

# Fuzzy Subalgebras and Fuzzy Ideals of BCI-Algebras with Operators

Yuli Hu, Shaoquan Sun

**Abstract**—The aim of this paper is to introduce the concepts of fuzzy subalgebras, fuzzy ideals and fuzzy quotient algebras of BCI-algebras with operators, and to investigate their basic properties.

**Keywords**—BCI-algebras, BCI-algebras with operators, fuzzy subalgebras, fuzzy ideals, fuzzy quotient algebras.

## I. INTRODUCTION

THE fuzzy set is a generalization of the classical set and it has been applied to many mathematical branches such as groups, rings, ideals and obtained many theories about fuzzy set since Zadeh [13] first raised the concept of fuzzy set in 1965.

BCK/BCI-algebras are two classes of logical algebras, which were introduced by Imai and Iseki [1], [2]. In 1991, Xi [3] applied the concept of fuzzy sets to BCK-algebras, since then fuzzy BCK/BCI-algebras have been extensively investigated by several researchers. Jun et al. [4], [5] introduced the concepts of fuzzy positive implicative ideals and fuzzy commutative ideals of BCK-algebras. Meng et al. [6] introduced the concept of fuzzy implicative ideals of BCK-algebras. Jun et al. [7] introduced the concept of commutative ideals of BCI-algebras, Liu and Meng [9], [10] introduced the concepts of fuzzy positive implicative ideals and fuzzy implicative ideals of BCI-algebras. In 1993, Zheng [8] defined operators in BCK-algebras and introduced the concept of BCI-algebras with operators and gave some isomorphism theorems of it. Next, Liu [12] introduced the universality property of direct products of BCI-algebras. In 2002, Liu [11] introduced the concept of the fuzzy quotient algebras of BCI-algebras.

In this paper, we introduce the definitions of fuzzy subalgebras, fuzzy ideals and fuzzy quotient algebras of BCI-algebras with operators, Moreover, the basic properties were discussed and many results have been obtained, which enriches the theory of BCK/BCI-algebras.

## II. PRELIMINARIES

We recall some definitions and propositions which will be needed.

An algebra  $\langle X; *, 0 \rangle$  of type (2,0) is called a BCI-algebra, if

Yuli Hu is with the College of Mathematics and Physics, Qingdao University of Science and Technology, Qingdao, China (phone: 178-06259520; e-mail: 1198260194@qq.com).

Shaoquan Sun is with the College of Mathematics and Physics, Qingdao University of Science and Technology, Qingdao, China (phone: 185-61681686; e-mail: qdsunshaoquan@163.com).

it satisfies the following conditions:

$$BCI-(1) ((x * y) * (x * z)) * (z * y) = 0,$$

$$BCI-(2) (x * (x * y)) * y = 0, \quad BCI-(3) x * x = 0,$$

$$BCI-(4) x * y = 0 \text{ and } y * x = 0 \text{ imply } x = y,$$

for all  $x, y, z \in X$ . We can define  $x * y = 0$  if and only if  $x \leq y$ , then the above conditions can be written as:

1.  $(x * y) * (x * z) \leq z * y$ ,
2.  $x * (x * y) \leq y$ ,
3.  $x \leq x$ ,
4.  $x \leq y$  and  $y \leq x$  imply  $x = y$ ,

for all  $x, y, z \in X$ . If a BCI-algebra satisfies the identity  $0 * x = 0$ , then it is called a BCK-algebra.

**Definition 1.** If  $\langle X; *, 0 \rangle$  is a BCI-algebra,  $A$  is a non-empty subset of  $X$ , and  $x * y \in A$  for all  $x, y \in A$ , then  $\langle A; *, 0 \rangle$  is called a subalgebra of  $\langle X; *, 0 \rangle$ .

**Definition 2.** [10] A fuzzy set in a set  $S$  is a function  $A$  from  $S$  into  $[0, 1]$ .

**Definition 3.** [4] If  $\langle X; *, 0 \rangle$  is a BCI-algebra, a fuzzy set  $A$  of  $X$  is called a fuzzy subalgebra of  $X$  if for all  $x, y \in X$ , it satisfies:

$$A(x * y) \geq A(x) \wedge A(y).$$

**Definition 4.** [5]  $\langle X; *, 0 \rangle$  is a BCI-algebra, a fuzzy subset  $A$  of  $X$  is called a fuzzy ideal of  $X$  if it satisfies:

1.  $A(0) \geq A(x), \forall x \in X$ ,
2.  $A(x) \geq A(x * y) \wedge A(y), \forall x, y \in X$ .

**Definition 5.** [6]  $\langle X; *, 0 \rangle$  is a BCI-algebra,  $M$  is a non-empty set, if there exists a mapping  $(m, x) \rightarrow mx$  from  $M \times X$  to  $X$  which satisfies

$$m(x * y) = (mx) * (my), \forall x, y \in X, m \in M.$$

then  $M$  is called a left operator of  $X$ ,  $X$  is called a BCI-algebra with left operator  $M$ , or  $M$ -BCI-algebra for short.

**Proposition 1.** Let  $\langle X; *, 0 \rangle$  be a  $M$ -BCI-algebra, if  $A$  is a

fuzzy ideal of it, and  $x * y \leq z$ , then  $A(x) \geq A(y) \wedge A(z)$  for all  $x, y, z \in X$ .

**Definition 6.** Let  $A$  and  $B$  be fuzzy sets of set  $X$ , then the direct product  $A \times B$  of  $A$  and  $B$  is a fuzzy subset of  $X \times X$ , define  $A \times B$  by

$$A \times B(x, y) = A(x) \wedge B(y), \forall x, y \in X.$$

**Definition 7.** [6] Let  $\langle X; *, 0 \rangle$  and  $\langle \bar{X}; *, 0 \rangle$  be two  $M$ -BCI-algebras, if  $f$  is a homomorphism from  $\langle X; *, 0 \rangle$  to  $\langle \bar{X}; *, 0 \rangle$ , and  $f(mx) = mf(x)$  for all  $x \in X, m \in M$ , then  $f$  is called a homomorphism with operators.

**Definition 8.**  $\langle X; *, 0 \rangle$  is a  $M$ -BCI-algebra, let  $B$  be a fuzzy set of  $X$ , and  $A$  be a fuzzy relation of  $B$ , if

$$A_B(x, y) = B(x) \wedge B(y) \text{ for all } x, y \in X,$$

then  $A$  is called a strong fuzzy relation of  $B$ . In the following parts,  $X$  always means an  $M$ -BCI-algebra unless otherwise specified.

### III. FUZZY SUBALGEBRAS OF BCI-ALGEBRAS WITH OPERATORS

**Definition 9.** If  $\langle X; *, 0 \rangle$  is an  $M$ -BCI-algebra,  $A$  is a non-empty subset of  $X$ , and  $mx \in A$  for all  $x \in A, m \in M$ , then  $\langle A; *, 0 \rangle$  is called an  $M$ -subalgebra of  $\langle X; *, 0 \rangle$ .

**Definition 10.**  $\langle X; *, 0 \rangle$  is a  $M$ -BCI-algebra,  $A$  is a fuzzy subalgebra of  $X$ , if  $A(mx) \geq A(x)$  for all  $x \in X, m \in M$ , then  $A$  is called an  $M$ -fuzzy subalgebra of  $X$ .

**Example 1.** If  $A$  is an  $M$ -fuzzy subalgebra of  $X$ , then  $X_A$  is an  $M$ -fuzzy subalgebra of  $X$ , define  $X_A$  by

$$X_A : X \rightarrow [0, 1], X_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

**Proof.** (1) For all  $x, y \in X$ , if  $x, y \in A$ , then  $x * y \in A$ , therefore

$$X_A(x * y) = 1 \geq X_A(x) \wedge X_A(y),$$

if there exists at least one which does not belong to  $A$  between  $x$  and  $y$ , for example  $x \notin A$ , thus

$$X_A(x * y) \geq 0 = X_A(x) \wedge X_A(y),$$

therefore  $X_A$  is a fuzzy subalgebra of  $X$ .

(2) For all  $x \in X, m \in M$ , if  $x \in A$ , then  $mx \in A$ , therefore

$$X_A(mx) = 1 \geq X_A(x),$$

if  $x \notin A$ , then

$$X_A(mx) \geq 0 = X_A(x),$$

therefore  $X_A$  is an  $M$ -fuzzy subalgebra of  $X$ .

**Proposition 3.**  $A$  is an  $M$ -fuzzy subalgebra of  $X$  if and only if  $A_t$  is an  $M$ -subalgebra of  $X$ , where  $A_t$  is a non-empty set, define  $X_A$  by

$$A_t = \{x \mid x \in X, A(x) \geq t\}, \forall t \in [0, 1].$$

**Proof.** Suppose  $A$  is an  $M$ -fuzzy subalgebra of  $X$ ,  $A_t$  is a non-empty set,  $t \in [0, 1]$ , then we have

$$A(x * y) \geq A(x) \wedge A(y).$$

If  $x \in A_t, y \in A_t$ , then

$$A(x) \geq t, A(y) \geq t,$$

thus

$$A(x * y) \geq A(x) \wedge A(y) \geq t,$$

thus we have

$$x * y \in A_t.$$

For all  $x \in X, m \in M$ , if  $A$  is an  $M$ -fuzzy subalgebra of  $X$ , hence

$$A(mx) \geq A(x) \geq t,$$

thus

$$mx \in A_t,$$

therefore  $A_t$  is an  $M$ -subalgebra of  $X$ . Conversely, suppose  $A_t$  is an  $M$ -subalgebra of  $X$ , then we have  $x * y \in A_t$ . Let  $A(x) = t$ , then

$$A(x * y) \geq t = A(x) \geq A(x) \wedge A(y).$$

For all  $x \in X, m \in M$ , if  $A_t$  is an  $M$ -subalgebra of  $X$ , then we have

$$A(mx) \geq t = A(x),$$

therefore  $A$  is an  $M$ -fuzzy subalgebra of  $X$ .

**Proposition 4.** Suppose  $X, Y$  are  $M$ -BCI-algebra,  $f$  is a mapping from  $X$  to  $Y$ , if  $A$  is an  $M$ -fuzzy subalgebra of

the  $Y$ , then  $f^{-1}(A)$  is an  $M$ -fuzzy subalgebra of  $X$ .

**Proof.** Let  $y \in Y$ , suppose  $f$  is an epimorphism, then there exists  $x$  in  $X$ , we have  $y = f(x)$ . If  $A$  is an  $M$ -fuzzy subalgebra of  $Y$ , then we have

$$A(x * y) \geq A(x) \wedge A(y), A(mx) \geq A(x).$$

For all  $x, y \in X, m \in M$ ,

$$(1) f^{-1}(A)(x * y) = A(f(x) * f(y)) \geq A(f(x)) \wedge A(f(y))$$

$$= f^{-1}(A)(x) \wedge f^{-1}(A)(y);$$

$$(2) f^{-1}(A)(mx) = A(f(mx)) = A(mf(x)) \geq A(f(x))$$

$$= f^{-1}(A)(x).$$

Therefore  $f^{-1}(A)$  is an  $M$ -fuzzy subalgebra of  $X$ .

#### IV. FUZZY IDEALS OF BCI-ALGEBRAS WITH OPERATORS

**Definition 11.**  $\langle X; *, 0 \rangle$  is an  $M$ -BCI-algebra,  $A$  is a fuzzy ideal of  $X$ , if  $A(mx) \geq A(x)$  for all  $x \in X, m \in M$ , then  $A$  is called an  $M$ -fuzzy ideal of  $X$ .

**Example 2.** If  $A$  is an  $M$ -fuzzy ideal of  $X$ , then  $X_A$  is an  $M$ -fuzzy ideal of  $X$ , define  $X_A$  by

$$X_A : X \rightarrow [0, 1], X_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

**Proof.** (1) For all  $x, y \in X$ , if  $x, y \in A$ , then  $x * y \in A$ , therefore

$$X_A(0) = 1 \geq X_A(x), X_A(x) = 1 \geq X_A(x * y) \wedge X_A(y),$$

if there exists at least one which does not belong to  $A$  between  $x$  and  $y$ , for example  $x \notin A$ , thus

$$X_A(0) = 1 \geq X_A(x), X_A(x) \geq X_A(x * y) \wedge X_A(y) = 0,$$

therefore  $X_A$  is a fuzzy ideal of  $X$ .

(2) For all  $x \in X, m \in M$ , if  $x \in A$ , then  $mx \in A$ , therefore

$$X_A(mx) = 1 \geq X_A(x).$$

If  $x \notin A$ , then

$$X_A(mx) \geq 0 = X_A(x),$$

therefore  $X_A$  is an  $M$ -fuzzy ideal of  $X$ .

**Proposition 5.**  $A$  is an  $M$ -fuzzy ideal of  $X$  if and only if  $A_t$  is an  $M$ -ideal of  $X$ , where  $A_t$  is non-empty set, define  $A_t$  by

$$A_t = \{x \mid x \in X, A(x) \geq t\}, \forall t \in [0, 1].$$

**Proof.** Suppose  $A$  is an  $M$ -fuzzy ideal of  $X$ ,  $A_t$  is non-empty set,  $t \in [0, 1]$ , then we have

$$A(0) \geq A(x) \geq t,$$

thus  $0 \in A_t$ . If  $x * y \in A_t, y \in A_t$ , then

$$A(x * y) \geq t, A(y) \geq t,$$

thus

$$A(x) \geq A(x * y) \wedge A(y) \geq t,$$

thus we have

$$x \in A_t.$$

For all  $x \in X, m \in M$ , if  $A$  is an  $M$ -fuzzy ideal of  $X$ , hence

$$A(mx) \geq A(x) \geq t,$$

thus

$$mx \in A_t,$$

therefore  $A_t$  is an  $M$ -ideal of  $X$ . Conversely, suppose  $A_t$  is an  $M$ -ideal of  $X$ , then we have  $0 \in A_t, A(0) \geq t$ . Let  $A(x) = t$ , thus  $x \in A_t$ , we have

$$A(0) \geq t = A(x),$$

suppose there is no

$$A(x) \geq A(x * y) \wedge A(y),$$

then there exist  $x_0, y_0 \in X$ , we have

$$A(x_0) < A(x_0 * y_0) \wedge A(y_0),$$

let  $t_0 = A(x_0 * y_0) \wedge A(y_0)$ , then

$$A(x_0) < t_0 = A(x_0 * y_0) \wedge A(y_0),$$

if  $x_0 * y_0 \in A_{t_0}, y_0 \in A_{t_0}$ , then we have

$$x_0 \in A_{t_0},$$

then

$$A(x_0) \geq t_0,$$

which is inconsistent with  $A(x_0) < t_0 = A(x_0 * y_0) \wedge A(y_0)$ , then we have

$$A(x) \geq A(x * y) \wedge A(y).$$

For all  $x \in X, m \in M$ , if  $A_t$  is an  $M$ -ideal of  $X$ , then we have

$$A(mx) \geq t = A(x),$$

therefore  $A$  is an  $M$ -fuzzy ideal of  $X$ .

**Proposition 6.** Suppose  $X, Y$  are  $M$ -BCI-algebras,  $f$  is a mapping from  $X$  to  $Y$ ,  $A$  is an  $M$ -fuzzy ideal of  $Y$ , then  $f^{-1}(A)$  is an  $M$ -fuzzy ideal of  $X$ .

**Proof.** Let  $y \in Y$ , suppose  $f$  is an epimorphism, then there exists  $x \in X$ , we have  $y = f(x)$ . If  $A$  is an  $M$ -fuzzy ideal of  $Y$ , then we have

$$A(0) \geq A(y) \text{ or } A(f(0)) \geq A(y).$$

For all  $x, y \in X, m \in M$ ,

$$(1) f^{-1}(A)(0) = A(f(0)) = A(0) \geq A(f(x)) = f^{-1}(A)(x);$$

$$(2) f^{-1}(A)(x) = A(f(x))$$

$$\geq A(f(x) * f(y)) \wedge A(f(y)) = A(f(x * y)) \wedge A(f(y)) \\ = f^{-1}(A)(x * y) \wedge f^{-1}(A)(y);$$

$$(3) f^{-1}(A)(mx) = A(f(mx)) = A(mf(x)) \geq A(f(x)) = f^{-1}(A)(x).$$

Therefore  $f^{-1}(A)$  is an  $M$ -fuzzy ideal of  $X$ .

#### V. FUZZY QUOTIENT BCI-ALGEBRAS WITH OPERATORS

**Definition 12.** Let  $A$  be an  $M$ -fuzzy ideal of  $X$ , for all  $a \in X$ , fuzzy set  $A_a$  on  $X$  defined as:

$$A_a : X \rightarrow [0, 1]$$

$$A_a(x) = A(a * x) \wedge A(x * a), \forall x \in X.$$

Denote  $X/A = \{A_a : a \in X\}$ .

**Proposition 7.** Let  $A_a, A_b \in X/A$ , then  $A_a = A_b$  if and only if  $A(a * b) = A(b * a) = A(0)$ .

**Proof.** Let  $A_a = A_b$ , then we have  $A_a(b) = A_b(b)$ , thus

$$A(a * b) \wedge A(b * a) = A(b * b) \wedge A(b * b) = A(0).$$

That is  $A(a * b) = A(b * a) = A(0)$ . Conversely, suppose that  $A(a * b) = A(b * a) = A(0)$ . For all  $x \in X$ , since

$$(a * x) * (b * x) \leq a * b, (x * a) * (x * b) \leq b * a.$$

It follows from Proposition 1 that

$$A(a * x) \geq A(b * x) \wedge A(a * b), A(x * a) \geq A(x * b) \wedge A(b * a).$$

Hence

$$A_a(x) = A(a * x) \wedge A(x * a) \geq A(b * x) \wedge A(x * b) = A_b(x).$$

That is  $A_a \geq A_b$ . Similarly, for all  $x \in X$ , since

$$(b * x) * A(a * x) \leq b * a, (x * b) * A(x * a) \leq a * b.$$

It follows from Proposition 1 that

$$A(b * x) \geq A(a * x) \wedge A(b * a), A(x * b) \geq A(x * a) \wedge A(a * b).$$

Hence

$$A_b(x) = A(b * x) \wedge A(x * b) \geq A(a * x) \wedge A(x * a) = A_a(x).$$

That is  $A_b \geq A_a$ . Therefore,  $A_a = A_b$ . we complete the proof.

**Proposition 8.** Let  $A_a = A_{a'}, A_b = A_{b'}$ , then  $A_{a * b} = A_{a' * b'}$ .

**Proof.** Since

$$\begin{aligned} ((a * b) * (a' * b')) * (a * a') &= ((a * b) * (a * a')) * (a' * b') \\ &\leq (a' * b) * (a' * b') \leq b' * b, \\ ((a' * b') * (a * b)) * (b * b') &= ((a' * b') * (b * b')) * (a * b) \\ &\leq (a' * b) * (a * b) \leq a' * a. \end{aligned}$$

Hence

$$A((a * b) * (a' * b')) \geq A(a * a') \wedge A(b' * b) = A(0),$$

$$A((a' * b') * (a * b)) \geq A(b * b') \wedge A(a' * a) = A(0).$$

Therefore

$$A((a * b) * (a' * b')) = A((a' * b') * (a * b)) = A(0),$$

it follows from Proposition 7 that  $A_{a * b} = A_{a' * b'}$ . we completed the proof.

Let  $A$  be an  $M$ -fuzzy ideal of  $X$ . The operation "\*" of  $R/A$  is defined as:

$$\forall A_a, A_b \in R/A, A_a * A_b = A_{a * b}.$$

By Proposition 7, the above operation is reasonable.

**Proposition 9.** Let  $A$  be an  $M$ -fuzzy ideal of  $X$ , then

$R/A = \{R/A; *, A_0\}$  is an  $M$ -BCI-algebra.

**Proof.** For all  $A_x, A_y, A_z \in R/A$ ,

$$\begin{aligned} ((A_x * A_y) * (A_x * A_z)) * (A_z * A_y) &= A_{((x*y)*(x*z))*(z*y)} = A_0; \\ (A_x * (A_x * A_y)) * A_y &= A_{(x*(x*y))*y} = A_0; \quad A_x * A_x = A_{x*x} = A_0; \end{aligned}$$

if  $A_x * A_y = A_0, A_y * A_x = A_0$ , then

$$A_{x*y} = A_0, A_{y*x} = A_0,$$

it follows from Proposition 7 that

$$A(x * y) = A(0), A(y * x) = A(0),$$

hence

$$A_x = A_y.$$

Therefore  $R/A = \{R/A; *, A_0\}$  is a BCI-algebra. For all  $A_x \in R/A, m \in M$ , we define  $mA_x = A_{mx}$ . Firstly, we verify that  $mA_x = A_{mx}$  is reasonable. If  $A_x = A_y$ , then we verify

$$mA_x = mA_y,$$

that is to verify

$$A_{mx} = A_{my}.$$

We have

$$A(mx * my) = A(m(x * y)) \geq A(x * y) = A(0)$$

and

$$A(my * mx) = A(m(y * x)) \geq A(y * x) = A(0),$$

so we have

$$A(mx * my) = A(my * mx) = A(0),$$

that is,  $A_{mx} = A_{my}$ . In addition, for all  $m \in M, A_x, A_y \in R/A$ ,

$$m(A_x * A_y) = mA_{x*y} = A_{m(x*y)} = A_{(mx)*(my)} = A_{mx} * A_{my} = mA_x * mA_y.$$

Therefore  $R/A = \{R/A; *, A_0\}$  is an  $M$ -BCI-algebra.

**Definition 13.** Let  $\mu$  be an  $M$ -fuzzy subalgebra of  $X$ , and  $A$  be an  $M$ -fuzzy ideal of  $X$ , we define a fuzzy set of  $X/A$  as:

$$\mu/A : X/A \rightarrow [0,1], \quad \mu/A(A_i) = \sup_{A_i=A} \mu(x), \forall A_i \in X/A.$$

**Proposition 10.**  $\mu/A$  is an  $M$ -fuzzy subalgebra of  $X/A$ .

**Proof.** For all  $A_x, A_y \in X/A$ ,

$$\begin{aligned} \mu/A(A_x * A_y) &= \mu/A(A_{x*y}) \\ &= \sup_{A_z=A_{x*y}} \mu(z) \geq \sup_{A_s=A_x, A_t=A_y} \mu(s * t) \geq \sup_{A_s=A_x, A_t=A_y} \mu(s) \wedge \mu(t) \\ &= \sup_{A_s=A_x} \mu(s) \wedge \sup_{A_t=A_y} \mu(t) = \mu/A(A_x) \wedge \mu/A(A_y). \end{aligned}$$

For all  $m \in M, A_x \in R/A$ ,

$$\mu/A(A_{mx}) = \sup_{A_{mc}=A_{mx}} \mu(mc) \geq \sup_{A_z=A_x} \mu(z) = \mu/A(A_x).$$

Therefore,  $\mu/A$  is an  $M$ -fuzzy subalgebra of  $X/A$ .

### VI. DIRECT PRODUCTS OF FUZZY IDEALS IN BCI-ALGEBRAS WITH OPERATORS

**Proposition 11.** Suppose  $A$  and  $B$  are  $M$ -fuzzy ideals of  $X$ , then  $A \times B$  is an  $M$ -fuzzy ideal of  $X \times X$ .

**Proof.** (1) Let  $(x, y) \in X \times X$ , then

$$A \times B(0, 0) = A(0) \wedge B(0) \geq A(x) \wedge B(y) = A \times B(x, y),$$

thus for all  $(x, y) \in X \times X, A \times B(0, 0) \geq A \times B(x, y)$ ;

(2) For all  $(x_1, x_2), (y_1, y_2) \in X \times X$ , we have

$$\begin{aligned} A \times B((x_1, x_2) * (y_1, y_2)) &\wedge A \times B(y_1, y_2) \\ &= A \times B(x_1 * y_1, x_2 * y_2) \wedge A \times B(y_1, y_2) \\ &= (A(x_1 * y_1) \wedge B(x_2 * y_2)) \wedge A(y_1) \wedge B(y_2) \\ &= (A(x_1 * y_1) \wedge A(y_1)) \wedge (B(x_2 * y_2) \wedge B(y_2)) \\ &\leq A(x_1) \wedge B(x_2) \\ &= A \times B(x_1, x_2), \end{aligned}$$

thus for all  $(x_1, x_2), (y_1, y_2) \in X \times X$ , we have

$$A \times B(x_1, x_2) \geq A \times B((x_1, x_2) * (y_1, y_2)) \wedge A \times B(y_1, y_2);$$

(3) For all  $(x, y) \in X \times X$ , we have

$$\begin{aligned} A \times B(m(x, y)) &= A \times B(mx, my) = A(mx) \wedge B(my) \\ &\geq A(x) \wedge B(y) = A \times B(x, y), \end{aligned}$$

thus for all  $\forall (x, y) \in X \times X$ , we have

$$A \times B(m(x, y)) \geq A \times B(x, y).$$

Therefore  $A \times B$  is an  $M$ -fuzzy ideal of  $X \times X$ .

**Proposition 12.** Suppose  $A$  and  $B$  are fuzzy sets of  $X$ , if

$A \times B$  is an  $M$ -fuzzy ideal of  $X \times X$ , then  $A$  or  $B$  is an  $M$ -fuzzy ideal of  $X$ .

**Proof.** Suppose  $A$  and  $B$  are  $M$ -fuzzy ideals of  $X$ , then for all  $(x_1, x_2), (y_1, y_2) \in X \times X$ , we have

$$\begin{aligned} A \times B(x_1, x_2) &\geq A \times B((x_1, x_2) * (y_1, y_2)) \wedge A \times B(y_1, y_2) \\ &= A \times B((x_1 * y_1), (x_2 * y_2)) \wedge A \times B(y_1, y_2), \end{aligned}$$

if  $x_1 = y_1 = 0$ , then

$$A \times B(0, x_2) \geq A \times B(0, x_2 * y_2) \wedge A \times B(0, y_2),$$

we have  $A \times B(0, x) = A(0) \wedge B(x) = B(x)$ , so  $B(x_2) \geq B(x_2 * y_2) \wedge B(y_2)$ . If  $A \times B$  is an  $M$ -fuzzy ideal of  $X$ , then

$$A \times B(m(x, y)) \geq A \times B(x, y), \forall (x, y) \in X \times X,$$

let  $x = 0$ , then

$$\begin{aligned} A \times B(m(x, y)) &= A \times B(mx, my) = A(mx) \wedge B(my) = B(my) \\ &\geq A(x) \wedge B(y) = A(0) \wedge B(y) = B(y), \end{aligned}$$

thus we have  $B(my) \geq B(y)$  for all  $y \in X, m \in M$ . Therefore  $B$  is an  $M$ -fuzzy ideal of  $X$ .

**Proposition 13.** If  $B$  is a fuzzy set,  $A$  is a strong fuzzy relation  $A_B$  of  $B$ , then  $B$  is a  $M$ -fuzzy ideal of  $X$  if only if  $A_B$  is an  $M$ -fuzzy ideal of  $X \times X$ .

**Proof.** If  $B$  is an  $M$ -fuzzy ideals of  $X$ , then for all  $(x, y) \in X \times X$ , we have

$$A_B(0, 0) = B(0) \wedge B(0) \geq B(x) \wedge B(y) = A_B(x, y);$$

for all  $(x_1, x_2), (y_1, y_2) \in X \times X$ , we have

$$\begin{aligned} A_B(x_1, x_2) &= B(x_1) \wedge B(x_2) \\ &\geq (B(x_1 * y_1) \wedge B(y_1)) \wedge (B(x_2 * y_2) \wedge B(y_2)) \\ &= (B(x_1 * y_1) \wedge B(x_2 * y_2)) \wedge (B(y_1) \wedge B(y_2)) \\ &= A_B(x_1 * y_1, x_2 * y_2) \wedge A_B(y_1, y_2) \\ &= A_B((x_1, x_2) * (y_1, y_2)) \wedge A_B(y_1, y_2); \end{aligned}$$

for all  $(x, y) \in X \times X, m \in M$ ,

$$\begin{aligned} A_B(m(x, y)) &= A_B(mx, my) = B(mx) \wedge B(my) \\ &\geq B(x) \wedge B(y) = A_B(x, y). \end{aligned}$$

Therefore, if  $B$  is an  $M$ -fuzzy ideal of  $X$ , then  $A_B$  is an  $M$ -fuzzy ideal of  $X \times X$ . Conversely, suppose  $A_B$  is an  $M$ -fuzzy ideal of  $X \times X$ , then  $\forall (x_1, x_2) \in X \times X$ , we have

$$B(0) \wedge B(0) = A_B(0, 0) \geq A_B(x, x) = B(x) \wedge B(x);$$

for all  $(x_1, x_2), (y_1, y_2) \in X \times X$ , we have

$$\begin{aligned} B(x_1) \wedge B(x_2) &= A_B(x_1, x_2) \\ &\geq A_B((x_1, x_2) * (y_1, y_2)) \wedge A_B(y_1, y_2) \\ &= A_B(x_1 * y_1, x_2 * y_2) \wedge A_B(y_1, y_2) \\ &= (B(x_1 * y_1) \wedge B(x_2 * y_2)) \wedge (B(y_1) \wedge B(y_2)) \\ &= (B(x_1 * y_1) \wedge B(y_1)) \wedge (B(x_2 * y_2) \wedge B(y_2)); \end{aligned}$$

let  $x_2 = y_2 = 0$ , then

$$B(x_1) \wedge B(0) \geq (B(x_1 * y_1) \wedge B(y_1)) \wedge B(0),$$

if  $A_B$  is an  $M$ -fuzzy ideal of  $X \times X$ , then

$$\begin{aligned} A_B(m(x, y)) &\geq A_B(x, y), \forall x, y \in X \times X, m \in M, \\ B(mx) \wedge B(my) &= A_B(mx, my) \geq A_B(x, y) = B(x) \wedge B(y), \end{aligned}$$

if  $x = 0$ , then

$$B(0) \wedge B(my) = A_B(0, my) \geq A_B(0, y) = B(0) \wedge B(y),$$

namely,  $B(my) \geq B(y)$ . Therefore  $B$  is an  $M$ -fuzzy ideal of  $X$ .

REFERENCES

- [1] Y. Imai and K. Iseki, "On axiom system of propositional calculus," Proc Aapan Academy, vol. 42, pp. 26-29, 1966.
- [2] K. Iseki, "On BCI-algebras," Math. Sem. Notes, vol. 8, pp.125-130, 1980.
- [3] O.G. Xi, "Fuzzy BCK-algebras," Math Japon, vol. 36, pp. 935-942, 1991.
- [4] Y.B. Jun, S.M. Hong, J. Meng and X.L. Xin, "Characterizations of fuzzy positives implicative ideals in BCK-algebras", Math. Japon, vol. 40, pp.503-507, 1994.
- [5] Y.B. Jun and E.H. Roh, "Fuzzy commutative ideals of BCK-algebras," Fuzzy Sets and Systems, vol. 64, pp. 401-405, 1994.
- [6] J. Meng, Y.B. Jun and H.S. Kim, "Fuzzy implicative ideals of BCK-Algebras," Fuzzy sets syst, vol. 89, pp. 243-248, 1997.
- [7] Y.B. Jun and J. Meng, "Fuzzy commutative ideals in BCI-algebras," Comm. Korean Math. Soc, vol. 9, pp. 19-25, 1994.
- [8] W. X. Zheng, "On BCI-algebras with operators and their isomorphism theorems," Journal of Qingdao University, vol. 6, pp. 17-22, 1993.
- [9] Y.L. Liu and J. Meng, "Fuzzy ideals in BCI-algebras," Fuzzy Sets and Systems, vol. 123, pp. 227-237, 2001.
- [10] J. Meng, "Fuzzy ideals of BCI-algebras," S EA Bull. math, vol. 18, pp. 401- 405, 1994.
- [11] Y. L. Liu, "Characterizations of some classes of quotient BCI-algebras" Journal of Quan zhou Normal College (Natural Science Edition), vol. 20, pp. 16-20, 2002.
- [12] J.L, "Universal property of direct products of BCI-Algebra" Journal of

Jiangnan University, vol. 18, pp. 36-38, 2001.

- [13] L.A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, pp. 338-353, 1965.