

# Generalized Method for Estimating Best-Fit Vertical Alignments for Profile Data

Said M. Easa and Shinya Kikuchi

**Abstract**—When the profile information of an existing road is missing or not up-to-date and the parameters of the vertical alignment are needed for engineering analysis, the engineer has to re-create the geometric design features of the road alignment using collected profile data. The profile data may be collected using traditional surveying methods, global positioning systems, or digital imagery. This paper develops a method that estimates the parameters of the geometric features that best characterize the existing vertical alignments in terms of tangents and the expressions of the curve, that may be symmetrical, asymmetrical, reverse, and complex vertical curves. The method is implemented using an Excel-based optimization method that minimizes the differences between the observed profile and the profiles estimated from the equations of the vertical curve. The method uses a ‘wireframe’ representation of the profile that makes the proposed method applicable to all types of vertical curves. A secondary contribution of this paper is to introduce the properties of the equal-arc asymmetrical curve that has been recently developed in the highway geometric design field.

**Keywords**—Optimization, parameters, data, reverse, spreadsheet, vertical curves.

## I. INTRODUCTION

THE parameters of an existing highway vertical alignment need to be estimated when the profile parameters according to which the roadway was built originally, are missing or are not up-to-date. This is a problem often faced when identifying the alignment parameters of the older roads. It is also necessary to check how a new (as-built) roadway adheres to the original specifications. Some provinces in Canada, for example, have collected GPS-based pitch and heading data every meter for thousands of kilometres of roads for the purpose of estimating alignment parameters. The profile parameters are necessary when examining the adequacy of the sight distance and recommending safety improvements, such as changes in speed limits. The estimation process involves collecting the profile data (distance and elevation of different points along the vertical profile) using traditional surveying methods, global positioning systems

(GPS), or digital imagery, and applying a mathematical method that identifies the vertical curve parameters [1, 2].

This paper proposes a method that estimates the parameters of a vertical alignment that best fit the collected profile data. The input data include the profile data and the bounds on the values of the unknown parameters. These parameters depend on the type of vertical curve, but they basically include the tangent grades and the vertical curvature of the parabolic curve. The proposed method derives the values of the parameters such that the squared sum or absolute value sum of the differences between the observed profile elevations and the elevations obtained from the estimated vertical curve alignment. The method is suited for estimating the parameters of vertical alignments involving any combination of vertical curve types.

Two basic types of vertical curves are described in surveying engineering literature. They are the symmetrical and asymmetrical curves [3–6]. The symmetrical curve consists of a parabolic curve connecting two tangents that are equal in length. It may be a crest or a sag curve, or a reverse curve that connects the crest and sag curves. There are two types of asymmetric curves: the traditional asymmetric curve [7] and the equal-arc asymmetrical curve [8]. The traditional asymmetric curve (TA) consists of two parabolic arcs that connect two unequal-length tangents, where for crest curves the point of common curvature (PCC) of the two arcs lies below the PVI and vice versa for sag curves. The two arcs have different curvatures. The curve length requirements to satisfy sight distance needs have been established [9, 10].

The equal-arc asymmetrical curve (EAA), not addressed in surveying engineering texts, has been developed to improve the features of the traditional asymmetric curve. The curve consists of two parabolic arcs with different curvatures, but its PCC lies in the middle of the curve. This curve enhances vertical alignment aesthetics, improves sight distance, reduces required curve length, increases vertical clearance, and improves driver comfort. The length requirements for the EAA curve have been established [11]. All or some of the three preceding curves may be present in the existing highway vertical alignment, and the proposed method identifies the alignment parameters from the existing profile data.

The following sections present previous work and motivation, the properties of vertical curves, and the problem and the proposed method. Application examples are then presented, followed by the concluding remarks.

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## II. PAST WORK AND MOTIVATION

Earlier research on estimating the vertical curves that best fit the existing profile data focused on symmetrical curves [12]. In this work, the estimation process was converted into a linear optimization (LP) model that involved four parameters: elevation of the point of vertical curvature (PVC), elevation of the point of vertical tangency (PVT), and the first and second tangent slopes (grades). The objective was to minimize the sum of the squared deviations between the observed and estimated elevations subject to symmetrical parabolic-curve geometric constraints. To obtain the global optimal solution, the LP model was solved repeatedly for various locations of the PVC and PVT using the LINGO software [13]. The estimation process was subsequently improved by automating the iterations using Visual Basic [14]. The method used the Solver software with spreadsheets that are familiar to most surveying professionals. The iterative method, however, requires a very long time to find the solution (e.g. about 10 hours for an increment of 0.1 m). To improve the solution time, an LP formulation that yields the global optimal solution in a minute was subsequently developed using binary variables that distinguish the endpoints between the curve and its tangents [15].

Another model that uses piecewise parametric equations has been developed and applied to GPS-based profile data of highway alignments [16]. In this model, each consecutive four data points are used to generate a B-spline curve and the spline had second-order continuity. This model is adequate for moderate accuracy levels only since the B-spline may have crest and sag shapes within the spline curve, resulting in a bumpy surface. Such a surface may adversely affect the accuracy of the calculated tangent slopes. It is clear that so far no method can estimate the parameters of complex vertical alignments simultaneously for all of its components (i.e. tangents and parabolic curves). Such a method is presented in this paper.

## III. PROPERTIES OF VERTICAL CURVES

The properties of three types of vertical curves are reviewed. A *symmetrical curve* consists of a parabolic arc connecting two equal-length tangents (see Fig. 1). The curve starts at PVC, ends at PVT, and its tangents intersect at the point of vertical intersection (PVI). The algebraic difference in slope of the curve (in decimals),  $A$ , is given by

$$A = g_1 - g_2 \quad (1)$$

where  $g_1$  = slope of the first tangent and  $g_2$  = slope of the second tangent. Note that  $A$  is positive for crest curves and negative for sag curves. The rate of change in slope,  $r$ , of the symmetrical curve equals  $A/L$ , where  $L$  is curve length.

The *asymmetrical curve* connects two tangents whose horizontal projections are not equal. It consists of two symmetrical curves (with different rates of change in slope) that have a common tangent at PCC. The curve length,  $L$ , is given by

$$L = L_1 + L_2 \quad (2)$$

where  $L_1$  is the shorter tangent and  $L_2$  is the longer tangent. The ratio between the shorter tangent length and the curve length is given by

$$R = L_1 / L \quad (3)$$

where  $R$  is called tangent ratio.

For the traditional asymmetrical curve, the PCC lies under the PVI. The rates of change in slope of the shorter (sharper) and longer (flatter) arcs,  $r_{1t}$  and  $r_{2t}$ , respectively, are given by:

$$r_{1t} = \frac{A(1-R)}{L R} \quad (4)$$

$$r_{2t} = \frac{A R}{L(1-R)} \quad (5)$$

For the equal-arc asymmetrical vertical curve, the PCC lies in the middle of the curve, thus minimizing the difference between the curvatures of the two arcs. The rates of change in slope for the shorter and longer arcs,  $r_{1e}$  and  $r_{2e}$ , are given by:

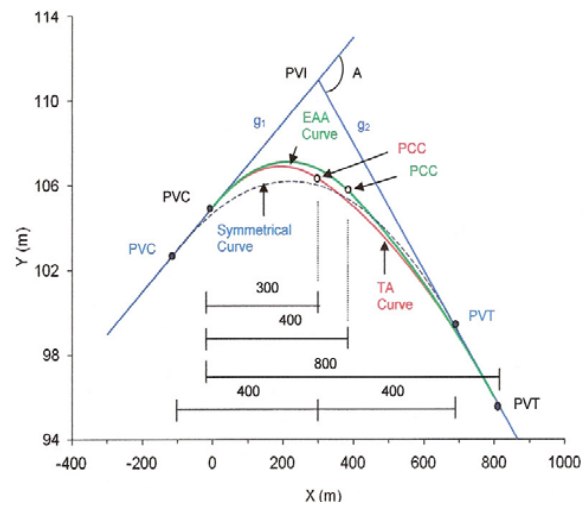


Fig. 1 Geometry of asymmetrical and symmetric vertical curves (TA = traditional asymmetrical and EAA = equal-arc asymmetrical)

$$r_{1e} = \frac{A(3-4R)}{L} \quad (6)$$

$$r_{2e} = \frac{A(-1+4R)}{L} \quad (7)$$

where

$$R > 0.25 \quad (8)$$

For the traditional asymmetrical curve,  $R$  should be greater than 0. Note that the symmetrical curve is a special case of the TA and EAA asymmetrical curves. For  $R = 0.5$ , (4) and (5) of the TA curve yield  $r_{1t} = r_{2t} = A/L$ , which is the rate of change in slope of a symmetrical curve, and similarly for (6) and (7) of the EAA curve.

A comparison of the two types of asymmetrical curves (TA and EAA) is shown also in Fig. 1 for  $L_1 = 300$  m,  $L_2 = 500$  m,  $g_1 = 0.02$ , and  $g_2 = -0.03$ . The PCC lies under PVI for the TA curve, while it lies in the middle of the curve for the EAA curve. Clearly, the first arc of the EAA curve is flatter than that of the TA curve, and vice

versa for the second arc. Thus, the EAA curve smoothes out the curvature of the TA curve.

The *reverse vertical curve*, which is especially useful for mountainous terrain and interchange ramps, connects two symmetrical vertical curves (one crest and the other sag) that have a common tangent at the PCC. The two curves may or may not have equal rates of change in slope and the first and second tangents may be parallel.

#### IV. PROBLEM AND PROPOSED METHOD

This section clarifies the problem at hand, formulates it as an optimization problem, and presents the proposed solution technique. In the process, a new concept, the 'wireframe' representation of the vertical alignment, is introduced. The wireframe represents a molded shape of the vertical alignment which is constructed using the least number of parameters. Since the symmetrical curve is a special case of the asymmetrical curve, the proposed method is described for only the asymmetrical curves.

##### A. The Problem

The situation at hand is as follows. The elevations along the vertical road profile are given at certain intervals. The problem is to determine the parameters of the mathematical expressions for the parabolic curves and the tangents that form the vertical alignment. The nature of this problem is in the category of adjustment, in which the mathematical expression's parameters are estimated from a given set of data. The parameters are determined so that the curve fits the observed profile elevations as much as possible. This is an optimization problem, in which the objective is to minimize the sum of the squared differences (SD) between the observed elevations and the elevations derived from the mathematical expressions:

$$\text{Minimize } z = \sum_{i=1}^N (y_i - Y_i)^2 \quad (\text{SD Criterion}) \quad (9)$$

where  $N$  = number of observations,  $y_i$  = observed  $y$ -coordinate, and  $Y_i$  = estimated  $y$ -coordinate.

Alternatively, the objective may be to minimize the absolute deviations (AD),

$$\text{Minimize } z = \sum_{i=1}^N |y_i - Y_i| \quad (\text{AD Criterion}) \quad (10)$$

where  $|\cdot|$  denotes absolute value. The AD criterion has certain merits in some cases, like detecting outliers and allowing a model to become linear in the objective function, and has been advocated by a number of researchers [17, 12, 14].

The constraints of the model depend on the type of vertical alignment and the specified parameters (decision variables) used for establishing the wireframe, which is explained next. For the asymmetrical curve, for example, the parameters may be selected as the  $x$ -coordinate of PVI and the  $x$  and  $y$ -coordinates of PVC, PVT, and a specified point on the first tangent. The constraints would then include relationships for: (1) the lower and upper bounds on the parameters, (2) the  $y$ -coordinate of PVI, (3) lengths of the two tangents and tangent ratio, (4) curve length and algebraic difference in slopes, (5) rates of change in slope ( $r_{1b}$ ,  $r_{2l}$  or  $r_{1e}$ ,  $r_{2e}$ ), and (6) slopes of the first, second, and common tangents (the latter depends on the type of asymmetrical curve). Alternatively, the parameters

may be selected as the slopes of the first and second tangents, the  $x$  and  $y$ -coordinates of PVC, and the  $x$ -coordinates of PVC and PVT. Similar constraints are then established to satisfy the properties of the asymmetrical curve.

##### B. Wireframe Representation

The wireframe is defined here as the geometric representation which uniquely establishes the vertical alignment with the minimum number of parameters. This number depends on the type of vertical alignment, but in general, the parameters whose bounds can be easily established should be selected. In the following subsections, the parameters of the wireframe are defined for the traditional asymmetrical curve, equal-are asymmetrical curve, reverse curve, and complex alignments.

*Traditional Asymmetrical Curves.* Consider the TA curve shown in Fig. 2a. The PVC, PVT, PVI, and PCC are denoted by A, B, C, and D, respectively. The common tangent at D intersects with the first and second tangents at C1 and C2, respectively. To facilitate establishing the lower and upper bounds on the parameters, the following seven parameters are selected:  $x_Q$ ,  $y_Q$ ,  $x_A$ ,  $y_A$ ,  $x_B$ ,  $y_B$ , and  $x_C$ , where the subscript  $Q$  = an arbitrary point on the first tangent, ( $x_Q$ ,  $y_Q$ ), ( $x_A$ ,  $y_A$ ), ( $x_B$ ,  $y_B$ ) =  $x$  and  $y$ -coordinates of Q, A, and B, respectively, and  $x_C$  =  $x$ -coordinate of C. The preceding seven variables completely define the wireframe. Note that different values in the parameters will only shift the placement of the wireframe, while the shape of the wireframe itself will remain intact. The basic relationships for establishing the wireframe (which become the constraints in the optimization model) are as follows.

The slope of the first tangent,  $s_A$ , is determined using the coordinates of Q and A,

$$s_A = (y_A - y_Q) / (x_A - x_Q) \quad (11)$$

The  $y$ -coordinate of C is then given by

$$y_C = y_A + s_A (x_C - x_A) \quad (12)$$

The slope of the second tangent is then determined using the coordinates of B and Q as

$$s_B = (y_B - y_Q) / (x_B - x_Q) \quad (13)$$

The length of the vertical curve  $L$  and the algebraic difference in slope  $A$  are given by

$$L = x_B - x_A \quad (14)$$

$$A = s_A - s_B \quad (15)$$

The lengths of the first and second tangents are given by

$$L_1 = x_C - x_A \quad (16)$$

$$L_2 = x_B - x_C \quad (17)$$

The tangent ratio,  $R$ , and the rates of change in slope,  $r_{1l}$  and  $r_{2b}$ , are calculated using (3)–(5), respectively. The slope of the common tangent,  $s_D$ , and the  $x$  and  $y$ -coordinates of D ( $x_D$ ,  $y_D$ ) are given by

$$s_D = s_A - r_{1l} L_1 \quad (18)$$

$$x_D = x_A + L_I \quad (19)$$

$$y_D = y_A + s_A L_I - r_{1I} L_I^2 / 2 \quad (20)$$

It should be noted that the order of the calculations of (11)–(20) is not important since they can be stored anywhere in the spreadsheet. The estimated coordinates,  $Y_i$ , are calculated using a conditional Excel formula that models the four regions of the TA curve, as follows

$$= \text{IF}(x \leq x_A, y = y_Q + s_A (x - x_Q), \text{IF}(\text{AND}(x \geq x_A, x \leq x_D), y = y_A + s_A (x - x_A) - r_{1I} (x - x_A)^2 / 2, \text{IF}(\text{AND}(x \geq x_D, x \leq x_B), y = y_D + s_D (x - x_D) - r_{2I} (x - x_D)^2 / 2, y = y_B + s_B (x - x_B)))) \quad (21)$$

which simply says that if  $x \leq x_A$ ,  $y$  is given by the equation of the first tangent; if  $x_A \leq x \leq x_D$ ,  $y$  is given by the equation of the first parabolic arc, if  $x_D \leq x \leq x_B$ ,  $y$  is given by the equation of the second parabolic arc; and if  $x \geq x_B$ ,  $y$  is given by the equation of the second tangent. The rates of change in slope,  $r_{1I}$  and  $r_{2I}$ , are calculated using (4) and (5). The deviation ( $y_i - Y_i$ ), the squared deviation, and the sum of the squared deviations are then calculated.

**Equal-Arc Asymmetrical Curve.** The EAA curve is shown in Fig. 2(b). The parameters and geometric characteristics of the curve are identical to those of the TA curve, except that (18)–(20) are given by

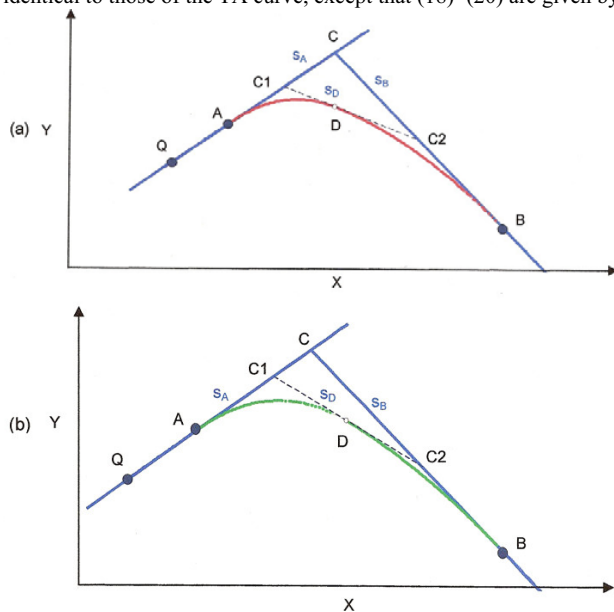


Fig. 2 Wireframe representation of asymmetrical vertical curves: (a) TA curve and (b) EAA curve

$$s_D = s_A - r_{1e} (L / 2) \quad (22)$$

$$x_D = x_A + (L / 2) \quad (23)$$

$$y_D = y_A + s_A (L / 2) - r_{1e} (L / 2)^2 / 2 \quad (24)$$

The rates of change in slope,  $r_{1e}$  and  $r_{2e}$ , are calculated using (6) and (7). The preceding individual formulations of the TA and EAA curves were unified using a variable that denotes the curve type. This variable allows the user to specify the type of vertical curve to be fitted to the data (TA, EAA, or symmetrical).

**Reverse Curves.** Since the intersection point of the two tangents of a reverse curve may lie very far from the profile data (or even at infinity for parallel tangents), the wireframe of the curve was represented in a different way. The parameters in this case were defined as the x and y-coordinates of Q, A, B, the x-coordinate of C, and the common slope at D, which define the wireframe shown in Fig. 3. Thus, the parameters are  $x_Q, y_Q, x_A, y_A, x_B, y_B, x_{C1}$ , and  $s_D$ . With these parameters, the elements of the wireframe can be calculated similar to those of the asymmetrical curves. The slope  $s_A$  is given by (11) and the coordinate  $y_{C1}$  is calculated by substituting for  $x_{C1}$  in (12). Then,  $x_D$  and  $y_D$  are given by

$$x_D = x_A + 2(x_{C1} - x_A) \quad (25)$$

$$y_D = y_{C1} + s_D (x_D - x_{C1}) \quad (26)$$

The coordinates of  $C_2$  and the slope of the second tangent are given by

$$x_{C2} = (x_D + x_B) / 2 \quad (27)$$

$$y_{C2} = y_D + s_D (x_{C2} - x_D) \quad (28)$$

$$s_B = y_B (-y_{C2}) / (x_B - x_{C2}) \quad (29)$$

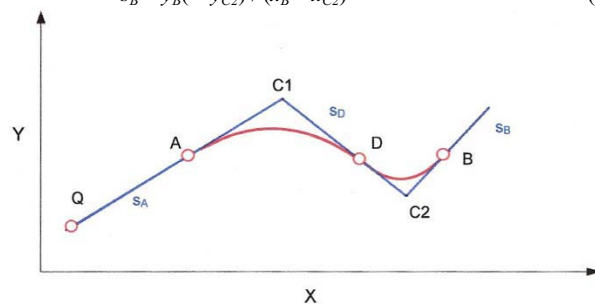


Fig. 3 Wireframe representation of reverse vertical curve

The lengths of the first and second symmetrical curves are given by  $L_1 = x_D - x_A$  and  $L_2 = x_B - x_D$  and the respective algebraic differences in slope are calculated based on (1) as  $A_1 = s_A - s_D$  and  $A_2 = s_D - s_B$ . The rate of change in slope are given by  $r_1 = L_1 / A_1$  and  $r_2 = L_2 / A_2$ . The estimated  $Y_i$  coordinates are calculated similar to (21). The calculation of the objective function is identical to that of the asymmetrical curves.

**Complex Alignments.** The proposed wireframe representation can be applied to more complex vertical alignments. Consider, for example, a reverse curve that consists of a crest asymmetrical curve followed by a sag symmetrical curve after an intermediate tangent. This would involve a simple modification to the formulation of the asymmetrical curves. In this case, the following sag symmetrical curve would require three additional parameters. Let the start, end, and the PVI of the symmetrical curve be denoted by E, F, and G. Then, the additional parameters will be  $x_E, x_F$ , and  $y_F$ . The y-coordinate of E is given by

$$y_E = y_B + s_B (x_E - x_B) \quad (30)$$

The coordinates of G ( $x_G, y_G$ ) and the slope of the second tangent of the symmetrical curve,  $s_F$ , are given by

$$x_G = (x_F + x_E) / 2 \quad (31)$$

$$y_G = y_B + s_B (x_G - x_B) \quad (32)$$

$$s_F = (y_F - y_G) / (x_F - x_G) \quad (33)$$

Other elements of the symmetrical curve are given by  $L = x_F - x_E$ ,  $A = s_B - s_F$ , and  $r = A / L$ . The estimated elevation of the symmetrical curve is then easily formulated using a conditional formula similar to (21).

### C. Solution Process

Once the objective and constraints of the optimization model are defined as shown above, the next step is to solve the model. This is a nonlinear, non-convex optimization model, can be solved using the Premium solver software [18]. This software implements the Generalized Reduced Gradient (GRG) technique that uses a multistart strategy for global optimization. The strategy generates candidate starting points with randomly selected values within the specified bounds of the parameters. These points are then grouped into clusters that are likely to lead to the same locally optimal solution. The solver is then run from a representative point in each cluster and continues with successively smaller clusters that are likely to capture each locally optimal solution. A Bayesian test is used to determine whether the process should continue [19], and ultimately the software converges in probability to a globally optimal solution. For more details on the software, the readers are encouraged to refer to [18].

A	B	C	D	E	F	G	H	I
1	$\Delta$ (%)	20		Type	T			
2								
3								
4	UB	24	2.88	120	5.64	264	4.68	186
5	Mid-point	20	2.4	100	4.7	220	3.9	155
6	LB	16	1.92	80	3.76	176	3.12	124
7								
8	Initial/Optimal Parameters	$x_0$	$y_0$	$x_A$	$y_A$	$x_B$	$y_B$	$x_C$
9		19.98578	2.384403	105.8184	4.899669	217.552	3.988046	149.291
10		$s_A$	$s_C$	$s_B$	$L$	$A$	$L_1$	$L_2$
11		0.029	6.17360	-0.03202	111.7336	0.06132	43.4726	68.2611
12		$R$	$r_1$	$r_2$	$s_0$	$s_D$	$y_D$	
13		0.38907	0.000862	0.000350	-0.00816	149.291	5.35929	
14		Observed Profile Data		Y Estimated	$v = \text{ABS}(y - Y)$	$v^2$	$R \geq 0$	
15		$x$	$y$				$R \leq 0.5$	
16		20	2.4	2.384819	0.015181	0.0002305		
17		40	2.9	2.970906	0.070906	0.0050276		
18		60	3.6	3.556992	0.043008	0.0018497		
19		80	4.2	4.143079	0.056921	0.00324		
20		100	4.7	4.729165	0.029165	0.0008506		
21		120	5.2	5.228593	0.028593	0.0008176		
22		140	5.4	5.397901	0.002099	4.404E-06		
23		160	5.3	5.251876	0.048124	0.0023159		
24		180	4.9	4.943933	0.043933	0.0019301		
25		200	4.5	4.496181	0.003819	1.459E-05		
26		220	3.9	3.909668	0.009668	9.346E-05		
27		240	3.3	3.269315	0.030685	0.0009416		
28		260	2.6	2.620962	0.020962	0.0008388		
29		280	2	1.98861	0.01139	0.0001297		
30				Sum	0.422454	0.0182846		

Fig. 4 Spreadsheet for estimating asymmetrical and symmetrical vertical curves (results are for the TA curve)

The software uses a spreadsheet in which the input data and the formulated model are presented. The spreadsheet for the unified formulation of vertical alignments is shown in Fig. 4. A detailed description of the spreadsheet is presented in Appendix 1.

### D. Key Features of Proposed Method

The following key features of the proposed method are worth noting:

- 1) *Applicability to both Crest and Sag Vertical Curves.* The wireframe representation allows the estimation of the best-fit

crest or sag vertical curves without any additional task. That is, the concept is applicable to both crest and sag asymmetrical curves, to both sag-crest and crest-sag reverse curves, and to any configuration of complex alignments.

- 2) *Ability to Estimate all Types of Vertical Curves.* The proposed method can be used to estimate the vertical curve (TA, EAA, and symmetrical) that best fits the profile data. This is done by running the software for each type individually and selecting the one that produces the least objective function.
- 3) *Parameter Bounds are easily Established.* Approximate values of the parameters can be determined using a graph of the profile data. The approximate values are input to the spreadsheet and used to establish the lower and upper bounds on the parameters automatically using a specified percentage deviation from the approximate values. This feature allows the user to change the bounds in both the spreadsheet and the Solver Parameters window by changing only the percentage deviation.
- 4) *Close Initial Approximations of the Parameters are not required.* Unlike traditional linearized least-squares problems, close initial approximations of the parameters are not required. The input initial values of the parameters can be set equal to zero and, in this case, the software will use the specified bounds to conduct the search for the optimal solution. It is suggested, however, that the approximate values used to establish the bounds be used as initial values of the parameters since this would speed up the finding of the optimal solution. Note that using reasonable values of the parameters is particularly useful during spreadsheet formulation since these values and the constraints can be used to plot the wireframe and check the correctness of the formulation.

## V. APPLICATION EXAMPLES

The proposed method is applied to three examples involving data for TA, EAA, and symmetrical curves (Example 1), reverse curve (Example 2), and complex alignment (Example 3). The assumed observed profile data for each example are shown in Table 1 (throughout, all units of the x and y-coordinates are in meters). In all examples, the optimization is based on the SD criterion of (9) which is normally used in surveying applications.

### A. Example 1: Asymmetrical and Symmetrical Curves

Consider the profile data shown in Fig. 5. The parameters and curve elements for the TA, EAA, and symmetrical curves are shown in Tables 2 and 3. For the TA curve, the initial values in the parameters were:  $x_0 = 25$ ,  $y_0 = 4.5$ ,  $x_A = 110$ ,  $y_A = 6$ ,  $x_B = 220$ ,  $y_B = 5$ , and  $x_C = 160$ , which correspond to the wireframe in Fig. 5(a). The approximate values of the parameters were first determined using a graphical plot of the data. Assuming  $\Delta = 20\%$ , the lower and upper bounds of the parameters are established, as shown in Fig. 4. All calculations and results are shown in the figure. By setting *TYPE* = T, a traditional asymmetrical curve was fitted to the profile data, as shown in Fig. 5(b).

The value of the objective function was  $z = 0.0183$ . Since there are 14 observations, the approximate average deviation between the observations and the estimated profile equals  $(z/14)^{0.5} = 0.036$  m. Clearly, this is excellent fit, but it reflects the hypothetical data used in this application example. In actual applications, the accuracy of the fit would depend on the method used for collecting the profile data. The optimal curve elements are  $L_1 = 41.74$  m,  $L_2 = 63.66$  m,  $L = 105.40$  m,  $R = 0.40$ ,  $s_A = 0.029$ ,  $s_B = -0.032$ ,  $r_1 = 0.00086$ , and  $r_2 = 0.00035$ .

TABLE I  
HYPOTHETICAL (OBSERVED) PROFILE DATA  
FOR APPLICATION EXAMPLES

Example 1: TA, EAA, Symm. Curves (x=20-280)		Example 2: Reverse Curve (x=20-280)		Example 3: Complex Alignment (x = 20-520)	
x (m)	y (m)	x (m)	y (m)	x (m)	y (m)
20	2.4	20	2.4	20	2.4
40	2.9	40	2.9	40	2.9
60	3.6	60	3.6	60	3.6
80	4.2	80	4.2	80	4.2
				300	1.4
				320	1
				340	0.7
				360	0.6

100	4.7	100	4.7	100	4.7	380	0.8
120	5.2	120	5.2	120	5.2	400	1.3
140	5.4	140	5.4	140	5.4	420	1.9
160	5.3	160	5.3	160	5.3	440	2.6
180	4.9	180	4.9	180	4.9	460	3.4
200	4.5	200	4.4	200	4.5	480	4.2
220	3.9	220	3.9	220	3.9	500	5
240	3.3	240	4	240	3.3	520	5.8
260	2.6	260	5	260	2.7	-	-
280	2	280	6	280	2	-	-

TABLE II  
OPTIMAL OBJECTIVE VALUE AND PARAMETERS FOR THE TA, EAA, AND  
SYMMETRICAL CURVES

Curve Type	Z	$x_Q$	$y_Q$	$x_A$	$y_A$	$x_B$	$y_B$	$x_C$
TA	0.0183	18.00	2.34	107.92	4.93	213.32	4.11	149.66
EAA	0.0151	18.03	2.34	105.19	4.86	209.80	4.22	149.31
Symmetrical	0.0471	18.85	2.35	90.00	4.47	208.60	4.29	149.30

TABLE III  
OPTIMAL ELEMENTS FOR THE TA, EAA, AND SYMMETRICAL CURVES

Curve Type	$s_A$	$s_B$	$y_C$	$x_D$	$y_D$	$s_D$	$L_1$	$L_2$	R
TA	0.029	-0.032	6.13	149.66	5.37	-0.008	41.74	63.66	0.40
EAA	0.029	-0.032	6.13	157.50	5.33	-0.011	44.12	60.49	0.42
Symmetrical	0.030	-0.033	6.23	149.30	5.31	-0.002	59.30	59.30	0.5

The EAA curve was fitted to the same profile data (Table 1, Example 1) by setting *TYPE* = E. The fitted curve is shown in Fig. 6. The objective value was  $z = 0.0151$  which is slightly better than that of the TA curve ( $z = 0.0183$ ). The elements corresponding to the optimal solution are  $L_1 = 44.12$  m,  $L_2 = 60.49$  m,  $L = 104.61$  m,  $R = 0.42$ ,  $s_A = 0.029$ ,  $s_B = -0.032$ ,  $r_1 = 0.00076$  and  $r_2 = 0.00040$ . As noted,  $r_1$  of the EAA curve is smaller than that of the TA curve and  $r_2$  is larger. This is expected since the EAA curve smoothes out the curvature of the TA curve, as previously mentioned (even though the start and end points of the two curves are slightly different). Also, the lengths and tangent ratios of the two curves are very close, and the tangent slopes are identical (for three decimals).

By setting *TYPE* = S, the best-fit symmetrical curve for the same profile data was estimated. The corresponding global optimal solution was  $z = 0.0471$ . Clearly, the TA and EAA curves fit the profile data better than the symmetrical curve, since they provide smaller  $z$  values, as shown in Table 2. This is expected since during the estimation of these curves, the global optimal solution would have corresponded to  $R = 0.5$  if this value provided the best fit. Based on the preceding results, the EAA curve would be selected as it provides the best fit because of the smallest  $z$  value.

#### B. Example 2: Reverse Curve

Consider the profile data for the reverse curve (Table 1, Example 2) which consist of a crest followed by a sag curve as shown in Fig. 7. Based on the formulation previously presented, the best-fit vertical curve is shown in the figure and corresponds to  $z = 0.0239$ . The optimal parameters were  $x_Q = 16.27$ ,  $y_Q = 2.26$ ,  $x_A = 97.15$ ,  $y_A = 4.67$ ,  $x_B = 252.00$ ,  $y_B = 4.55$ ,  $x_C = 150.93$ , and  $s_D = -0.038$ . The tangent slopes were  $s_A = 0.0298$  and  $s_B = 0.0523$ .

#### C. Example 3: Complex Alignment

To illustrate the case of a complex vertical alignment, consider a reverse vertical curve consisting of a crest asymmetrical curve followed by a sag symmetrical curve after an intermediate tangent. The corresponding data are given in Table 1 under Example 3. The formulas previously presented for reverse curves were added to the spreadsheet of Fig. 4 along with the observed profile data for the symmetrical curve which followed the existing data on the spreadsheet. The optimal curve is shown in Fig. 8 and corresponds to  $z = 0.0353$ . The values of the parameters are  $x_Q = 19.54$ ,  $y_Q = 2.37$ ,  $x_A = 105.98$ ,  $y_A = 4.90$ ,  $x_B = 215.22$ ,  $y_B = 4.06$ ,  $x_C = 148.47$ ,  $x_E = 296.19$ ,  $x_F = 425.02$ , and  $y_F = 4.54$ . The tangent slopes are  $s_A = 0.0293$ ,  $s_B = -0.0313$ , and  $s_F = 0.0393$ . Other curve elements can be easily



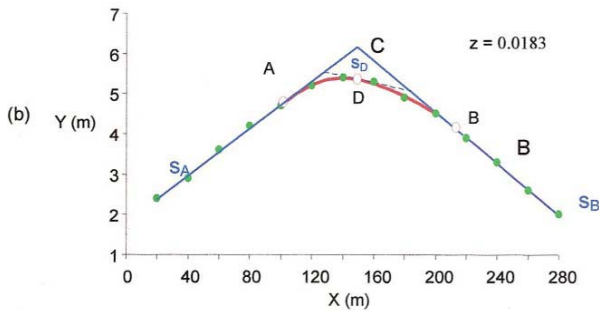
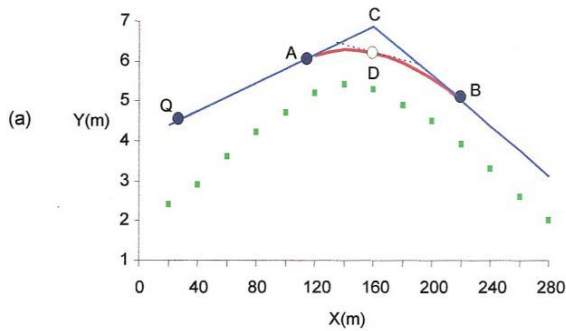


Fig. 5 Best-fit TA vertical curve: (a) Wireframe location before optimization and (b) Fitted curve after optimization

calculated. Clearly, the wireframe representation is working well for all types of vertical curves, including complex alignments.

## VI. CONCLUDING REMARKS

This paper has presented a generalized method for estimating the optimal vertical alignments (parameters for the parabolic curve and tangents) for given vertical profile data. The proposed wireframe concept allows the user to estimate the parameters of different types of vertical curves (asymmetrical, symmetrical, reverse, and complex) easily and efficiently. The concept can be used to model a complex vertical alignment consisting of any number and order of vertical curves and tangents along the roadway. The optimization model uses

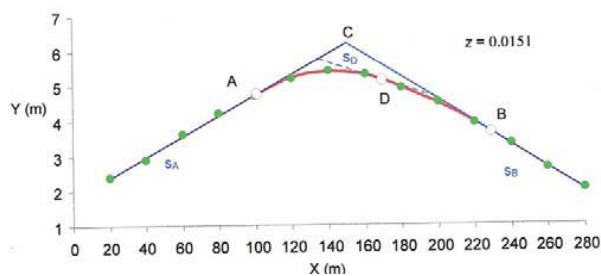


Fig. 6 Best-fit EAA vertical curve

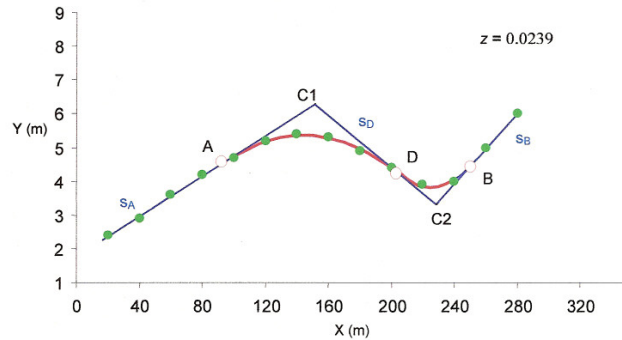


Fig. 7 Best-fit reverse vertical curve

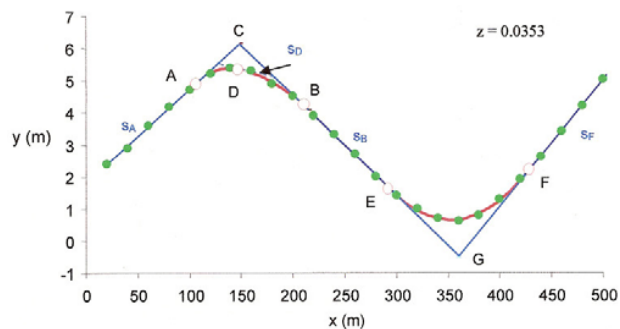


Fig. 8 Best-fit complex vertical alignment consisting of a TA crest curve followed by a symmetrical curve after an intermediate tangent

the Generalized Reduced Gradient technique that converges to a globally optimal solution quickly. The proposed method implements a spreadsheet that is familiar to most surveying professionals.

The bounds on the parameters are established using approximate values input by the user initially. It is essential that the bounds be large enough to capture the actual global optimal solution. This can be checked by comparing each optimal value in the parameters with the respective lower and upper bounds. If an optimal value equals the lower or upper bound, this would indicate that the respective bound is binding the optimal solution, and that the global solution may lie outside that bound. In this respect, the formulated spreadsheet allows the user to easily relax all bounds by changing a simple variable.

It is unlikely that the profile data for any constructed EAA curve will be missing since the curve has only been recently developed. However, the curve may be considered in estimating old traditionally-designed asymmetrical curves with missing profile data. The reason for this is that the curve may provide a better fit to the data than the traditional asymmetrical curve owing to the differential settlements that may have occurred in the roadway profile over the years. In addition, when as-built roadways are checked for compliance with the original specifications, the EAA curve may be part of the alignment and would therefore require to be considered in the estimation process.

The EAA curve represents a useful tool in the design of asymmetrical curves since it provides important benefits. It is hoped that the curve will be introduced in the surveying engineering texts to help provide surveying engineering students and professionals with all the tools of vertical curve design.

# APPENDIX 1: DESCRIPTION OF SPREADSHEET FORMULATION

The spreadsheet used as input to the Premium Solver software is shown in Fig. 4. The spreadsheet includes the unified formulation and corresponds to the results of the TA curve. The entries of the spreadsheet are as follows:

- 1) Store the type of vertical curve *TYPE* in F2, where *TYPE* = T, E, or S for the TA, EAA, and symmetrical curves, respectively.
- 2) The lower and upper bounds on the parameters are required. The user need only provide in the spreadsheet the approximate initial values of the parameters (C5–I5) and the desired percentage deviation,  $\Delta$ , of the lower and upper bounds from these values (C2). These initial values can be determined from a graphical plot of the profile data. The spreadsheet uses  $\Delta$  to calculate the lower and upper bounds of the parameters which are stored in C6–I6 and C4–I4, respectively. These bounds are added in the Solver Parameters window in terms of cell addresses rather than numerical values. For example, the bounds on  $x_A$  are written as,  $E9 \geq E6$  and  $E9 \leq E4$ . In this way, the user can evaluate different bounds by changing only  $\Delta$  in the spreadsheet.
- 3) The optimal parameters are stored in C9–J9. Before the solution starts, the mid-point values in C5–I5 can be input as the initial values of the parameters. Such values would speed up the optimization process.
- 4) The variables  $s_A$ ,  $y_C$ ,  $s_B$ ,  $L$ ,  $A$ ,  $L_1$ ,  $L_2$ , and  $R$  are calculated using (11)–(17) and (3), respectively, and are stored in C12–I12 and C15. All the preceding variables are the same for both the TA and EAA curves.
- 5) The variables that are different for the TA and EAA curves are  $r_1$ ,  $r_2$ ,  $s_D$ ,  $x_D$ , and  $y_D$ . These variables are calculated using (18)–(20) of the TA curve or (22)–(24) of the EAA curve, and are stored in D15–H15. The conditional formulas for making these calculations, which are based on the curve type, are listed in Table 4.
- 6) The constraints on  $R$  are stored in I17 and I18 which represent the lower and upper bounds on  $R$ , respectively. The upper bound is 0.5 and the lower bound is given by

$$=IF(F2="T",0,IF(F2="E",0.25,0.5)) \quad (\text{lower bound}) \quad (34)$$

Equation (34) states that if the curve type is T (TA curve) the lower bound is zero, if the curve type is E (EAA curve) the lower bound is 0.25 and otherwise the lower bound is 0.5 (symmetrical curve). Thus, for the symmetrical curve the upper and lower bound constraints will be  $R \leq 0.5$  and  $R \geq 0.5$ , which are equivalent to  $R = 0.5$  as required. Two constraints are then added in the Solver Parameters window:  $C15 \geq I17$  and  $C15 \leq I18$ . For the TA curve, the user may specify a lower bound on  $R$  by replacing the zero value in (34) with the desired limit.

- 7) The observed  $x$  and  $y$ -coordinates are added in C19–C32 and D19–D32, respectively.

TABLE IV  
CONDITIONAL FORMULAS FOR ASYMMETRICAL AND SYMMETRICAL VERTICAL CURVE ESTIMATION

Variable	Cell	Conditional Formula
$r_{1t}, r_{1e}, \text{ or } r$	D15	$=IF(G5="T",L2*(1-D19)/(G16*D19),IF(G5="E",L2*(3-4*D19)/G16,G12/F12))$
$r_{2t}, r_{2e}, \text{ or } r$	E15	$=IF(G5="T",L2*D19/(G16*(1-D19)),IF(G5="E",L2*(4*D19-1)/G16,G12/F12))$
$s_D$	F15	$=IF(G5="T",D16-E19*I16,IF(G5="E",D16-E19*G16/2,D16-E19*G16/2))$
$x_D$	G15	$=IF(G5="T",F13+I16,IF(G5="E",F13+G16/2,F13+G16/2))$
$y_D$	H15	$=IF(G5="T",G13+D16*I16-E19*I16^2/2,IF(G5="E",G13+D16*G16/2-E19*(G16/2)^2/2,G13+D16*G16/2-E19*(G16/2)^2/2))$

- 8) The estimated  $Y_i$  coordinates are stored in E19–E32 based on (21).
- 9) The absolute and squared deviations are calculated based on the observed and estimated elevations ( $y_i$  and  $Y_i$ ) and are stored in F19–F32 and G19–G32, respectively. The sums of the absolute and squared deviations are stored in F33 and G33, respectively, and the specific cell to be minimized is then marked in the Solver Parameters window.

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