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Fuzzy T-Neighborhood Groups Acting on Sets

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***Abstract*—**In this paper, The T-G-action topology on a set acted on by a fuzzy T-neighborhood (T-neighborhood, for short) group is defined as a final T-neighborhood topology with respect to a set of maps. We mainly prove that this topology is a T-regular T-neighborhood topology.

***Keywords*—**Fuzzy set, Fuzzy topology, Triangular norm, Separation axioms.

I. Introduction

A T-neighborhood topology on a set can be defined by several method e.g., via closures, interiors, filters, etc. Sometimes a T-neighborhood topology constructed out of given T-neighborhood topologies may be useful. In the classical theory of topological groups, when a topological group G acts on a set X, it confers a topology on X, called the G-action topology on X. In this paper we develop a fuzzy extension of that notion, in the case G is a T-neighborhood group. Varity of useful characterizations of this T-neighborhood topology are considered. We show that the T-G-action topology τ_X^{T-G} coincides with the final T-neighborhood topology τ_f introduced on X by a set of functions $\left\{\hat{g}\right\}$;

$$\hat{g} : G \to X .$$

II. Definition and Preliminaries

Definition 2.1. [8] A topological group G acts on a non-empty set X, if to each $g \in G$ and each $x \in X$ there corresponds a unique element gx such that

$g_2(g_1x) = (g_2g_1)x \quad \forall\ x \in X$ and $g_1, g_2 \in G$

$ex = x$.

When G acts on a set X, two families of functions can be defined as follows:

To each $g \in G$, we define $\hat{g} : X \to X$,

$\hat{g}(x) = gx$.

To each $x \in X$, we define $\hat{x} : G \to X$,

$\hat{x}(g) = gx$.

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We will use two important theorems which introduced in [7]. The first gives necessary and sufficient conditions for a group structure and T-neighborhood system to be compatible, and the second gives necessary and sufficient conditions for a filter to be the T-neighborhood filter of e in a T-neighbourhood group.

Theorem 2.1 [7] Let $(G, .)$ be a group and β a T-neighborhood base on G. Then $(G, .,t(\beta))$ is a T-neighborhood group if and only if the following are fulfilled:

(a) For every $a \in G$ we have

$$\beta(a) = \{\zeta_a(\mu) \ /\ \mu \in \beta(e)\}$$

$$(\text{res}.\ \beta(a) = \{R_a(\mu) / \mu \in \beta(e)\} \text{ and}$$

$\beta(a) = \{\zeta_a(\mu) \ /\ \mu \in \beta(e)\}$ is a T-neighborhood base at a.

(b) For all $\mu \in \beta(e)$ and for all $\varepsilon \in I_0$ there exists $v \in \beta(e)$ such that $v - \varepsilon \leq \mu^{-1}$, i.e., r is continuous at e.

(c) For all $\mu \in \beta(e)$ and for all $\varepsilon \in I_0$ there exists $v \in \beta(e)$ such that $v.\ v - \varepsilon \leq \mu$, i.e., m is continuous at (e, e).

(d) For all $\mu \in \beta(e)$, for all $\varepsilon \in I_0$ and for all $x \in G$ there exist $v \in \beta(e)$ such that $1_x.\ v.1_x^{-1} - \varepsilon \leq \mu$, i.e., int_x is continuous at e.

Where $\zeta_x : G \to G : z \mapsto x z$ (resp. $R_x : G \to G : z \mapsto z x$) is the left (resp. right) translation.

Theorem 2.2 [7] Let $(G, .)$ be a group and $\Im$ a family of fuzzy subset of G such that the following hold:

(a) $\Im$ is a filterbasis, such that $\mu(e) = 1$ for all $\mu \in \Im$.

(b) For all $\mu \in \Im$ and for all $\varepsilon \in I_0$ there exists $v \in \Im$ such that $v - \varepsilon \leq \mu^{-1}$.

(c) For all $\mu \in \Im$ and for all $\varepsilon \in I_0$ there exists $v \in \Im$ such that $v.\ v - \varepsilon \leq \mu$.

(d) For all $\mu \in \Im$, for all $\varepsilon \in I_0$ and for all $x \in G$ there exists $v \in \Im$ such that $1_x.\ v.1_{x-1} - \varepsilon \leq \mu$.

Then there exists a unique T-neighborhood system β such that $\Im$ is T-neighborhood basis for the T-neighbourhood system at e, $\beta(e)$ and β is compatible with the group structure. This T-neighbourhood system is given by

$$\beta(x) = \{1_x.\mu \ /\ \mu \in \Im\}^{-1} = \{\mu .1_x / \mu \in \Im\}^{-1},\ x \in G.$$

III. T-Neighborhood Topologies Induced by T-Neighborhood Group Actions on Set

Definition 3.1. Let $(G, .)$ be a group acting on a set X, then for all $\Gamma \in I^G$, $\mu \in I^X$, $g \in G$ and $x \in X$ we define for all $y \in X$

$$\Gamma\mu(y) = \sup \{\Gamma(g) T \mu(x) : (g, x) \in G \times X \text{ and } gx = y\} \quad (1)$$

Proposition 3.1. Let $(G, .)$ be a group acting on a set X and $\Psi, \Gamma \in I^G$, $\mu \in I^X$. Then

(a) $\Psi(\Gamma\mu) \leq (\Psi.\Gamma)\mu$ In particular
$\Psi(\Gamma\mu)(y) \leq (\Psi.\Gamma)\mu(y)$
(b) $\Gamma 1_M = \bigvee_{x\in M} \Gamma 1_x$
(c) $\Gamma 1_M(y) = \sup\{\Gamma(g): g \in G \text{ and } g^{-1}y \in M\}$
(d) $\Gamma 1_x(y) = \sup\{\Gamma(g): g \in G \text{ and } gx = y\}$
(e) $1_g\mu(y) = \sup\{\mu(x): x \in X \text{ and } gx = y\}$
$= \mu(g^{-1}y)$

Proof: (b)-(e) follow immediately from Definition 3.1.
(a) For any $y \in X$:
$\Psi(\Gamma\mu)(y) = \sup\{\Psi(g)\, T\, \Gamma\mu(x): (g, x) \in G\times X, gx = y\}$
$= \sup\{\Psi(g)\, T \sup\{\Gamma(h)\, T\, \mu(z): hz = x\}: gx = y\}$
$= \sup\{\Psi(g)\, T\, \Gamma(h)\, T\, \mu(z): ghz = y\}$
$(\Psi \odot_T \Gamma)\mu(y) = \sup\{(\Psi \odot_T \Gamma(k)\, T\, \mu(z): kz = y\}$
$= \sup\{\sup\{\Psi(g)\, T\, \Gamma(h): gh = k\}\, T\, \mu(z): kz = y\}$
$= \sup\{\Psi(g)\, T\, \Gamma(h)\, T\, \mu(z): (g, h, z) \in G\times G\times X \text{ and } ghz = y\}$

Hence $\Psi(\Gamma\mu)(y) = (\Psi \odot_T \Gamma)\mu(y) \leq (\Psi.\Gamma)\mu(y)$.

If both Γ, μ are crisp, then $\Gamma\mu$ is also crisp and is given by $\Gamma\mu = \{gx: g \in \Gamma \text{ and } x \in \mu\}$.

Note that $\Gamma\mu$, $\Gamma 1_x$, $1_g\mu \in I^X$ and $\Gamma 1_x(y) = 0$ if $y \notin$ orbit of x.

Theorem 3.1. Let G be a T-neighborhood group acting on a set X, and let $\Re$ be a fundamental system of G at e. For each $x \in X$, let $\beta_x = \{\Gamma 1_x: \Gamma \in \Re\} \in I^X$. Then $\{\beta_x\}_{x\in X}$ is a T-neighborhood basis on X. The resulting T-neighborhood space is denoted by τ_X^{T-G}. Its fuzzy closure operator $^- : I^X \to I^X$ is given by: For all $\eta \in I^X$, $x \in X$:

$$\bar{\eta}(x) = \inf_{\Gamma\in\Re} \sup_{g\in G} \Gamma(g)\, T\, \eta(gx) \qquad (2)$$

Proof. First, we verify that $\{\beta_x\}_{x\in X}$ is a T-neighborhood basis in X. Let $x \in X$, Γ, $\Psi \in \Re$, $\mu = \Gamma 1_x \in \beta_x$, $\lambda = \Psi 1_x \in \beta_x$
(i) $\mu(x) = \Gamma 1_x(x) = \sup\{\Gamma(g): g \in G \text{ and } gx = x\}$
$\geq \Gamma(e) = 1$ (Because $ex = x$).
(ii) There exists $\Lambda \in \Re: \Gamma \wedge \Psi \geq \Lambda$. Hence
$\mu \wedge \lambda = \Gamma 1_x \wedge \Psi 1_x \geq \Lambda 1_x$,
which is in β_x.
(iii) T-kernel condition:
Recall that $\{\Re 1_g\}_{g\in G}$ is a T-neighborhood basis of the T-neighborhood group G Theorem 2.2 . Let, as before, $\mu = \Gamma 1_x \in \beta_x$. By the T-kernel condition for $\Gamma \in \Re$, for all $\varepsilon \in I_0$ there exists a family $\{\Gamma_g 1_g \in \Re_g\}_{g\in G}$ such that for all $g, k \in G$

$$\Gamma_e(k)\, T\, (\Gamma_k 1_k)(g) \leq \Gamma(g) + \varepsilon. \qquad (3)$$

We take $v_x = \Gamma_e 1_x$. For each $y \in X$, if $y \notin$ orbit of x, take for v_y any element of $\beta_y = \Re_y$.

If $y \in$ orbit of x, choose some $h \in G$ such that $y = hx$, and

$$\delta + \Gamma_e(h) \geq \sup\{\Gamma_e(k): kx = y\} \qquad (4)$$

where $\delta \in I_0$ is a real number that satisfies
$(b + \delta)\, T\, (c + \delta) \leq (b\, T\, c) + \varepsilon$
for all $b, c \in I$. Such δ exists by the uniform continuity of T. Take $v_y = \Gamma_h 1_y \in \beta_y$. Then, if $y \notin$ orbit of x, we find for all $z \in X$ that

$$2\varepsilon + \mu(z) \geq \upsilon_x(y)\, T\, \upsilon_y(z)$$

because then $v_x(y) = (\Gamma_e 1_x)(y) = 0$. And when $y \in$orbit of x, we find for all $z \in X$:
$2\varepsilon + \mu(z) = 2\varepsilon + (\Gamma 1_x)(z)$
$= \varepsilon + \sup\{\varepsilon + \Gamma(g): gx = z\}$
$\geq \varepsilon + \sup\{\Gamma_e(h)\, T\, (\Gamma_h 1_h)(g): gx = z\}$ by (3)
$\geq (\Gamma_e(h) + \delta)\, T \sup\{(\Gamma_h 1_h)(g): gx = z\}$
$\geq \sup\{\Gamma_e(k): kx = y\}\, T \sup\{(\Gamma_h)(gh^{-1}): (gh^{-1})(hx) = z\}$
by (4) Since $hx = y$, then
$2\varepsilon + \mu(z) \geq (\Gamma_e 1_x)(y)\, T \sup\{(\Gamma_h)(t): ty = z\}$
$= (\Gamma_e 1_x)(y)\, T\, (\Gamma_h 1_y)(z)$
$= v_x(y)\, T\, v_y(z)$.

Thus, the kernel condition holds for $\mu \in \beta_x$ in both cases of y. Finally, for all $\eta \in I^X$

$$\bar{\eta}(x) = \inf_{\mu\in\beta} \sup_{y\in X} \mu(y)\, T\, \eta(y)$$
$$= \inf_{\Gamma\in\Re} \sup_{y\in X} \eta(y)\, T\, (\Gamma 1x)(y)$$
$$= \inf_{\Gamma\in\Re} \sup_{y\in orbitex} \eta(y)\, T \sup\{\Gamma(g): g \in G \text{ and } gx = y\}.$$

Because if $y \notin$ orbit x, then $(\Gamma 1_x)(y) = 0$. Thus,

$$\bar{\eta}(x) = \inf_{\Gamma\in\Re} \sup_{g\in G} \eta(gx)\, T\, \Gamma(g),$$

Rendering (2).

Proposition 3.2. Let $\Gamma \in I^G$, $\wp \subset I^G$, $g \in G$, $x \in X$ then
$(\Gamma.1_g)1_x = (\Gamma 1_{gx}) \in I^X$, and hence
$(\wp.1_g\})1_x = \wp 1_{gx} \subset I^X$.

Proof:
$((\Gamma.1_g)1_x)(y) = \sup\{(\Gamma.1_g)(k): k \in G \text{ and } kx = y\}$
$= \sup\{\Gamma(kg^{-1}): k \in G \text{ and } kg^{-1}gx = y\}$
$= \sup\{\Gamma(t): t \in G \text{ and } tgx = y\}$
$= (\Gamma 1_{gx})(y)$.

This completes the proof.

Proposition 3.3. For each filterbasis F in I^G and for $x \in X$.

$$\{\Gamma 1_x: \Gamma \in \mathrm{F}^{\sim}\} \subset \{\Psi 1_x : \Psi \in \mathrm{F}\}^{\sim} \subset I^X \qquad (5)$$

Proof: Let $\Gamma \in \mathrm{F}^{\sim}$ Then for all $\varepsilon > 0$ there exists $\Gamma_\varepsilon \in \mathrm{F}$ such that $\Gamma + \varepsilon \geq \Gamma_\varepsilon$. Then for all $y \in X$ we have
$\varepsilon + (\Gamma 1_x)(y) = \varepsilon + \sup\{\Gamma(g): gx = y\}$
$= \sup\{\varepsilon + \Gamma(g): gx = y\}$

$\geq sup\ \{\Gamma_\varepsilon(g): gx = y\}$
$= (\Gamma_\varepsilon 1_x)(y)$.

Thus, $\varepsilon + \Gamma 1_x \geq \Gamma_\varepsilon 1_x \in \{\Psi 1_x : \Psi \in \mathrm{F}\}$. Hence $\Gamma 1_x \in \{\Psi 1_x: \Psi \in \mathrm{F}\}^{\sim}$. This proves (5).

Proposition 3.4. The fuzzy closure operator on X defined in (2) does not depend on the particular choice of a fundamental system $\Re$ of e.

Proof: All fundamental systems $\Re$ of G at e have the same saturation $\Re^{\sim}$. Also, for each $x \in X$

$$\beta_x = \{\Gamma 1_x: \Gamma \in \Re\}$$
$$\subset \{\Gamma 1_x: \Gamma \in \Re^{\sim}\}$$
$$\subset \{\Gamma 1_x: \Gamma \in \Re\}^{\sim} = \beta_x^{\sim}.$$

As $\{\beta_x\}$, $\{\beta_x^{\sim}\}$ induce the same fuzzy closure operator on X, then the fuzzy closure operator defined in (2) is also given by

$$\bar{\eta}(x) = \inf_{\Gamma \in \tilde{\Re}} \sup_{g \in G} \Gamma(g) T \eta(gx) \qquad (6)$$

Which is independent of the particular choice of a fundamental system $\Re$ of e.

The following definition is well phrased by virtue of Theorem 3.1, and Proposition 3.4;

Definition 3.2. Let G be a T-neighborhood group acting on a set X. A T-G-action-topology on X denoted by τ_X^{T-G} is introduced through its closure operator $^{-}$, defined in (2).

Proposition 3.5. Let $\Re$ be a fundamental system at e of G, $\mu \in \Re$. Then

$$1_g . \Re . 1_{g^{-1}} \subset \Re^{\sim} \qquad (7)$$

Proof: From condition (d) in Theorem 2.1, for all $\varepsilon > 0$ there exists $v_\varepsilon \in \Re$ such that

$v_\varepsilon - \varepsilon \leq 1_g . \mu . 1_{g^{-1}}$.

This proves that $1_g . \mu . 1_{g^{-1}} \in \Re^{\sim}$

Notion: In T-G-action topology

(1) We denote the T-neighborhood system at $x \in X$ by $\aleph(x)$.

(2) Let $\Re$ be the T-neighborhood system of G at e, $x \in X$. We denote $\Re 1_x$ by $Ç(x)$. Recall that $Ç^{\sim} = \aleph$; i.e $Ç(x)$ is a T-neighborhood basis at x for this space.

Definition 3.3. Let $(X, ., t(\beta))$ be a T-neighborhood space, M *be* a non-empty set in X. Then $\mu \in I^X$ is said to be *a* T-neighborhood of M if μ is a T-neighborhood of all points x in M. It follows that the set of all T-neighborhoods of M (called the T-neighborhood system of M) is the set $\bigwedge_{x \in M} \aleph(x)$.

Proposition 3.6. Let $\Gamma \in I^G$, $g \in G, z \in X$ then

$1_g^{-1}(\Gamma 1_z) = (1_g^{-1} . \Gamma) 1_z$

Proof:

$1_{g^{-1}}(\Gamma 1_z)(y) = (\Gamma 1_z)(gy)$
$= sup\ \{\Gamma(h): h \in G, hz = gy\}$
$= sup\ \{\Gamma(gk): k \in G, kz = y\}$
$(1_{g^{-1}} . \Gamma) 1_z(y) = sup\ \{(1_{g^{-1}}.\Gamma)(k): kz = y\}$
$= sup\ \{\Gamma(gk): k \in G, kz = y\}$

Then

$1_{g^{-1}}(\Gamma 1_z) = (1_{g^{-1}} . \Gamma) 1_z$

Theorem 3.2. Under this T-neighborhood topology the functions $\{\hat{g}\}$ are homeomorphisms on X.

Proof: Without loss of generality, we take $\Re$ the whole T-neighborhood system at e. Then from Proposition 3.5, $1_g . \Re . 1_{g-1} \subset \Re$. Given $x \in X$, $g \in G$, $\Re 1_{gx}$ is a T-neighborhood basis at $\hat{g}x$. Let $\mu \in \Re 1_{gx}$ we have $\hat{g}^{-1}(\mu)(y) = \mu(gy) = 1_{g-1}\ \mu(y)$, then $\hat{g}^{-1}(\mu) = 1_{g-1}\mu \in 1_g^{-1} \Re 1_{gx}$, and from Proposition 3.6

$1_g^{-1}(\Re 1_{gx}) = (1_g^{-1} . \Re) 1_{gx}$,
$= (1_g^{-1} . \Re . 1_g) 1_x$ by Proposition 3.2
$\subset \Re^{\sim} 1_x$ by Proposition 3.5
$\subset \aleph(x)$.

i.e., $\hat{g}^{-1}(\mu)$ is a T- neighborhood of x. So by Theorem 5.1 in [5] $\hat{g}$ is continuous at x for all x, and hence it is continuous. Since $(g^{-1})^{\wedge} = (\hat{g})^{-1}$. Then $(\hat{g})^{-1}$ is also continuous. Thus $\hat{g}$ is a homeomorphism.

Proposition 3.7. For any symmetric T-neighborhood Δ of e, and any $M \subset X$; $x, z \in X$

$(\Delta 1_x)(z) = (\Delta 1_z)(x)$

$(\Delta 1_M)(x) = \sup_{y \in X} 1_M(y)\ T\ (\Delta 1_x)(y)$.

Proposition 3.8. For any subset M of X and any T-neighborhood Γ of e, $\Gamma 1_M$ is a T-neighborhood of M, and

$$(1_M)^{-} \leq \Gamma 1_M \in I^X. \qquad (8)$$

Proof: Since $\Gamma 1_M = \bigvee_{x \in M} \Gamma 1_x$, then $\Gamma 1_M$ is a T-neighborhood of all points of M, hence $\Gamma 1_M$ is a T-neighborhood of M.

Next, let Γ be a T-neighborhood of e. Then Γ contains a symmetric T-neighborhood Δ of e. For any $x\in X$

$$(1_M)^{-}(x) = \inf_{\lambda\in Ç}\sup_{y\in X} 1_M(y)T\lambda(y)$$
$$\le \sup_{y\in X} 1_M(y)\, T\, \Delta\, 1_x(y)$$
$$= \sup_{y\in M} \Delta\, 1_x(y)$$
$$= \sup_{y\in M} \Delta\, 1_y(x) \quad \text{by Proposition 3.7}$$
$$= (\Delta\, 1_M)(x)$$
$$\le (\Gamma\, 1_M)(x).$$ This proves (8).

Proposition 3.9. Let $\Re$ be a fundamental system of T-neighborhoods of e. For any subset M of X

$$(1_M)^{-} = \bigwedge_{\Gamma\in\Re} \Gamma 1_M$$

Proof: From Proposition 3.8, $(1_M)^{-} \le \Gamma 1_M$ for every $\Gamma\in\Re$ .Then

$$(1_M)^{-} \le \bigwedge_{\Gamma\in O} \Gamma 1_M.$$

Next we prove that

$$\bigwedge_{\Gamma\in\Re} \Gamma 1_M = \bigwedge_{\Gamma\in\Re^{\sim}} \Gamma 1_M.$$

Since $\Re \subset \Re^{\sim}$, then

$$\bigwedge_{\Gamma\in\Re} \Gamma 1_M \ge \bigwedge_{\Gamma\in\Re^{\sim}} \Gamma 1_M$$

Also, let $\Gamma\in\Re^{\sim}$, for all $\varepsilon > 0$ there exists $\Gamma_\varepsilon\in\Re$ such that $\varepsilon + \Gamma \ge \Gamma_\varepsilon$,

$$\varepsilon + \Gamma 1_M \ge (\varepsilon + \Gamma)1_M \ge \Gamma_\varepsilon 1_M \ge \bigwedge_{\Gamma\in\Re} \Gamma 1_M$$

Since this holds for all $\varepsilon > 0$, then

$$\Gamma 1_M \ge \bigwedge_{\Gamma\in\Re} \Gamma 1_M.$$

This inequality holds for all $\Gamma\in\Re^{\sim}$.
Consequently,

$$\bigwedge_{\Gamma\in\Re^{\sim}} \Gamma 1_M \ge \bigwedge_{\Gamma\in\Re} \Gamma 1_M$$

Hence, equality holds.
It is clear that if O is the set of symmetric elements in $\Re^{\sim}$ then.

$$\bigwedge_{\Gamma\in\Re} \Gamma 1_M = \bigwedge_{\Gamma\in\Re^{\sim}} \Gamma 1_M \le \bigwedge_{\Delta\in O} \Delta 1_M$$

Conversely, let O is the set of symmetric elements in $\Re^{\sim}$. Then O is a fundamental system at e:

$$(\bigwedge_{\Delta\in O} \Delta 1_M)(x) = \inf_{\Delta\in O} (\Delta 1_M)(x)$$
$$= \inf_{\Delta\in O}\sup_{y\in X} 1_M(y)\, T\, (\Delta 1x)(y) \text{ by Proposition 3.7}$$
$$= (1_M)^{-}(x)$$

because the set $\{\Delta 1_x : \Delta \in O\}$ is a T-neighborhood basis at x.

Theorem 3.3. A T-G-action topology on X is a T-regular T-neighborhood topology.

Proof: Let $M \subset X$ and $x \in X$. We establish condition (N^4-T-regularity) of Theorem 3.2 in [6], which is equivalent to the T-regularity of X. For all
$M \subset X$, $x \in X$ such that

$$Inf\ hgt\ (\rho\ T\ v\colon \rho \in Ç(M),\ v \in O1_x)$$
$$\le \inf_{\Delta\in O} hgt\ (\Delta 1_M T \Delta 1_x)$$
$$\le \inf_{\Delta\in O} sup\ \{((\Delta 1_M) \wedge \Delta 1_x)(y)\colon y \in X\}$$
$$= \inf_{\Delta\in O} sup\ \{((\Delta 1_M)(y) \wedge (\Delta 1_x)(y))\colon y \in X\}$$
$$= \inf_{\Delta\in O} sup\ \{sup\ \{\Delta(h)\colon h \in G,\ hy \in M\} \wedge sup\{\Delta(k)\colon k \in G,\ y = kx\}\colon y \in X\}$$
$$= \inf_{\Delta\in O} \sup\{sup\ \{\Delta(h)\colon h \in G,\ hy \in M\} \wedge sup\{\Delta(k)\colon k \in G,\ y = kx\}\colon y \in orbit\ x\}$$

So. (call $y \in kx$)

$$Inf\ hgt\ (\rho\ T\ v\colon \rho \in Ç(M),\ v \in O1_x)$$
$$\le \inf_{\Delta\in O} \sup\{sup\ \{\Delta(h)\colon hkx \in M\} \wedge \Delta(k)\colon k \in G\}$$
$$= \inf_{\Delta\in O} sup\ \{\Delta(h) \wedge \Delta(k)\colon h, k \in G \text{ and } hkx \in M\}$$
$$= \inf_{\Delta\in O} sup\ \{(\Delta\Delta)(g)\colon g \in G \text{ and } gx \in M\}$$
$$= \inf_{\Delta\in O} sup\ ((\Delta\Delta)(1_M))(x)$$

But by Theorem 2.2 in [7], for every $\Delta \in O$, $\varepsilon \ge 0$ there exists $\Delta_1 \in O$ such that $\Delta_1 \Delta_1 \le \Delta + \varepsilon$. Hence,

$$Inf\ hgt\ (\rho\ T\ v\colon \rho \in Ç(M),\ v \in O1_x)$$
$$\le \inf_{\Delta\in O,\varepsilon>0} ((\Delta + \varepsilon)\, 1_M)(x)$$
$$= \inf_{\Delta\in O} (\Gamma 1_M)(x)$$
$$= \bigwedge_{\Gamma\in O} (\Gamma 1_M)(x) = (1_M)^{-}(x) \quad \text{by Proposition 3.9.}$$

The opposite inequality is always valid.

Theorem 3.4. A T-G-action-topology τ_X^{T-G} coincides with the final T-neighborhood topology τ_f on X defined by the set of functions

$$\{\hat{x} : G \to X\colon x\in X\},\ \hat{x}(g) = gx$$

Proof: For any $x\in X$, the function
$\hat{x} : G \to (X,\ \tau_X^{T-G})$ is continuous, because for all $g \in G$ and for each neighborhood $\Gamma(1_{gx})$ in the fundamental system $\Re\, 1_{gx}$ of $\hat{x}(g) = gx$, where $\Gamma\in\Re$, we have

$$\hat{x}\ (\Gamma .\ 1_g)(y) = sup\ \{(\Gamma .\ 1_g)(h)\colon h \in G,\ \hat{x}(h) = y\}$$
$$= sup\ \{(\Gamma . 1_g)(h)\colon h \in G,\ hx = y\}$$
$$= (\Gamma .\ 1_g)1_x(y)$$

then $\hat{x}\ (\Gamma . 1_g) = (\Gamma . 1_g)1_x = \Gamma 1_{gx}$ and $\Gamma . 1_g$ is a T-neighborhood of g by Theorem 2.3 in [7]. Therefore

$\tau_X^{T-G} \subset \tau_f$ since τ_f is the finest T-neighborhood topology making all $\hat{x}$ continuous.

Next, let x $\in$ X, μ a T-neighborhood of x in τ_f. Then $\hat{x}^{-1}(\mu)$ a T-neighborhood of e in G; i.e. $(\hat{x}^{-1}(\mu))1_x$ is a T-neighborhood of x in τ_X^{T-G}.

But $(\hat{x}^{-1}(\mu))1_x = \hat{x}(\hat{x}^{-1}(\mu)) = \mu \wedge 1_{range\hat{x}} \leq \mu$. This proves that μ is a T-neighborhood of x in τ_X^{T-G}. Then $\tau_f \subset \tau_X^{T-G}$. Hence, equality holds.

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