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Generalization Kernel for Geopotential Approximation by Harmonic Splines

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Abstract—This paper presents a generalization kernel for gravitational potential determination by harmonic splines. It was shown in [10] that the gravitational potential can be approximated using a kernel represented as a Newton integral over the real Earth body. On the other side, the theory of geopotential approximation by harmonic splines uses spherically oriented kernels. The purpose of this paper is to show that in the spherical case both kernels have the same type of representation, which leads us to conclusion that it is possible to consider the kernel represented as a Newton integral over the real Earth body as a kind of generalization of spherically harmonic kernels to real geometries.

Keywords—Geopotential, Reproducing Kernel, Approximation, Regular Surface

I. Introduction

The actual problem of gravitational theory is the determination of a harmonic function (regular at infinity), to certain linear functionals, for example, discrete boundary data on the Earth's surface or discrete satellite data from space. In consequence, gravitational field theory canonically leads to interpolation based on a specific linear functionals, usually functional values or derivatives in certain (discretely given) points. In the conventional geodetic approach due to [7], [11], it was proposed, that the class of approximating functions should conveniently be structured as a Hilbert space with reproducing kernel. Interpolation of the Earth's gravitational potential field in terms of reproducing kernels immediately leads to a spline formulation.

Considering spherical approximations to the shape of the Earth, this can be seen by the well–known theory of spherical harmonic splines. There is an extensive list of publications in geomathematics considering this spherical approach. Numerous applications from theory of spherical harmonic splines has been used with very good results. Interested reader is referred to the list of publications of the AG Geomathematik at the TU Kaiserslautern. On the other hand, following the work of [11] it was shown in [10] that it is also possible to develop the real Earth body methods for geopotential determination by using a reproducing kernel expressed as a Newton integral over the real body of the Earth. Here it will be shown that this kernel represents a generalization to spherically oriented kernels to real geometries.

II. Spherical Harmonic Splines

Mathematical methods for approximation of gravitational potential, like spherical harmonic splines have in their foundation the Runge approach, which means that they are considering the Runge (or Bjerhammar sphere), which is completely situated in the Earth's interior. Next we introduce the geometrical concepts behind this theory.

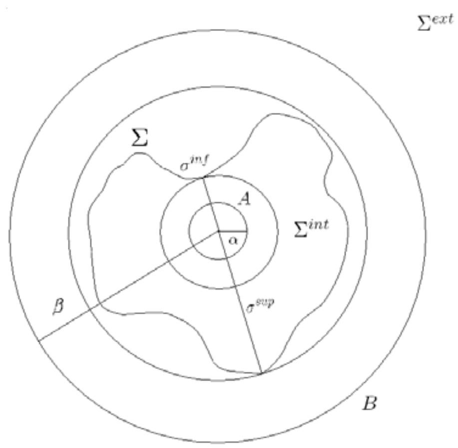

Fig. 1. The geometric concept of a regular surface

Definition 1: A surface $\Sigma \subset \mathbb{R}^3$ is called *a $C^{(k)}$-regular surface*, if it satisfies the following properties:

(i) Σ divides $\mathbb{R}^3$ into the bounded region Σ^{int} (inner space) and unbounded region Σ^{ext} (outer space) defined by $\Sigma^{ext} = \mathbb{R}^3 \setminus \overline{\Sigma^{int}}$, $\overline{\Sigma^{int}} = \Sigma^{int} \cup \Sigma$.
(ii) Σ is a closed and compact surface free of double points.
(iii) The origin 0 is contained in Σ^{int}.
(iv) Σ is locally a $C^{(k)}$-surface (i.e., every point $x \in \Sigma$ has an open neighborhood $\mathcal{U} \subset \mathbb{R}^3$ such that $\Sigma \cap \mathcal{U}$ has a parametrization which is k-times continuously differentiable).

Given a regular surface, there exist a positive constants α, β, such that

$$\alpha < \sigma^{inf} = \inf_{x\in\Sigma} |x| \leq \sup_{x\in\Sigma} |x| = \sigma^{sup} < \beta. \qquad (1)$$

By A^{int}, B^{int} (resp. A^{ext}, B^{ext}) we denote the inner (resp. outer) space of the sphere A resp. B around the origin with radius α resp. β. A is a so–called *'Runge sphere'* for Σ^{ext}.

The theory of spherical harmonic interpolation is well–known (see [1], [2], [4], [5], [8], [9]). However, we need to present in short some basic elements of this theory.

In the following we denote by Σ the real Earth surface, and by Σ^{int} and Σ^{ext} its interior and exterior respectively.

We first define a class of potentials, namely Pot(Σ^{ext}) as

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the space of all functions U in $C^{(2)}(\Sigma^{ext})$ satisfying the Laplace equation in the outer space Σ^{ext} and being regular at infinity (that is, $|U(x)| = O(|x|^{-1})$, $|\nabla U(x)| = O(|x|^{-2})$ for $|x| \to \infty$ uniformly with respect to all directions $\xi = \frac{x}{|x|}$).
For $k = 0, 1, \ldots$ we denote by $\mathrm{Pot}^{(k)}(\overline{\Sigma^{ext}})$ the space of all functions $U \in C^{(k)}(\overline{\Sigma^{ext}})$ such that $U|_{\Sigma_{ext}}$ is of class $\mathrm{Pot}(\Sigma_{ext})$. In shorthand notation,

$$\mathrm{Pot}^{(k)}(\overline{\Sigma^{ext}}) = \mathrm{Pot}(\Sigma^{ext}) \cap C^{(k)}(\overline{\Sigma^{ext}}). \tag{2}$$

Let U be of class $\mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}})$. Then the maximum/minimum principle for the outer space Σ^{ext} gives

$$\sup_{x \in \overline{\Sigma^{ext}}} |U(x)| \le \sup_{x \in \Sigma} |U(x)|. \tag{3}$$

We next introduce the most commonly used harmonic functions for representing scalar functions on a spherical surface, namely the spherical harmonics. They form a complete orthonormal system in the Hilbert space $L^2(\Omega)$, (Ω denotes a unit sphere) and thus can be used for the construction of Fourier series in $L^2(\Omega)$. Spherical harmonics of different degrees are orthogonal in the sense of the $L^2(\Omega)$- inner product

$$(Y_n, Y_m)_{L^2(\Omega)} = \int_\Omega Y_n(\xi) Y_m(\xi) d\omega(\xi) = 0, \quad n \ne m. \tag{4}$$

For the Laplace operator Δ in $\mathbb{R}^3$ we have the representation

$$\Delta = \left(\frac{\partial}{\partial r}\right)^2 + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta^*_\xi, \tag{5}$$

where Δ^* is the Beltrami operator on the unit sphere Ω. For explicit representations in polar coordinates see [6].
Any spherical harmonic $Y_n, n \in \mathbb{N}_0$, is an infinitely often differentiable eigenfunction of the Beltrami operator, corresponding to the eigenvalue $-n(n+1), n \in \mathbb{N}_0$. A special class of functions, in close connection to spherical harmonics, are the Legendre polynomials. They can be defined via the Legendre operator

$$L_t = (d/dt)(1 - t^2)(d/dt),$$

which is the 'longitude-independent part' of the Beltrami operator. The Legendre polynomial
$P_n : [-1, +1] \to \mathbb{R}$ of degree n is the (uniquely defined) infinitely often differentiable eigenfunction of the Legendre operator L_t, corresponding to the eigenvalue $-n(n+1)$. It is well–known that the Legendre polynomials are orthogonal with respect to the $L^2([-1, +1])-$inner product, i.e.,

$$\int_{-1}^{+1} P_n(t) P_m(t) dt = \frac{2}{2n+1}\delta_{n,m}, \tag{6}$$

where $\delta_{n,m}$ is the Kronecker symbol. The system $\{P_n\}_{n \in \mathbb{N}_0}$ of all Legendre polynomials is closed and complete in $L^2([-1, +1])$, with respect to
$\| \cdot \|_{L^2[-1,+1]}$.
For $t \in [-1, 1]$ and all $h \in (-1, 1)$

$$\sum_{n=0}^{\infty} P_n(t) h^n = \frac{1}{\sqrt{1 + h^2 - 2ht}}. \tag{7}$$

Also, for $0 \le h < 1$ and $t \in [-1, 1]$ the following series representation can be derived from (7)

$$\sum_{n=0}^{\infty} (2n+1) P_n(t) h^n = \frac{1 - h^2}{(1 + h^2 - 2ht)^{3/2}}. \tag{8}$$

The following theorem, known as the addition theorem for spherical harmonics, relates functions on the unit sphere (spherical harmonics) of degree n to the univariate functions defined on the interval $[-1, +1]$ (Legendre polynomials).

Theorem 2: (Addition Theorem for Spherical Harmonics) Let $\{Y_{n,k}\}_{k=1,\ldots,2n+1}$ be an orthonormal system of spherical harmonics with respect to $(\cdot, \cdot)_{L^2(\Omega)}$ in $Harm_n(\Omega)$. Then

$$\sum_{k=1}^{2n+1} Y_{n,k}(\xi) Y_{n,k}(\eta) = \frac{2n+1}{4\pi} P_n(\xi \cdot \eta), \quad \xi, \eta \in \Omega. \tag{9}$$

Next we consider a sphere $\Omega_R \subset \mathbb{R}^3$ around the origin with radius $R > 0$. Denote by Ω_R^{int} and Ω_R^{ext} the inner and the outer space of Ω_R, respectively. By virtue of the isomorphism $\Omega \ni \xi \mapsto R\xi \in \Omega_R$ we can assume functions $F : \Omega \to \mathbb{R}$ to be defined on Ω_R. With the surface measure $d\omega_R$ of Ω_R,

$$d\omega_R = R^2 d\omega, \tag{10}$$

we are able to introduce the $L^2(\Omega_R)$ - inner product $(\cdot, \cdot)_{L^2(\Omega_R)}$ and the associated norm $\| \cdot \|_{L^2(\Omega_R)}$, as usual. Obviously, an $L^2(\Omega)$ - orthonormal system of spherical harmonics forms an orthogonal system on Ω_R (with respect to $(\cdot, \cdot)_{L^2(\Omega_R)}$).

The function spaces defined on Ω have their natural generalizations as spaces of functions defined on Ω_R. We have for example, $C(\Omega_R), L^p(\Omega_R)$, etc.
The system of spherical harmonics $\{Y^R_{n,k}\}_{\substack{n=0,1,\ldots \\ k=1,\ldots,2n+1}}$, where

$$Y^R_{n,k}(x) = \frac{1}{R} Y_{n,k}\left(\frac{x}{|x|}\right), \quad x \in \Omega_R, \tag{11}$$

is orthonormal in $L^2(\Omega_R)$-sense.

The system $\{H^\alpha_{-n-1,k}\}_{\substack{n=0,1,\ldots \\ k=1,\ldots,2n+1}}$ of *outer harmonics* of degree n and order k defined by

$$H^\alpha_{-n-1,k}(x) = \left(\frac{\alpha}{|x|}\right)^{n+1} Y^\alpha_{n,k}(x), \quad x \in \mathbb{R}^3 \setminus \{0\}, \tag{12}$$

(where $Y^\alpha_{n,k}$ is system of spherical harmonics for the Runge sphere), satisfies the following properties:

- $H^\alpha_{-n-1,k}$ is of class $C^{(\infty)}(\mathbb{R}^3 \setminus \{0\})$,
- $\Delta H^\alpha_{-n-1,k}(x) = 0, \quad x \in \mathbb{R}^3 \setminus \{0\}$,
- $H^\alpha_{-n-1,k}$ is regular at infinity, i.e.,
$$\left|H^\alpha_{-n-1,k}(x)\right| = O\left(\frac{1}{|x|}\right), \quad |x| \to \infty,$$
$$\left|\nabla H^\alpha_{-n-1,k}(x)\right| = O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty,$$
- $H^\alpha_{-n-1,k}|_A = Y^\alpha_{n,k}$,
- $\left(H^\alpha_{-n-1,k}, H^\alpha_{-p-1,q}\right)_{L^2(A)} = \delta_{n,p}\,\delta_{k,q}$.

A. *The Hilbert Spaces* $\mathcal{H}\left(\{A_n\};\overline{A^{ext}}\right)$

In the following we introduce the (Sobolev-like) Hilbert spaces $\mathcal{H}(\{A_n\};\overline{A^{ext}})$ of harmonic functions which serve as reference spaces for spherically harmonic spline theory. As already mentioned, from the mathematical point of view, functions in $\mathcal{H}(\{A_n\};\overline{A^{ext}})$ can be seen as series expansions in terms of outer harmonics with certain assumptions on the growth of the coefficients. Let $\mathcal{A} = \{\{A_n\}_{n\in\mathbb{N}_0} | A_n \in \mathbb{R}^+ \text{ for all } n \in \mathbb{N}_0\}$ denote the set of all sequences of positive real numbers. Given a sequence $\{A_n\}_{n\in\mathbb{N}_0} \in \mathcal{A}$, we consider the linear space $\mathcal{E} = \mathcal{E}(\{A_n\};\overline{A^{ext}})$, $\mathcal{E} \subset \text{Pot}^{(\infty)}(\overline{A^{ext}})$ of all potentials F of the form

$$F = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} F^{\wedge}(n,j)H^{\alpha}_{-n-1,j} \tag{13}$$

whose Fourier coefficients (with respect to $L^2(A)$)

$$F^{\wedge}(n,j) = \int_A F(x)H^{\alpha}_{-n-1,j}d\omega_\alpha(x) \tag{14}$$

satisfy

$$\sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} A_n^2(F^{\wedge}(n,j))^2 < \infty \tag{15}$$

The last sum is imposed as a norm for $\mathcal{E}$

$$||F||_{\mathcal{H}(\{A_n\};\overline{A^{ext}})} = \left(\sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} A_n^2(F^{\wedge}(n,j))^2\right)^{1/2} \tag{16}$$

Definition 3: The Sobolev space $\mathcal{H}\left(\{A_n\};\overline{A^{ext}}\right)$ is defined by

$$\mathcal{H}(\{A_n\};\overline{A^{ext}}) = \overline{\mathcal{E}(\{A_n\};\overline{A^{ext}})}^{||\cdot||_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}}. \tag{17}$$

It is a Hilbert space equipped with the inner product

$$(F,G)_{\mathcal{H}(\{A_n\};\overline{A^{ext}})} = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} A_n^2 F^{\wedge}(n,j)G^{\wedge}(n,j) \tag{18}$$

for $F, G \in \mathcal{H}(\{A_n\};\overline{A^{ext}})$, where $F^{\wedge}(n,j)$ and $G^{\wedge}(n,j)$ are Fourier coefficients of F and G with respect to $L^2(A)$. Every element F of the space $\mathcal{H}(\{A_n\};\overline{A^{ext}})$ is uniquely determined by its Fourier coefficients $F^{\wedge}(n,j)$ that satisfy

$$||F||^2_{\mathcal{H}(\{A_n\};\overline{A^{ext}})} = \Big(\sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} A_n^2(F^{\wedge}(n,j))^2\Big) < \infty, \tag{19}$$

and F can be formally represented by the expansion

$$F = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} F^{\wedge}(n,j)H^{\alpha}_{-n-1,j}, \tag{20}$$

which has to be understood in 'distributional sense' (at least on A). Condition (19) determines the maximal possible growth behavior of the Fourier coefficients. It follows directly from the definition of $\mathcal{H}(\{A_n\};\overline{A^{ext}})$ that the set $\{A_n^{-1}H^{\alpha}_{-n-1,k}\}_{n\in\mathbb{N}_0,k=1,\ldots,2n+1}$ is a complete orthonormal system in $\mathcal{H}(\{A_n\};\overline{A^{ext}})$.

Remark: In particular, we let

$$\mathcal{H}_s(\overline{A^{ext}}) = \mathcal{H}(\{(n+1/2)^s\};\overline{A^{ext}}), s \in \mathbb{R}. \tag{21}$$

Especially for $s = 0$ we have

$$\mathcal{H}_0(\overline{A^{ext}}) = \mathcal{H}(\{1\};\overline{A^{ext}}). \tag{22}$$

The space $\mathcal{H}_0(\overline{A^{ext}})$ may be understood as the space of all harmonic functions in $\overline{A^{ext}}$, regular at infinity, corresponding to $L^2(A)$–restrictions. Its norm $||\cdot||_{\mathcal{H}_0(\overline{A^{ext}})}$ can be understood as the $L^2(A)$–norm. Loosely spoken, the topology of $\mathcal{H}_0(\overline{A^{ext}})$ is led back to the topology of $L^2(A) = \mathcal{H}_0(\overline{A^{ext}})|_A$ and $\mathcal{H}_0(\overline{A^{ext}})$ forms the harmonic continuations of $L^2(A)$-functions.

According to our construction, the space $\text{Pot}^{\infty}(\overline{A^{ext}})$ is a dense subspace of $\mathcal{H}_s(\overline{A^{ext}})$ for each s. Moreover, if $t < s$, then $||F||_{\mathcal{H}_t(\overline{A^{ext}})} \leq ||F||_{\mathcal{H}_s(\overline{A^{ext}})}$.

When we associate to a potential $F \in \text{Pot}^{\infty}(\overline{A^{ext}})$ the series (20), it is of fundamental importance to know if the series converges uniformly on $\overline{A^{ext}}$. The answer is provided by an analogue of the Sobolev lemma. In order to present this lemma, we first introduce the concept of summable sequences.

Definition 4: A sequence $\{A_n\}_{n\in\mathbb{N}_0} \in \mathcal{A}$ is called summable if it satisfies the summability condition

$$\sum_{n=0}^{\infty}\frac{2n+1}{4\pi}\frac{1}{A_n^2} < \infty. \tag{23}$$

Lemma 5: (Sobolev Lemma) If a sequence $\{A_n\}_{n\in\mathbb{N}_0} \in \mathcal{A}$ is summable, then each $F \in \mathcal{H}(\{A_n\};\overline{A^{ext}})$ corresponds to a potential of class $\text{Pot}^{(0)}(\overline{A^{ext}})$.

Theorem 6: Let $\{A_n\}_{n\in\mathbb{N}_0} \in \mathcal{A}$ be a summable sequence. Then $\mathcal{H}(\{A_n\};\overline{A^{ext}})$ is a reproducing kernel Hilbert space with the reproducing kernel given by

$$\begin{aligned} K_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(x,y) &= \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1}\frac{1}{A_n}H^{\alpha}_{-n-1,j}(x)\frac{1}{A_n}H^{\alpha}_{-n-1,j}(y) \\ &= \sum_{n=0}^{\infty}\frac{1}{A_n^2}\frac{2n+1}{4\pi\alpha^2}\left(\frac{\alpha^2}{|x||y|}\right)^{n+1}P_n\Big(\frac{x}{|x|}\cdot\frac{y}{|y|}\Big) \end{aligned} \tag{24}$$

where $x, y \in \overline{A^{ext}}$.

B. $\mathcal{H}(\{A_n\};\overline{A^{ext}})$*–Splines*

The Sobolev spaces of harmonic functions $\mathcal{H}(\{A_n\};\overline{A^{ext}})$ allow the definition of harmonic splines (see [1], [3] for the original papers or the text books [4], [6]). These splines are introduced with respect to a set of linear bounded functionals which provide interpolation conditions. The choise of the solution space $\mathcal{H}(\{A_n\};\overline{A^{ext}})$, i.e., the corresponding sequence $\{A_n\}_{n\in\mathbb{N}_0} \in \mathcal{A}$, is dictated by the specifics of the functional under consideration.

Definition 7: Let $\{\mathcal{L}_1, ...\mathcal{L}_N\}$ be a set of N linearly independent bounded linear functionals on the Sobolev-type Hilbert space $\mathcal{H}(\{A_n\};\overline{A^{ext}})$. Then any function S of the form

$$S(x) = \sum_{i=1}^{N} a_i \mathcal{L}_i K_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\cdot, x), x \in \overline{A^{ext}}, \tag{25}$$

with a set of real numbers $\{a_1, ..., a_N\} \subset \mathbb{R}$ is called a $\mathcal{H}(\{A_n\};\overline{A^{ext}})$-spline relative to $\{\mathcal{L}_1, ...\mathcal{L}_N\}$.

The function space of all $\mathcal{H}(\{A_n\};\overline{A^{ext}})$–splines relative to $\{\mathcal{L}_1, ...\mathcal{L}_N\}$ is denoted by $S_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\mathcal{L}_1, ...\mathcal{L}_N)$.

$\mathcal{H}(\{A_n\};\overline{A^{ext}})$–spline interpolation problem

Let $F \in \mathcal{H}(\{A_n\};\overline{A^{ext}})$, and let $\{\mathcal{L}_1, ...\mathcal{L}_N\}$ be a set of N linearly independent bounded linear functionals on the Hilbert space $\mathcal{H}(\{A_n\};\overline{A^{ext}})$. As usual we denote the representer of $\mathcal{L}_i$, by $\mathcal{L}_i K_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\cdot, \cdot)$, $i = 1, ..., N$. Denoting the space of all interpolating functions in $\mathcal{H}(\{A_n\};\overline{A^{ext}})$ for F relative to $\mathcal{L}_1, ..., \mathcal{L}_N$ by $\mathcal{I}_{\mathcal{L}_1,...,\mathcal{L}_N}$, the $\mathcal{H}(\{A_n\};\overline{A^{ext}})$–spline interpolation problem is to determine a function $S^F_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\mathcal{L}_1, ...\mathcal{L}_N)$ in $S_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\mathcal{L}_1, ...\mathcal{L}_N) \cap \mathcal{I}_{\mathcal{L}_1,...,\mathcal{L}_N}$, i.e., to determine a spline $S^F_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\mathcal{L}_1, ...\mathcal{L}_N)$ which fulfills the interpolation conditions

$$\mathcal{L}_i S^F_{\mathcal{H}(\{A_n\};\overline{A^{ext}})}(\mathcal{L}_1, ...\mathcal{L}_N) = \mathcal{L}_i F,$$

for all $i = 1, ..., N$.

The solution to the interpolation problem corresponding to $\mathcal{H}(\{A_n\};\overline{A^{ext}})$-splines relative to a finite set of linear bounded functionals, relates the interpolation conditions to a system of linear equations which needs to be solved to obtain the spline coefficients. Together with the set of linear bounded functionals and the Sobolev space $\mathcal{H}(\{A_n\};\overline{A^{ext}})$ (or the corresponding representers) these coefficients define the interpolating spline. For this spline the minimum norm properties are valid.

III. The Reproducing Kernel via the Newton Potential

Spherical and ellipsoidal models are widely used in geosciences as approximations to the shape of the Earth. However, the technological progress and the increasing observational accuracy require adequate mathematical methods observing geophysically more realistic reference surfaces than sphere and ellipsoid. This was the idea behind our approach. Following the work of [11] we proposed that it is reasonable to use a reproducing kernel given as a Newton integral over the real Earth body:

$$\mathcal{K}(x, y) = \int_{\Sigma^{int}} \frac{dz}{|x - z||y - z|}, \;\; x \in \overline{\Sigma^{ext}}. \tag{26}$$

Using this kernel, a real Earth based spline formulation for the solution of interpolation problem of geopotential determination is given in [10]. However, investigations on this kernel in spherical case showed a remarkable result. Replacing a regular surface Σ with the Runge sphere $A = \Omega_\alpha$ this kernel takes the form of the reproducing kernel of type (24). Indeed, the kernel gets the following representation

$$\mathcal{K}(x, y) = \int_{A^{int}} \frac{dz}{|x - z||y - z|}. \tag{27}$$

Now using the known expansions in spherical harmonics for fundamental solutions (of the Laplace's equation) appearing in the integral we can write

$$\frac{1}{|x - z|} = \sum_{n=0}^{\infty} \frac{|z|^n}{|x|^{n+1}} P_n\left(\frac{x}{|x|} \cdot \frac{z}{|z|}\right). \tag{28}$$

and

$$\frac{1}{|y - z|} = \sum_{m=0}^{\infty} \frac{|z|^m}{|y|^{m+1}} P_m\left(\frac{y}{|y|} \cdot \frac{z}{|z|}\right). \tag{29}$$

Substituting this expressions in (27) we get

$$\begin{aligned} \mathcal{K}(x, y) &= \int_{A^{int}} \frac{1}{|x - z|} \frac{1}{|y - z|} dz \\ &= \int_0^\alpha \int_{\Omega_r} \sum_{n=0}^{\infty} \frac{|z|^n}{|x|^{n+1}} P_n\left(\frac{x}{|x|} \cdot \frac{z}{|z|}\right) \\ &\quad \cdot \sum_{m=0}^{\infty} \frac{|z|^m}{|y|^{m+1}} P_m\left(\frac{y}{|y|} \cdot \frac{z}{|z|}\right) d\omega_r\left(\frac{z}{|z|}\right) dr \end{aligned} \tag{30}$$

Using the addition theorem for spherical harmonics the last expression can be written as

$$\begin{aligned} &\int_0^\alpha \sum_{n=0}^{\infty} \frac{r^{2n+2}}{(|x||y|)^{n+1}} \left(\frac{4\pi}{2n+1}\right) P_n\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right) dr \\ &= \sum_{n=0}^{\infty} \frac{4\pi}{2n+1} \cdot \frac{1}{(|x||y|)^{n+1}} P_n\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right) \int_0^\alpha r^{2n+2} dr \\ &= \sum_{n=0}^{\infty} \frac{4\pi}{2n+1} \frac{1}{(|x||y|)^{n+1}} P_n\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right) \frac{\alpha^{2n+3}}{2n+3} \\ &= \sum_{n=0}^{\infty} \frac{4\pi\alpha}{(2n+1)(2n+3)} \cdot \left(\frac{\alpha^2}{|x||y|}\right)^{n+1} P_n\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right) \end{aligned} \tag{31}$$

Altogether we have

$$\mathcal{K}(x, y) = \sum_{n=0}^{\infty} \frac{1}{A_n^2} \frac{2n+1}{4\pi\alpha^2} \left(\frac{\alpha^2}{|x||y|}\right)^{n+1} P_n\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right), \tag{32}$$

where $A_n = 4\pi(2n+1)(2n+3)^{1/2}\alpha^{-3/2}$.

This means that in case of $\Sigma = A$, the kernel (26) corresponds to the type of kernels defined by $\mathcal{H}\Big(\{A_n\};\overline{A^{ext}}\Big)$, where A_n is the summable sequence

$$A_n = 4\pi(2n+1)(2n+3)^{1/2}\alpha^{-3/2}, \tag{33}$$

Following figures represent the reproducing kernel (32), calculated for $x, y \in \Omega$ and different values of α using the Clenshaw algorithm.

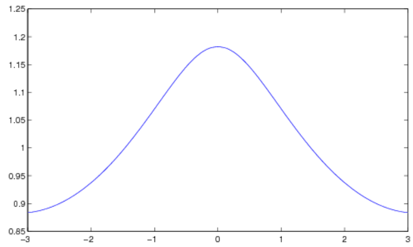

Fig. 2. Kernel $\mathcal{K}$ on Ω with $\alpha = 0.7$.

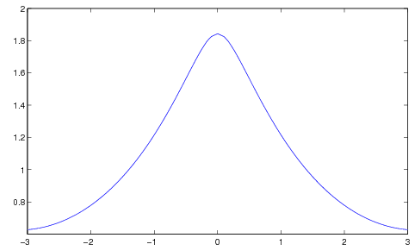

Fig. 3. Kernel $\mathcal{K}$ on Ω with $\alpha = 0.9$.

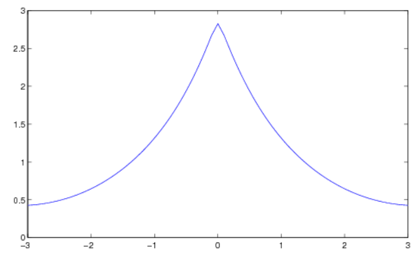

Fig. 4. Kernel K on Ω with $\alpha = 0.99$.

IV. Numerical Results

For some special classes of summable sequences $\{A_n\}_{n\in\mathbb{N}_0}$ we can find closed representations of the reproducing kernel as an elementary function by the use of the addition theorem (9) as well as (7) or (8), respectively. Taking the advantage of closed representations, numerical computations using spherical

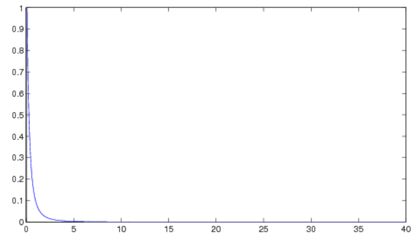

Fig. 5. Coefficients A_n^{-1} of $\mathcal{K}$ with $\alpha = 0.7$.

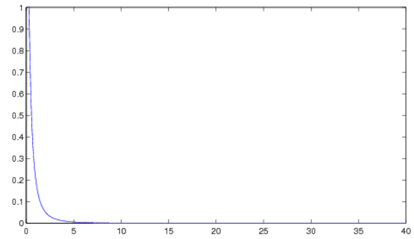

Fig. 6. Coefficients A_n^{-1} of $\mathcal{K}$ with $\alpha = 0.9$.

kernels is mainly done by using type of kernels like Abel-Poisson or the singularity of kernel.

(i) Kernels of Abel–Poisson type: $A_n = h^{-n/2}$ for $h \in (0,1)$

$$\begin{aligned} &K_{\mathcal{H}\left(\{h^{-n/2}\};\overline{A^{ext}}\right)}(x,y) \\ &= \sum_{n=0}^{\infty} h^n \frac{2n+1}{4\pi\alpha^2}\left(\frac{\alpha^2}{|x||y|}\right)^{n+1} P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right) \\ &= \frac{1}{4\pi}\frac{|x|^2|y|^2 - h^2\alpha^4}{\left(|x|^2|y|^2 + h^2\alpha^4 - 2h\alpha^2(x\cdot y)\right)^{3/2}} \end{aligned} \tag{34}$$

with $x, y \in \overline{A^{ext}}$.

(ii) Kernels of Singularity type: $A_n = \left(n+\frac{1}{2}\right)h^{-n/2}$ for $h \in (0,1)$

$$\begin{aligned} &K_{\mathcal{H}\left(\{\left(n+\frac{1}{2}\right)h^{-n/2}\};\overline{A^{ext}}\right)}(x,y) \\ &= \sum_{n=0}^{\infty} \frac{h^n}{\left(n+\frac{1}{2}\right)}\frac{2n+1}{4\pi\alpha^2}\left(\frac{\alpha^2}{|x||y|}\right)^{n+1} P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right) \\ &= \frac{1}{2\pi}\frac{1}{\left(|x|^2|y|^2 + h^2\alpha^4 - 2h\alpha^2(x\cdot y)\right)^{1/2}} \end{aligned}$$

with $x, y \in \overline{A^{ext}}$.

Considering the particular sequence (45) we are interested in the existence of a closed expression for the kernel $\mathcal{K}$. We have

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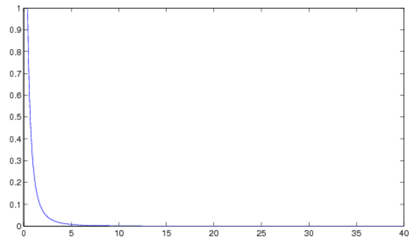

Fig. 7. Coefficients A_n^{-1} of $\mathcal{K}$ with $\alpha = 0.99$.

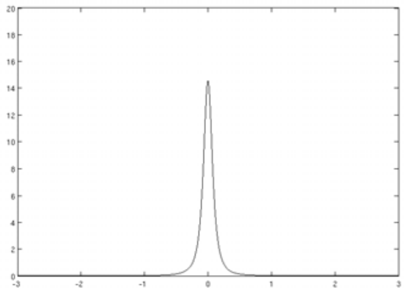

Fig. 8. Kernel of Abel-Poisson type on Ω with $h = 0.9$

$$\mathcal{K}(x,y) = \frac{4\pi\alpha^3}{|x||y|}\sum_{n=0}^{\infty}\frac{\alpha^{2n}}{(|x||y|)^n}\frac{1}{(2n+1)(2n+3)}P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right) \quad (35)$$

Writing $h_1 = \frac{\alpha^2}{|x||y|} = \left(\frac{\alpha}{\sqrt{|x||y|}}\right) = h^2$, and using partial fraction we get

$$\begin{aligned}\mathcal{K}(x,y) &= \frac{4\pi\alpha^3}{|x||y|}\sum_{n=0}^{\infty}h^{2n}\frac{1}{(2n+1)(2n+3)}P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)\\ &= \frac{2\pi\alpha^3}{|x||y|}\underbrace{\sum_{n=0}^{\infty}h_1^n\frac{1}{2n+1}P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)}_{=S_1}\\ &\quad -\frac{2\pi\alpha^3}{|x||y|}\underbrace{\sum_{n=0}^{\infty}h_1^n\frac{1}{2n+3}P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)}_{=S_2}.\end{aligned} \quad (36)$$

For the sum S_1 we get from (7) for $t = \frac{x}{|x|}\cdot\frac{y}{|y|}$

$$\sum_{n=0}^{\infty}h_1^n P_n(t) = \sum_{n=0}^{\infty}(h^2)^n P_n(t) = \frac{1}{\sqrt{1+h^4-2th^2}}. \quad (37)$$

Integrating both sides with respect to h, we get

$$\sum_{n=0}^{\infty}\frac{h^{2n+1}}{2n+1}P_n(t) = \int\frac{1}{\sqrt{1+h^4-2th^2}}dh = \mathcal{F}(h,t), \quad (38)$$

where

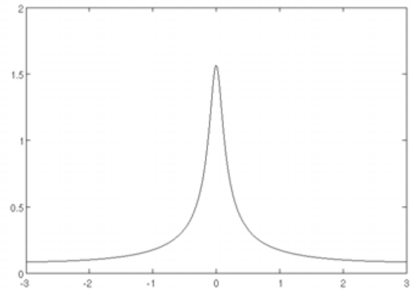

Fig. 9. Singularity Kernel on Ω with $h = 0.9$

$\mathcal{F}(h,t)$ includes an elliptic integral of I kind.

In conclusion we have for the sum S_1

$$S_1 = \sum_{n=0}^{\infty}\frac{h_1^n}{2n+1}P_n(t) = \frac{1}{\sqrt{h_1}}\mathcal{F}(\sqrt{h_1},t). \quad (39)$$

In a similar way we calculate the sum S_2. Equation (7) yields for $t = \frac{x}{|x|}\cdot\frac{y}{|y|}$ and for the sum

$$\sum_{n=0}^{\infty}h^{2n+2}P_n(t) = h^2\sum_{n=0}^{\infty}(h^2)^n P_n(t) = \frac{h^2}{\sqrt{1+h^4-2th^2}}. \quad (40)$$

Again integrating both sides with respect to h, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{h^{2n+3}}{2n+3}P_n(t) &= \int\frac{h^2}{\sqrt{1+h^4-2th^2}}dh\\ = h^3\sum_{n=0}^{\infty}\frac{h^{2n}}{2n+3}P_n(t) &= \mathcal{G}(h,t) = \mathcal{G}(\sqrt{h_1},t),\end{aligned} \quad (41)$$

where $\mathcal{G}(h,t)$ is defined via elliptic integrals of I and II kind respectively.

In conclusion we have for the sum S_2

$$S_2 = \sum_{n=0}^{\infty}\frac{h_1^n}{2n+3}P_n(t) = \frac{1}{h_1^{3/2}}\mathcal{G}(\sqrt{h_1},t). \quad (42)$$

Finally for the reproducing kernel (32) we have

$$\begin{aligned}\mathcal{K}(x,y) &= \int_{A^{int}}\frac{1}{|x-z|}\frac{1}{|y-z|}dz\\ &= \frac{2\pi\alpha^3}{|x||y|}\left(\frac{1}{\sqrt{h_1}}\mathcal{F}(\sqrt{h_1},t) + \frac{1}{h_1^{3/2}}\mathcal{G}(\sqrt{h_1},t)\right).\end{aligned} \quad (43)$$

For elliptic integrals of I and II kind is known that there exist closed expression only in the case $t = -1$ or $t = 1$. For other values of t, namely for which we are interested, the integral must be calculated numerically. This means that the closed expression for this kernel does not exist even in the case of spherical boundary.

V. Conclusion

The result from the previous section shows us that when going over from the sphere to the regular surface Σ we can consider the kernel

$$\mathcal{K}(x,y) = \int_{\Sigma^{int}} \frac{dz}{|x-z||y-z|}, \;\; x \in \overline{\Sigma^{ext}}. \qquad (44)$$

a generalization to spherically oriented kernels. Moreover, the spherical representation of this kernel

$$\mathcal{K}(x,y) = \sum_{n=0}^{\infty} \frac{4\pi\alpha^{2n+3}}{(2n+1)(2n+3)(|x||y|)^{n+1}} P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)$$

associated with the summable sequence

$$A_n = 4\pi(2n+1)(2n+3)^{1/2}\alpha^{-3/2}, \qquad (45)$$

corresponds to the spherically oriented kernels described in Section II.

This is of significant result especially today when due to the technological advances spherical models are no longer satisfactory. Modern sciences that contribute to the study of the Earth processes are more and more interested in boundaries such as the real Earth surface or the real Earth body. Our result from the previous section opens the door for future investigations in approximations involving non-spherical boundaries.

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