)$$

In terms of stream function the boundary conditions and conserved quantity become

$$y = 0: \quad \psi_x = 0,\ \psi_{yy} = 0, \quad (18)$$

$$y = \pm\infty: \quad \psi_y = 0, \quad (19)$$

$$J = 2\rho\int_0^\infty \frac{1}{x}\psi_y^2dy. \quad (20)$$

The similarity solution for the third-order partial differential equation (4) for the stream function with finite fluid velocity at the orifice was derived in [1], [9]. We will derive the group invariant solution for the third-order partial differential equation (4) subject to conditions (18)-(20) having finite velocity at the orifice. In [12] authors showed that the conserved vector

$$T^1 = \frac{1}{x}\psi_y^2,\ T^2 = -\frac{1}{x}\psi_x\psi_y - \nu\psi_{yy} \quad (21)$$

gives the conserved quantity (20) for the radial free jet. The symmetry associated with the conserved vector (21) will give the group invariant solution for the third-order partial differential equation (4) subject to conditions (18)-(20). The symmetries associated with a known conserved vector can be determined by using (see Kara and Mahomed [13])

$$X^{[m]}(T^i) + D_k(\xi^k)T^i - D_k(\xi^i)T^k = 0, \quad (22)$$

where $X^{[m]}$ is the m^{th} prolongation of X if T^i depend upon m^{th} derivatives. Equation (22) gives rise to following two equations

$$X^{[1]}(T^1) + T^1D_y(\xi^2) - T^2D_y(\xi^1) = 0, \quad (23)$$

$$X^{[1]}(T^2) + T^2D_x(\xi^1) - T^1D_x(\xi^2) = 0, \quad (24)$$

which yield

$$3T^1[c_3 - c_1] = 0,\ 3T^2[c_3 - c_1] = 0. \quad (25)$$

Equations (23) and (24) are satisfied when $c_1 = c_3$. Thus

$$X = [c_1x + \frac{c_2}{x^2}]\frac{\partial}{\partial x} + [(c_1 - \frac{c_2}{x^3})y + k(x)]\frac{\partial}{\partial y}$$

$$+[c_1\psi + c_4]\frac{\partial}{\partial \psi}, \quad (26)$$

is the Lie point symmetry generator associated with the conserved vector (21).

Now, $\psi = \phi(x,y)$ is a group invariant solution of the equation (4) if

$$X(\psi - \phi(x,y)) \mid_{\psi=\phi} = 0, \quad (27)$$

where operator X is given in equation (26). The equation (27) yields

$$[c_1x+\frac{c_2}{x^2}]\phi_x+[(c_1-\frac{c_2}{x^3})y+k(x)]\phi_y=c_1\phi+c_4. \quad (28)$$

The solution of equation (28) for $\psi=\phi(x,y)$ is of the form

$$\psi=\left(x^3+\frac{c_2}{c_1}\right)^{1/3}g(\xi)-\frac{c_4}{c_1},\ c_1\neq 0, \quad (29)$$

$$\xi=\frac{xy}{(x^3+\frac{c_2}{c_1})^{2/3}}-K(x), \quad (30)$$

where

$$K(x)=\frac{1}{c_1}\int^x\frac{x^3k(x)}{(x^3+\frac{c_2}{c_1})^{5/3}}dx. \quad (31)$$

The conserved quantity given by equation (20) becomes

$$J=2\rho\int_{-K(x)}^{\infty}\left(\frac{dg}{d\xi}\right)^2d\xi, \quad (32)$$

and is independent of x provided $K(x)$ is a constant. We choose constant to be zero and to make $K(x)=0$ we choose $k(x)=0$. Since the stream function is determined up to an arbitrary constant, c_4 can be chosen to be zero.

Now, the substitution of equation (29), with $c_4=0$ and $k(x)=0$, into equation (4) results in a third-order ordinary differential equation for $g(\xi)$:

$$\nu\frac{d^3g}{d\xi^3}+g\frac{d^2g}{d\xi^2}+\left(\frac{dg}{d\xi}\right)^2=0. \quad (33)$$

To solve equation (33), introduce

$$\eta=\frac{A\xi}{\nu},\ Af=g, \quad (34)$$

where A is arbitrary constant. Equation (33) transforms to

$$f'''+ff''+f'^2=0, \quad (35)$$

where prime denotes differentiation with respect to η. The boundary conditions (18) and (19) and conserved quantity (32), in terms of $f(\eta)$, become

$$f(0)=0,\quad f''(0)=0,\quad f'(\pm\infty)=0, \quad (36)$$

$$J=\frac{2A^3\rho}{\nu}\int_0^{\infty}f'^2d\eta. \quad (37)$$

The solution of equation (35) subject to (36) and condition $f(\infty)=1$ is (see [3], [7], [9])

$$f(\eta)=\tanh\left(\frac{\eta}{2}\right). \quad (38)$$

The value of A in terms of J is

$$A=\left(\frac{3\nu J}{2\rho}\right)^{1/3}. \quad (39)$$

Substitution of (38) and (39) and $Af=g$ in (29) yields

$$\psi=\left[\frac{3\nu J}{2\rho}\left(x^3+\frac{c_2}{c_1}\right)\right]^{1/3}f(\eta), \quad (40)$$

$$u(x,y)=\left[\frac{9J^2}{4\rho^2\nu(x^3+\frac{c_2}{c_1})}\right]^{1/3}f'(\eta), \quad (41)$$

and

$$\eta=\left[\frac{3Jx^3}{2\rho\nu^2(x^3+\frac{c_2}{c_1})^2}\right]^{1/3}y. \quad (42)$$

The symmetry which generated the group invariant solution is

$$X=\left(x+\frac{c_2}{c_1x^2}\right)\frac{\partial}{\partial x}+\left(1-\frac{c_2}{c_1x^3}\right)y\frac{\partial}{\partial y}+\psi\frac{\partial}{\partial\psi}. \quad (43)$$

The ratio of constants c_2/c_1 can be taken as l^3 where l is a characteristic length associated with the boundary-layer development. Equations (40)-(42) with $c_2/c_1=l^3$ agree with the results derived by Scwarz [1] and Riley [9]. In, [1], [9] the similarity solution method was used to solve (4) and the form of stream function was assumed not derived whereas in group invariant method the form is derived. Now $u(x,0)$ is finite at $x=0$ and so our solution may has some significance even near the axis. Scwarz in [1] discussed the methods how to determine l. By taking $c_2=0$ the results for infinite velocity at orifice [1], [5] can be rediscovered.

IV. Group invariant solution for radial wall jet

Equations (1) and (2) are the governing equations for the radial wall jet. The boundary conditions and conserved quantity for radial wall jet are

$$y=0:\quad u=0,\ v=0, \quad (44)$$

$$y=\infty:\quad u=0, \quad (45)$$

and

$$F=\int_0^{\infty}xu\left(\int_y^{\infty}xu^2dy^*\right)dy. \quad (46)$$

For the stream function ψ equations (44)-(46) take the following form:

$$y=0:\quad \psi_x=0,\ \psi_y=0, \quad (47)$$

$$y=\infty:\quad \psi_y=0, \quad (48)$$

$$F=\frac{1}{x}\int_0^{\infty}\psi_y\left(\int_y^{\infty}\psi_{y^*}^2dy^*\right)dy. \quad (49)$$

Riley [8], [9] derived the similarity solution for the radial wall jet with finite velocity at the orifice. We will derive the group invariant solution for the radial wall jet having finite velocity at orifice. The conserved vector

$$T^1=\frac{1}{x}\psi\psi_y^2,\ T^2=-\frac{1}{x}\psi\psi_x\psi_y+\frac{\nu}{2}\psi_y^2-\nu\psi\psi_{yy} \quad (50)$$

gives the conserved quantity (49) for radial wall jet (see [12]). Equations (23) and (24), for conserved vector (50) result in

$$T^1[4c_3-3c_1]+\frac{\partial T^1}{\partial\psi}c_4=0,\ T^2[4c_3-3c_1]+\frac{\partial T^2}{\partial\psi}c_4=0, \quad (51)$$

and thus

$$\frac{c_3}{c_1}=\frac{3}{4},\ c_4=0. \quad (52)$$

The Lie point symmetry generator associated with the conserved vector (50) is

$$X=[c_1x+\frac{c_2}{x^2}]\frac{\partial}{\partial x}+[(\frac{5}{4}c_1-\frac{c_2}{x^3})y+k(x)]\frac{\partial}{\partial y}$$

$$+\frac{3}{4}c_1\psi\frac{\partial}{\partial\psi}. \quad (53)$$

Now, $\psi = \phi(x, y)$ is a group invariant solution of the equation (4) if

$$X(\psi - \phi(x, y)) \mid_{\psi=\phi} = 0, \quad (54)$$

where X is given in (53). The equation (54) finally yields the group invariant solution for $\psi = \phi(x, y)$ of the form

$$\psi = \left(x^3 + \frac{c_2}{c_1}\right)^{1/4} g(\xi), \quad \xi = \frac{xy}{(x^3 + \frac{c_2}{c_1})^{3/4}} - K(x), \quad (55)$$

where

$$K(x) = \frac{1}{c_1}\int^x \frac{x^3 k(x)}{(x^3 + \frac{c_2}{c_1})^{7/4}}dx. \quad (56)$$

The conserved quantity

$$F = \int_{-K(x)}^{\infty} \frac{dg}{d\xi}\left(\int_\xi^\infty \left(\frac{dg}{d\xi^*}\right)^2 d\xi^*\right)d\xi \quad (57)$$

is independent of x only if $K(x) = 0$ which gives $k(x) = 0$. The insertion of equation (55) into equation (4) gives rise to a third-order ordinary differential equation for $g(\xi)$:

$$\nu\frac{d^3g}{d\xi^3} + \frac{3}{4}g\frac{d^2g}{d\xi^2} + \frac{3}{2}\left(\frac{dg}{d\xi}\right)^2 = 0. \quad (58)$$

Letting $\eta = \frac{3A}{4\nu}\xi$ and $g = Af$, then equation (58) transforms to

$$f''' + ff'' + 2f'^2 = 0, \quad (59)$$

where prime denotes differentiation with respect to η. The boundary conditions (47) and (48) and conserved quantity (57) become

$$f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 0, \quad (60)$$

and

$$F = \frac{3A^4}{4\nu}\int_0^\infty f'\left(\int_\eta^\infty f'^2 d\eta^*\right)d\eta. \quad (61)$$

Equation (55) results in

$$\psi = A\left(x^3 + \frac{c_2}{c_1}\right)^{1/4} f(\eta), \quad (62)$$

and

$$\eta = \frac{3Axy}{4\nu(x^3 + \frac{c_2}{c_1})^{3/4}}. \quad (63)$$

Integrating equation (59) twice, we have (see Glauret [6])

$$f^{-1/2}f' + \frac{2}{3}f^{3/2} = \text{constant}. \quad (64)$$

Glauret in [6] selected a solution with $f(\infty) = 1$ and value of constant was determined to be $2/3$. Equation (64) becomes

$$\frac{dh}{d\eta} = \frac{1}{3}(1 - h^3), \ \text{ where } h^2 = f, \ 0 \le h \le 1, \quad (65)$$

which gives

$$\eta = \log\frac{\sqrt{1 + h + h^2}}{1 - h} + \sqrt{3}\tan^{-1}\frac{\sqrt{3}h}{2 + h}. \quad (66)$$

The conserved quantity will be used to determine the unknown constant A. The integration in (61) with respect to η^* and η is transformed to integration with respect to h by using (65) and thus we have

$$F = \frac{2A^4}{\nu}\int_0^1 h\left(\int_h^1 h^{*2}(1 - h^{*3})dh^*\right)dh, \quad (67)$$

which yields

$$F = \frac{3A^4}{40\nu}. \quad (68)$$

Hence we obtain

$$\psi = \left[\frac{40F\nu}{3}(x^3 + \frac{c_2}{c_1})\right]^{1/4} f(\eta), \quad (69)$$

$$u(x, y) = \left[\frac{15F}{2\nu(x^3 + \frac{c_2}{c_1}))}\right]^{1/2} f'(\eta), \quad (70)$$

$$\eta = \left[\frac{135Fx^4}{32\nu^3(x^3 + \frac{c_2}{c_1})^3}\right]^{1/4} y. \quad (71)$$

The symmetry

$$X = \left(x + \frac{c_2}{c_1x^2}\right)\frac{\partial}{\partial x} + \left(\frac{5}{4} - \frac{c_2}{c_1x^3}\right)y\frac{\partial}{\partial y} + \frac{3}{4}\psi\frac{\partial}{\partial\psi} \quad (72)$$

generates the group invariant solution for radial wall jet. Equations (69)-(71) with $c_2/c_1 = l^3$ agree with the results derived by the similarity solution method (see [8], [9]). A method whereby l is roughly determined is given in [14]. By choosing $c_2 = 0$, the results for infinite velocity at orifice [6] can be rediscovered.

V. Group invariant solution for radial liquid jet

The governing equations for radial liquid jet are (1) and (2). The boundary conditions and conserved quantity are

$$y = 0: \quad u = 0, \ v = 0, \quad (73)$$

$$y = \phi(x): \quad u_y = 0, \quad (74)$$

and

$$M = \int_0^{\phi(x)} xudy. \quad (75)$$

The conserved vector

$$T^1 = xu, \ T^2 = xv \quad (76)$$

gives the conserved quantity (75) for the radial liquid jet (see [12]).

In terms of stream function the boundary conditions, conserved quantity and conserved vector become

$$y = 0: \quad \psi_x = 0, \ \psi_y = 0, \quad (77)$$

$$y = \phi(x): \quad \psi_{yy} = 0, \quad (78)$$

$$M = \int_0^{\phi(x)} \psi_y dy, \quad (79)$$

and

$$T^1 = \psi_y, \ T^2 = -\psi_x. \quad (80)$$

Riley [9] discussed the similarity solution for third-order partial differential equation for the stream function governing the flow in radial liquid jet having finite velocity at the orifice.

Watson in [10] studied the similarity solution for system of equations for the velocity components. We will derive the group invariant solution for third-order partial differential equation for the stream function for radial liquid jet having finite fluid velocity at the orifice.

Equations (23) and (24), for conserved vector (80) give

$$T^1c_3 = 0,\ T^2c_3 = 0, \tag{81}$$

and thus

$$c_3 = 0. \tag{82}$$

The Lie point symmetry generator associated with the conserved vector (80) is

$$X = [c_1x + \frac{c_2}{x^2}]\frac{\partial}{\partial x} + [(2c_1 - \frac{c_2}{x^3})y + k(x)]\frac{\partial}{\partial y} + c_4\frac{\partial}{\partial \psi}, \tag{83}$$

and generates the group invariant solution for radial liquid jet. The group invariant solution of the equation (4) subject to boundary conditions (77) and (78) is of the form

$$\psi = g(\xi) + \ln(x^3 + \frac{c_2}{c_1})^{c_4/3c_1}, \quad \xi = \frac{xy}{x^3 + \frac{c_2}{c_1}} - K(x), \tag{84}$$

where

$$K(x) = \frac{1}{c_1}\int^x \frac{x^3k(x)}{(x^3 + \frac{c_2}{c_1})^2}dx. \tag{85}$$

The condition that the conserved quantity is independent of x is satisfied only if $K(x) = 0$ which gives $k(x) = 0$. Since $\psi_x(x, 0) = 0$ and therefore $\psi(x, 0) =$ constant. Thus the stream function contains an additive constant and we can choose $c_4 = 0$ and therefore

$$\psi = g(\xi), \quad \xi = \frac{xy}{x^3 + \frac{c_2}{c_1}}, \tag{86}$$

which after substitution into equation (4) results in

$$\nu\frac{d^3g}{d\xi^3} + 3\left(\frac{dg}{d\xi}\right)^2 = 0. \tag{87}$$

Letting $\eta = \frac{A}{\nu}\xi$ and $g = Af$ in (87) yields

$$f''' + 3f'^2 = 0 \tag{88}$$

and the boundary conditions (77) and (78) are

$$f(0) = 0, \quad f'(0) = 0, \quad f''(1) = 0, \tag{89}$$

where the free surface is chosen to be $\eta = 1$. The stream function ψ becomes

$$\psi = Af(\eta), \quad \eta = \frac{Axy}{\nu(x^3 + \frac{c_2}{c_1})}. \tag{90}$$

Equation (88) yields (see [7], [9], [11])

$$f'' = [2(k_1 - f'^3)]^{\frac{1}{2}}. \tag{91}$$

The boundary condition $f''(1) = 0$ and assumption $f'(1) = 1$ give $k_1 = 1$. Define $t = f'$, (91) becomes

$$\frac{dt}{d\eta} = [2(1 - t^3)]^{\frac{1}{2}}. \tag{92}$$

The final form of solution of equation (88) in parametric form is (see [11])

$$\eta = \frac{2}{3\sqrt{2}}\left[{}_2F_1[\frac{1}{2}, \frac{2}{3}, \frac{3}{2}, 1] - (1 - t^3)^{\frac{1}{2}} \times {}_2F_1[\frac{1}{2}, \frac{2}{3}, \frac{3}{2}, 1 - t^3]\right], \tag{93}$$

where ${}_2F_1$ is the Hypergeometric function of first kind. Thus from equation (93) we may tabulate the values of η for given values of parameter $t = f'$. The constant A can be determined from the conserved quantity M. The conserved quantity (79) with the help of (92) becomes

$$M = \int_0^1 Af'd\eta = \frac{A}{\sqrt{2}}\int_0^1 t(1 - t^3)^{-1/2}dt, \tag{94}$$

which yields (see [9])

$$M = \frac{A\pi}{3\sqrt{3}}. \tag{95}$$

Therefore

$$\psi(x, y) = \frac{3\sqrt{3}M}{\pi}f(\eta), \tag{96}$$

$$u(x, y) = \frac{27M^2}{\nu\pi^2(x^3 + \frac{c_2}{c_1})}f'(\eta), \tag{97}$$

$$\eta = \frac{3\sqrt{3}Mx}{\nu\pi(x^3 + \frac{c_2}{c_1})}y. \tag{98}$$

The Lie-point symmetry that generates the group invariant solution is

$$X = \left(x + \frac{c_2}{c_1x^2}\right)\frac{\partial}{\partial x} + \left(2 - \frac{c_2}{c_1x^3}\right)y\frac{\partial}{\partial y}. \tag{99}$$

Equations (96)-(98) with $c_2/c_1 = l^3$ agree with the results derived by Riley [9] and Watson [10] by the similarity solution method. Watson [10] discussed how to compute l. By taking $c_2 = 0$, we get solution for infinite velocity at orifice.

VI. General form of group invariant solution for radial jets

The governing equations for radial free, wall and liquid jet are (1) and (2). By introducing the stream function ψ, system (1)-(2) reduces to a single third-order partial differential equation (4). For each of radial jets we have a conserved quantity and certain boundary conditions. The Lie-point symmetry generator is given in equation (14). The symmetry associated with the conserved vector that gives conserved quantity for jet flow generates the group invariant solution. We first calculated that symmetry and we obtain

$$c_3 = (2 - \alpha)c_1, \tag{100}$$

where $\alpha = 1$ for radial free jet, $\alpha = 5/4$ for radial wall jet $\alpha = 2$ for radial liquid jet. For the radial wall jet we obtain $c_4 = 0$ also. We choose $c_4 = 0$ for free and liquid jets because the stream function is determined up to an arbitrary additive constant and we specify that constant by choosing to zero. The group invariant solution is of the form

$$\psi = \left(x^3 + \frac{c_2}{c_1}\right)^{\frac{2-\alpha}{3}} g(\xi), \tag{101}$$

$$\xi = \frac{xy}{(x^3 + \frac{c_2}{c_1})^{\frac{\alpha+1}{3}}} - K(x), \qquad (102)$$

where

$$K(x) = \frac{1}{c_1}\int^x \frac{x^3 k(x)}{(x^3 + \frac{c_2}{c_1})^{\frac{\alpha+4}{3}}} dx. \qquad (103)$$

The condition that conserved quantity is independent of x provided $K(x)$ is a constant. We choose the constant to be zero. To make $K(x) = 0$ choose $k(x) = 0$, in each of free, wall and liquid jets. Equation (101) transforms equation (4) to a third-order ordinary differential equation for $g(\xi)$:

$$\nu\frac{d^3 g}{d\xi^3} + (2-\alpha)g\frac{d^2 g}{d\xi^2} + (2\alpha - 1)\left(\frac{dg}{d\xi}\right)^2 = 0. \qquad (104)$$

To solve equation (104), define the transformation

$$\eta = (2-\alpha)\frac{A}{\nu}\xi \quad \text{radial free and wall jets}$$

$$\eta = \frac{A}{\nu}\xi \quad \text{radial liquid jet}, \quad g = Af, \qquad (105)$$

where A is a constant. The final form of group invariant solution is

$$\psi = \left(x^3 + \frac{c_2}{c_1}\right)^{\frac{2-\alpha}{3}} Af(\eta), \qquad (106)$$

$$\eta = (2-\alpha)\frac{Axy}{\nu(x^3 + \frac{c_2}{c_1})^{\frac{\alpha+1}{3}}} \quad \text{radial free and wall jets}$$

$$\eta = \frac{Axy}{\nu(x^3 + \frac{c_2}{c_1})^{\frac{\alpha+1}{3}}} \quad \text{radial liquid jet}. \qquad (107)$$

Watson [10] suggested the same form for the stream function and η for the similarity solution. He just assumed the form whereas by group methods this form is derived. For radial free and wall jets equation (104), with the help of (106) and (107), is transformed to

$$f''' + ff'' + \frac{2\alpha - 1}{2 - \alpha}f'^2 = 0, \qquad (108)$$

and for radial liquid jet we obtain

$$f''' + 3f'^2 = 0. \qquad (109)$$

The symmetry that generates the group invariant solution is

$$X = (x + \frac{c_2}{c_1 x^2})\frac{\partial}{\partial x} + (\alpha - \frac{c_2}{c_1 x^3})y\frac{\partial}{\partial y} + (2-\alpha)\psi\frac{\partial}{\partial \psi}. \quad (110)$$

VII. Velocity profiles for free, wall and liquid jets

The velocity profile $f'(\eta)$ for radial free jet having finite or infinite fluid velocity is the same and is shown in Figure 1 (see [7], [9]). In Figure 2, the variations of η with $f'(\eta)$ for the radial wall jet are illustrated (see [6]). For radial liquid jet the velocity profile is shown in Figure 3 which agrees with Riley [7], [9].

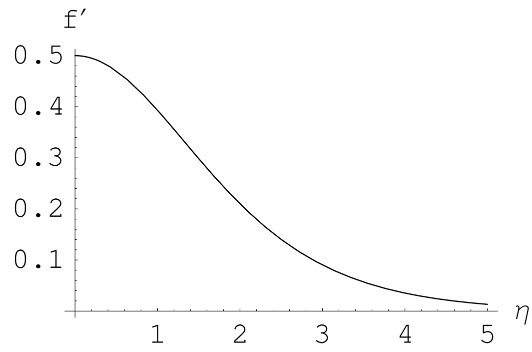

Fig. 1. The velocity profile $f'(\eta)$ for free jets.

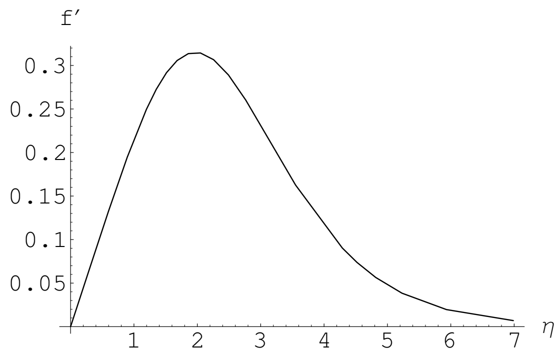

Fig. 2. The velocity profile $f'(\eta)$ for wall jets.

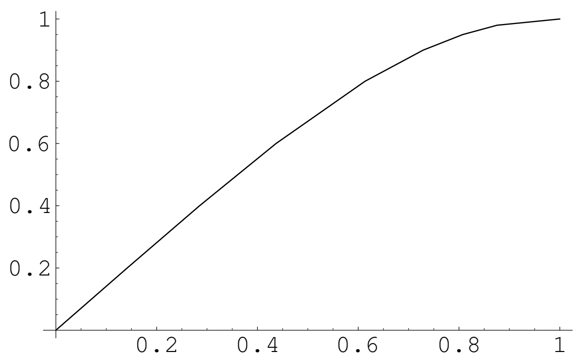

Fig. 3. The velocity profile $f'(\eta)$ for liquid jets.

VIII. Conclusions

The Lie-point symmetry generators for third-order partial differential equation for the stream function for radial jets were derived. The radial free, wall and liquid jets satisfy the same partial differential equation but boundary conditions and conserved quantity for each jet were different. For each jet a Lie point symmetry was associated with the conserved vector that generated the conserved quantity for each jet. The group invariant solution for the third-order partial equation for stream function subject to boundary conditions for each jet was generated by that Lie point symmetry. The velocity of fluid at the orifice was finite. The third-order partial differential equation was reduced to the third-order ordinary differential equation.

The velocity profile for radial free jet having finite or infinite velocity at orifice was same and was shown in Figure 1. The velocity profile, as shown in Figure 2, for radial wall jet having finite or infinite velocity at orifice was same. Also for the radial liquid jet the velocity profile having finite or infinite velocity at orifice was same and was shown if Figure 3.

The general form of group invariant solution for the radial jet flows was derived whereas for the similarity solution this form was just assumed and not derived.

REFERENCES

[1] W. H. Schwarz, *The radial free jet*, Chem. Eng. Sci. 18 (1963) 779-786.
[2] H. Schlichting, *Laminare Strahlausbreitung*, Z. Angew. Math. Mech. 13 (1933) 260-263.
[3] W. G. Bickley, *The plane jet*, Phil. Mag. 23 (1937) 727-731.
[4] D. P. Mason, *Group invariant solution and conservation law for free laminar two-dimensional jet*, Journal Nonlinear Math. Phys. 9 (2002) 92-101.
[5] R. Naz, F. M. Mahomed, D. P. Mason, *Conservation laws via the partial Lagrangian and group invariant solutions for radial and two-dimensional free jets*, Submitted to Journal of Nonlinear Analysis: Real world applications.
[6] M. B. Glauert, *The wall jet*, J. Fluid Mech. 1 (1956) 625-643.
[7] N. Riley, *Asymptotic expansions in radial jets*, J.Math. and Phys. 41 (1962) 132-146.
[8] N. Riley, *An Application of Mangler's transformation*, Quart. J. Mech. Applied. Math. 14 (1961) 197-202 .
[9] N. Riley, *Radial jets with swirl*, Part I. Incompressible Flow, Quart. Journ. mech. and Applied Math. 15, (1962) 435-458.
[10] E. J. Watson, *The radial spread of a liquid jet over a horizontal plane*, J. Fluid Mech. 20 (1964) 481-499.
[11] R. Naz, F. M. Mahomed and D. P. Mason, *Symmetry solutions of a third-order ordinary differential equation which arises from Prandtl boundary layer equations*, Journal Nonlinear Math. Phys. Accepted for publication.
[12] R. Naz, D. P. Mason and F. M. Mahomed, *Conservation laws and physical conserved quantities for laminar two-dimensional and radial jets*, Accepted for publication in journal of Nonlinear Analysis: Real World Applications.
[13] A. H. Kara and F. M. Mahomed, *Relationship between symmetries and conservation laws*, Int. J. Theor. Phys. 39 (2000) 23-40.
[14] M. B. Glauert, *symposium uber Grenzschichtforschung*, edited by H. Gortler (Springerverlag 1957).

Imran Naeem PhD student at School of Com- putational and Applied Mathematics, Centre for Differential Equations, Continuum Mechanics and Applications, University of the Witwatersrand, Jo- hannesburg, Wits 2050, South Africa.

Rehana Naz PhD student at School of Computational and Applied Mathematics, Centre for Differential Equations, Continuum Mechanics and Applications, University of the Witwatersrand, Johannesburg, Wits 2050, South Africa.