

Fuzzy Ideals in Near-subtraction Semigroups

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Abstract—In this paper, we introduce a notion of fuzzy ideals in near-subtraction semigroups and study their related properties.

Keywords—subtraction algebra, subtraction semigroup, an ideal, near—subtraction semigroup, fuzzy level set, fuzzy ideal, fuzzy homomorphism.

I. INTRODUCTION

THE systems of the form Φ , where $(\Phi; \circ, \backslash)$, considered by B. M. Schein [7], is a set of functions closed under the composition “ \circ ” of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction “ \backslash ” (and hence $(\Phi; \backslash)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [9] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. B. Jun et al. [3] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In [4], Y. B. Jun and H. S. Kim established the ideal generated by a set, and discussed related results. Near-ring theory has been developed by Pilz [6]. Based on near-ring theory, Dheena et al. [2], introduced the near-subtraction semigroups and strongly regular near-subtraction semigroups.

The concept of fuzzy subset was introduced by L.A. Zadeh [8]. Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. K.J. Lee and C.H. Park [5] introduced the notion of a fuzzy ideal in subtraction algebras, and give some conditions for a fuzzy set to be a fuzzy ideal in subtraction algebras. In this paper, we introduce the notion of fuzzy ideal in near-subtraction semigroup and have studied their related properties.

II. PRELIMINARIES

Definition 2.1: A non-empty set X together with a binary operation “ $-$ ” is said to be a subtraction algebra if it satisfies the following:

- (1) $x - (y - x) = x$.
- (2) $x - (x - y) = y - (y - x)$.
- (3) $(x - y) - z = (x - z) - y$, for all $x, y, z \in X$.

Example 2.2: Let $X = \{0, a, b, 1\}$ in which “ $-$ ” is defined by

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$-$	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	b	a	0

Then $(X, -)$ is a subtraction algebra.

In a subtraction algebra the following holds:

(P1) $x - 0 = x$ and $0 - x = 0$.

(P2) $(x - y) - x = 0$.

(P3) $(x - y) - y = x - y$.

(P4) $(x - y) - (y - x) = x - y$, where $0 = x - x$ is an element that does not depend on the choice of $x \in X$.

Following [9], we have the following definition of subtraction semigroup.

Definition 2.3: A non-empty set X together with the binary operations “ $-$ ” and “ \cdot ” is said to be a subtraction semigroup if it satisfies the following:

(SS1) $(X; -)$ is a subtraction algebra.

(SS2) $(X; \cdot)$ is a semigroup.

(SS3) $x(y - z) = xy - xz$ and $(x - y)z = xz - yz$, for all $x, y, z \in X$.

Example 2.4: [2] Let $X = \{0, a, b, 1\}$ in which “ $-$ ” and “ \cdot ” are defined by

$-$	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	b	a	0

\cdot	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

Then $(X, -, \cdot)$ is a subtraction semigroup.

Now we have the following definition of near-subtraction semigroup.

Definition 2.5: A non-empty set X together with the binary operations “ $-$ ” and “ \cdot ” is said to be a near-subtraction semigroup if it satisfies the following:

(NS1) $(X; -)$ is a subtraction algebra.

(NS2) $(X; \cdot)$ is a semigroup.

(NS3) $(x - y)z = xz - yz$, for all $x, y, z \in X$.

It is clear that $0x = 0$, for all $x \in X$. Similarly we can define a near-subtraction semigroup (left). Hereafter a near-subtraction semigroup means it is a near-subtraction semigroup (right) only.

Example 2.6: [2] Let $X = \{0, a, b, 1\}$ in which “ $-$ ” and “ \cdot ” are defined by

$-$	0	a	b	1
0	0	0	0	0
a	a	0	1	b
b	b	0	0	b
1	1	0	1	0

\cdot	0	a	b	1
0	0	0	0	0
a	a	a	a	a
b	a	0	1	b
1	0	a	b	1

Then $(X, -, \cdot)$ is a near-subtraction semigroup.

Definition 2.7: A near-subtraction semigroup X is said to be zero-symmetric if $x0 = 0$ for every $x \in X$.

Definition 2.8: A near-subtraction semigroup X is said to have an identity if there exists an element $1 \in X$ such that $1.x = x.1 = x$, for every $x \in X$.

Definition 2.9: A non-empty subset S of a subtraction algebra X is said to be a subalgebra of X , if $x - y \in S$, whenever $x, y \in S$.

Definition 2.10: Let $(X, -, \cdot)$ be a near-subtraction semigroup. A non-empty subset I of X is called
(I1) a *left ideal* if I is a subalgebra of $(X, -)$ and $xi - x(y - i) \in I$ for all $x, y \in X$ and $i \in I$.
(I2) a *right ideal* if I is a subalgebra of $(X, -)$ and $IX \subseteq I$.
(I3) an *ideal* if I is both a left and right ideal. $IX \subseteq I$.

Remark 2.11: (i) Suppose if X is a subtraction semigroup and I is a left ideal of X , then for $i \in X$ and $x, y \in X$, we have $xi - x(y - i) = xi - (xy - xi) = xi \in I$ by Property 1 of subtraction algebra. Thus we have $XI \subseteq I$.

(ii) If X is a zero symmetric near-subtraction semigroup, then for $i \in I$ and $x \in X$, we have $xi - x(0 - i) = xi - 0 = xi \in X$.

For the sake of completeness, now we study some concepts of fuzzy theory.

A mapping $\mu : X \rightarrow [0, 1]$ is called *fuzzy set* of X and the complement of a fuzzy set μ , denoted by μ' is the fuzzy set in X given by $\mu'(x) = 1 - \mu(x)$ for all $x \in X$. The *level set* of a fuzzy set μ of X is defined as $U(\mu; t) = \{x \in X | \mu(x) \geq t\}$, for all $0 \leq t \leq 1$.

III. FUZZY IDEALS

In what follows, let X denote a near-subtraction semigroup, unless otherwise specified.

Definition 3.1: A fuzzy set μ in X is called a *fuzzy ideal* of X if it satisfies the following conditions:

- (FI1) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$,
(FI2) $\mu(ax - a(b - x)) \geq \mu(x)$ for all $a, b, x \in X$ and
(FI3) $\mu(xy) \geq \mu(x)$, for all $x, y \in X$.

Note that μ is a *fuzzy left ideal* of X if it satisfies (FI1) and (FI2), and μ is a *fuzzy right ideal* of X if it satisfies (FI1) and (FI3).

Example 3.2: Let $X = \{0, a, b, 1\}$ in which “-” and “.” are defined by

-	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

.	0	a	b
0	0	0	0
a	0	a	0
b	a	0	b

Then $(X, -, \cdot)$ is a near-subtraction semigroup. Let μ be a fuzzy set on X defined by $\mu(0) = 0.8, \mu(a) = 0.5$ and $\mu(b) = 0.3$. Then by routine calculation, it is easy to prove that μ is a fuzzy ideal of X .

Theorem 3.3: Let μ be a fuzzy left (resp. right) of X . Then the set

$$X_\mu = \{x \in X | \mu(x) = \mu(0)\}$$

is a left (resp. right) ideal of X .

Proof: Suppose μ is a fuzzy left ideal of X and let $x, y \in X_\mu$. Then

$$\mu(x - y) \geq \min\{\mu(x), \mu(y)\} = \mu(0).$$

Thus $x - y \in X_\mu$.

For every $a, b \in X$ and $x \in X_\mu$, we have

$$\mu(ax - a(b - x)) \geq \mu(x) = \mu(0).$$

Thus $ax - a(b - x) \in X_\mu$. Hence, X_μ is a left ideal of X . Similarly, we have the desired result for the right case. ■

Theorem 3.4: Let A be a non-empty subset of X and μ_A be a fuzzy set in X defined by

$$\mu_A(x) = \begin{cases} s, & \text{if } x \in A, \\ t, & \text{otherwise.} \end{cases}$$

for all $x \in X$ and $s, t \in [0, 1]$ with $s > t$. Then μ_A is a fuzzy ideal of X if and only if A is an ideal of X . Moreover $X_{\mu_A} = A$.

Proof: Suppose μ_A is a fuzzy ideal of X . Let $x, y \in A$. Then

$$\mu(x - y) \geq \min\{\mu(x), \mu(y)\} = s.$$

Thus, $x - y \in A$.

For every $a, b \in X$ and $x \in A$, we have

$$\mu(ax - a(b - x)) \geq \mu(x) = s.$$

Thus $ax - a(b - x) \in A$.

For all $x, y \in A$. Then

$$\mu(xy) \geq \mu(x) = s.$$

Thus, $xy \in A$. Hence, μ_A is an ideal of X .

Conversely, assume that A is an ideal of X . Let $x, y \in X$. If at least one of x and y does not belong to A , then

$$\mu_A(x - y) \geq t = \min\{\mu_A(x), \mu_A(y)\}.$$

If $x, y \in A$ then $x - y \in A$, we have

$$\mu_A(x - y) \geq s = \min\{\mu_A(x), \mu_A(y)\}.$$

Let $a, b, x \in X$ and if $x \in A$ such that $ax - a(b - x) \in A$, we have

$$\mu_A(ax - a(b - x)) \geq s = \mu_A(x).$$

If $x \notin A$ such that $ax - a(b - x) \notin A$, we have

$$\mu_A(ax - a(b - x)) \geq t = \mu_A(x).$$

For all $x, y \in A$ then $xy \in A$, we have

$$\mu_A(xy) \geq s = \mu(x).$$

Suppose $x \notin A$ we have

$$\mu_A(xy) \geq t = \mu(x).$$

Hence μ_A is a fuzzy ideal of X . Moreover

$$\begin{aligned} X_{\mu_A} &= \{x \in X | \mu_A(x) = \mu_A(0)\} \\ &= \{x \in X | \mu_A(x) = s\} \\ &= \{x \in X | x \in A\} \\ &= A. \end{aligned}$$

Corollary 3.5: Let χ_A be the characteristic function of a subset $A \subseteq X$. Then χ_A is a fuzzy left (resp. right) ideal if and

only if A is a left(*resp.* right) ideal.

Theorem 3.6: Let μ be a fuzzy subset of X . Then μ is a fuzzy ideal of X if and only if each non-empty level subset $U(\mu; t)$ of μ is an ideal of X .

Proof: Assume that μ is a fuzzy ideal of X and $U(\mu; t)$ is a non-empty level subset of X .

(i) Since $U(\mu; t)$ is a non-empty level subset of μ , there exists $x, y \in U(\mu; t)$, $\mu(x-y) \geq \min\{\mu(x), \mu(y)\} = t$. Thus $x-y \in U(\mu; t)$.

(ii) Let $a, b, x \in U(\mu; t)$, we have $\mu(ax - a(b-x)) \geq \mu(x) \geq t$. Thus $ax - a(b-x) \in U(\mu; t)$.

(iii) Let $x, y \in U(\mu; t)$, such that $\mu(xy) \geq \mu(x) \geq t$. Thus $xy \in U(\mu; t)$. Hence, $U(\mu; t)$ is an ideal of R .

Conversely, suppose that $U(\mu; t)$ is an ideal of X .

(i) Let if possible, $\mu(x_0 - y_0) < \min\{\mu(x_0), \mu(y_0)\}$, for some $x_0, y_0 \in U(\mu; t)$, then by taking

$$t_0 = \frac{1}{2} \{\mu(x_0 - y_0) + \min\{\mu(x_0), \mu(y_0)\}\},$$

we have $\mu(x_0 - y_0) > t_0$, for $\mu(x_0) \geq t_0, \mu(y_0) \geq t_0$. Thus $x_0 - y_0 \notin U(\mu; t)$, for some $x_0, y_0 \in U(\mu; t)$. This is a contradiction, and so $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in U(\mu; t)$.

(ii) Let if possible, for some $x_0 \in U(\mu; t)$ $\mu(ax - (a(b-x))) < \mu(x_0)$, for all $a, b \in X$ and, then by taking

$$t_0 = \frac{1}{2} \{\mu(ax_0 - a(b-x_0)) + \mu(x_0)\},$$

we have $\mu(ax_0 - a(b-x_0)) > t_0$, for $\mu(x_0) \geq t_0, \mu(y_0) \geq t_0$. Thus $ax_0 - a(b-x_0) \notin U(\mu; t)$, for some $x_0 \in U(\mu; t)$ and for all $a, b \in X$. This is a contradiction, and so $\mu(ax - a(b-x)) \geq \mu(x)$, for all $x \in U(\mu; t)$ and $a, b \in X$.

(iii) Let if possible, $\mu(x_0 y_0) < \mu(x_0)$, for some $x_0, y_0 \in U(\mu; t)$, then by taking

$$t_0 = \frac{1}{2} \{\mu(x_0 y_0) + \mu(x_0)\},$$

we have $\mu(x_0 y_0) > t_0$, for $\mu(x_0) \geq t_0, \mu(y_0) \geq t_0$. Thus $x_0 y_0 \notin U(\mu; t)$, for some $x_0, y_0 \in U(\mu; t)$. This is a contradiction, and so $\mu(xy) \geq \mu(x)$, for all $x, y \in U(\mu; t)$. Hence $U(\mu; t)$ is a fuzzy ideal of X . ■

Definition 3.7: Let X be a near-subtraction semigroup and a family of fuzzy sets $\{\mu_i | i \in I\}$ in X . Then the intersection

$\left(\bigwedge_{i \in I} \mu_i\right)$ of $\{\mu_i | i \in I\}$ is defined by

$$\left(\bigwedge_{i \in I} \mu_i\right)(x) = \inf \{\mu_i(x) | i \in I\}$$

Theorem 3.8: If $\{\mu_i | i \in I\}$ is a family of fuzzy ideal of X , then $\left(\bigwedge_{i \in I} \mu_i\right)(x)$ is a fuzzy ideal of X .

Proof: Let $\{\mu_i | i \in I\}$ be a family of fuzzy ideal of X .

(i) For all $x, y \in X$, we have

$$\begin{aligned} \left(\bigwedge_{i \in I} \mu_i\right)(x-y) &= \inf \{\mu_i(x-y) | i \in I\} \\ &\geq \inf \{\min(\mu_i(x), \mu_i(y)) | i \in I\} \\ &= \min \{\inf(\mu_i(x) | i \in I), \inf(\mu_i(y) | i \in I)\} \\ &= \min \left\{ \left(\bigwedge_{i \in I} \mu_i\right)(x), \left(\bigwedge_{i \in I} \mu_i\right)(y) \right\} \end{aligned}$$

(i) For all $a, b, x \in X$, we have

$$\begin{aligned} \left(\bigwedge_{i \in I} \mu_i\right)(ax - a(b-x)) &= \inf \{\mu_i(ax - a(b-x)) | i \in I\} \\ &\geq \inf \{\mu_i(x) | i \in I\} \\ &= \{\inf(\mu_i(x) | i \in I)\} \\ &= \left(\bigwedge_{i \in I} \mu_i\right)(x). \end{aligned}$$

(iii) For all $x, y \in X$, we have

$$\begin{aligned} \left(\bigwedge_{i \in I} \mu_i\right)(xy) &= \inf \{\mu_i(xy) | i \in I\} \\ &\geq \inf \{\min(\mu_i(x)) | i \in I\} \\ &= \left(\bigwedge_{i \in I} \mu_i\right)(x) \end{aligned}$$

Hence $\left(\bigwedge_{i \in I} \mu_i\right)$ is a fuzzy ideal of X . ■

Definition 3.9: Let $f : X \rightarrow X'$ be a mapping, where X and X' are non-empty sets and μ is a fuzzy subset of X . The preimage of μ under f written μ^f , is a fuzzy subset of X defined by $\mu^f = \mu(f(x))$, for all $x \in X$.

Theorem 3.10: Let $f : X \rightarrow X'$ be a homomorphism of near-subtraction semigroups. If μ is a fuzzy ideal of X' , then μ^f is a fuzzy ideal of X .

Proof: Suppose μ is a fuzzy ideal of X' , then

(i) For all $x, y \in X$, we have

$$\begin{aligned} \mu^f(x-y) &= \mu(f(x-y)) = \mu(f(x) - f(y)) \\ &\geq \min\{\mu(f(x)), \mu(f(y))\} \\ &= \min\{\mu^f(x), \mu^f(y)\}. \end{aligned}$$

(ii) For all $a, b, x \in X$, we have

$$\begin{aligned} \mu^f(ax - a(b-x)) &= \mu(f(ax - a(b-x))) \\ &= \mu(f(ax) - f(a(b-x))) \\ &= \mu(f(a)f(x) - f(a)(f(b) - f(x))) \\ &\geq \mu(f(x)) \\ &= \mu^f(x). \end{aligned}$$

(iii) For all $x, y \in X$, we have

$$\begin{aligned}\mu^f(xy) &= \mu(f(xy)) \\ &= \mu(f(x)f(y)) \\ &\geq \mu(f(y)) \\ &= \mu^f(y).\end{aligned}$$

Hence μ^f is a fuzzy ideal of X .

Theorem 3.11: Let $f: X \rightarrow X'$ be a homomorphism of near-subtraction semigroup. If μ^f is a fuzzy ideal of X , then μ is fuzzy ideal of X' .

Proof: Suppose μ is a fuzzy ideal of X' , then

(i) Let $x', y' \in X'$, there exists $x, y \in X$ such that $f(x) = x'$ and $f(y) = y'$, we have

$$\begin{aligned}\mu(x' - y') &= \mu(f(x) - f(y)) \\ &= \mu(f(x - y)) \\ &= \mu^f(x - y) \\ &\geq \min\{\mu^f(x), \mu^f(y)\} \\ &= \min\{\mu(f(x)), \mu(f(y))\} \\ &= \min\{\mu(x'), \mu(y')\}.\end{aligned}$$

(ii) Let $a', b', x' \in X'$, there exists $a, b, x \in X$ such that $f(a) = a'$, $f(b) = b'$ and $f(x) = x'$, we have

$$\begin{aligned}\mu(a'x' - b(a' - x')) &= \mu(f(a)f(x) - f(b)(f(a) - f(x))) \\ &= \mu(f(ax) - f(b)f(a - x)) \\ &= \mu(f(ax) - f(b(a - x))) \\ &= \mu(f(ax - b(a - x))) \\ &= \mu^f(ax - b(a - x)) \\ &\geq \mu^f(x) \\ &= \mu(f(x)) \\ &= \mu(x').\end{aligned}$$

(iii) Let $x', y' \in X'$, there exists $x, y \in X$ such that $f(x) = x'$ and $f(y) = y'$, we have

$$\begin{aligned}\mu(x'y') &= \mu(f(x)f(y)) = \mu(f(xy)) \\ &= \mu^f(xy) \\ &\geq \mu^f(x) \\ &= \mu(f(x)) \\ &= \mu(x').\end{aligned}$$

Hence μ is a fuzzy ideal of X' . ■

Definition 3.12: Let f be a mapping defined on X . If ν is a fuzzy subset in $f(X)$, then the fuzzy subset $\mu = \nu \circ f$ in X (i.e., the fuzzy subset defined by $\mu(x) = \nu(f(x))$ for all $x \in X$) is called the *preimage* of ν under f .

Proposition 3.13: An onto homomorphic preimage of a fuzzy ideal of X is a fuzzy ideal.

Proof: Straight forward. ■

Let μ be a fuzzy subset in X and f be a mapping defined on X . Then the fuzzy subset μ^f in $f(X)$ defined by $\mu^f(y) = \sup_{x \in f^{-1}(y)} \mu(x)$ for all $y \in f(X)$ is called the *image*

of μ under f . A fuzzy subset μ in X is said to have an *sup property* if for every subset $N \subseteq X$, there exists $n_0 \in N$ such that $\mu(n_0) = \sup_{n \in N} \mu(n)$.

Proposition 3.14: An onto homomorphic image of a fuzzy ideal with sup property is fuzzy ideal.

Proof: Let $f: X \rightarrow X'$ be an onto homomorphism of near-subtraction semigroup and let μ be a fuzzy ideal of X with the sup property.

(i) Given $x', y' \in X'$, we let $x_0 \in f^{-1}(x')$ and $y_0 \in f^{-1}(y')$ be such that

$$\mu(x_0) = \sup_{n \in f^{-1}(x')} \mu(n), \quad \mu(y_0) = \sup_{n \in f^{-1}(y')} \mu(n)$$

respectively. Then, we have

$$\begin{aligned}\mu^f(x' - y') &= \sup_{z \in f^{-1}(x' - y')} \mu(z) \\ &\geq \min\{\mu(x_0), \mu(y_0)\} \\ &= \min\left\{\sup_{n \in f^{-1}(x')} \mu(n), \sup_{n \in f^{-1}(y')} \mu(n)\right\} \\ &= \min\{\mu^f(x'), \mu^f(y')\}\end{aligned}$$

(ii) Given $a', b', x' \in X'$, we let $a_0 \in f^{-1}(a')$, $b_0 \in f^{-1}(b')$, $x_0 \in f^{-1}(x')$ be such that

$$\begin{aligned}\mu^f(a'x' - a'(b' - x')) &= \sup_{z \in f^{-1}(a'x' - a'(b' - x'))} \mu(z) \\ &\geq \mu(x_0) \\ &= \sup_{n \in f^{-1}(x')} \mu(n) \\ &= \mu^f(x').\end{aligned}$$

(iii) Given $x', y' \in X'$, we let $x_0 \in f^{-1}(x')$ and $y_0 \in f^{-1}(y')$ be such that

$$\mu(x_0) = \sup_{n \in f^{-1}(x')} \mu(n), \quad \mu(y_0) = \sup_{n \in f^{-1}(y')} \mu(n)$$

respectively. Then, we have

$$\begin{aligned}\mu^f(x'y') &= \sup_{z \in f^{-1}(x'y')} \mu(z) \\ &\geq \mu(x_0) \\ &= \sup_{n \in f^{-1}(x')} \mu(n) \\ &= \mu^f(x')\end{aligned}$$

Hence, μ^f is a fuzzy ideal of X' . ■

IV. CHAIN CONDITIONS

Proposition 4.1: Let μ and ν be a fuzzy subset of X . If they are fuzzy ideal of X , then so $\mu \cap \nu$, where $\mu \cap \nu$ is defined by

$(\mu \cap \nu)(x) = \min\{\mu(x), \nu(x)\}$ for all $x \in X$.

Proof: (i) For all $x, y \in X$, we have

$$\begin{aligned} (\mu \cap \nu)(x - y) &= \min\{\mu(x - y), \nu(x - y)\} \\ &\geq \min\{\min\{\mu(x), \mu(y)\}, \\ &\quad \min\{\nu(x), \nu(y)\}\} \\ &= \min\{(\mu \cap \nu)(x), (\mu \cap \nu)(y)\}. \end{aligned}$$

(ii) For all $x, y \in X$, we have

$$\begin{aligned} &(\mu \cap \nu)(ax - a(b - x)) \\ &= \min\{\mu(ax - a(b - x)), \nu(ax - a(b - x))\} \\ &\geq \min\{\mu(x), \nu(x)\} \\ &= (\mu \cap \nu)(x). \end{aligned}$$

(iii) For all $x, y \in X$, we have

$$\begin{aligned} (\mu \cap \nu)(xy) &= \min\{\mu(xy), \nu(xy)\} \\ &\geq \min\{\mu(y), \nu(y)\} \\ &= (\mu \cap \nu)(y). \end{aligned}$$

Hence, $\mu \cap \nu$ is a fuzzy ideal of X .

Theorem 4.2: Let μ be a fuzzy subset in X and $Im(\mu) = \{\alpha_0, \alpha_1, \dots, \alpha_k\}$, where $\alpha_i < \alpha_j$ whenever $i > j$. Let $\{A_n | n = 0, 1, \dots, k\}$ be a family of ideals of X such that

(i) $A_0 \subseteq A_1 \subseteq \dots \subseteq A_k = X$,

(ii) $\mu(A_n^*) = \alpha_n$, where $A_n^* = A_n \setminus A_{n-1}$, $A_{-1} = \phi$ for all $n = 0, 1, \dots, k$.

Then μ is a fuzzy ideal of X .

Proof: Suppose $\{A_n | n = 0, 1, \dots, k\}$ be a family of ideals of X .

(i) For all $x, y \in X$, Then we discuss the following cases: If $x \in A_n$ and $y \in A_n$ such that $x - y \in A_n$, since A_n is an ideal of X , thus

$$\mu(x - y) \geq \alpha_n = \min\{\mu(x), \mu(y)\}.$$

If $x \notin A_n^*$ and $y \notin A_n^*$, then the following four cases arise:

$$1) \ x \in X \setminus A_n \text{ and } y \in X \setminus A_n$$

$$2) \ x \in A_{n-1} \text{ and } y \in A_{n-1}$$

$$3) \ x \in X \setminus A_n \text{ and } y \in A_{n-1}$$

$$4) \ x \in A_{n-1} \text{ and } y \in R \setminus A_n$$

But, in either cases, we know that

$$\mu(x - y) \geq \min\{\mu(x), \mu(y)\}.$$

If $x \in X \setminus A_n^*$ and $y \notin A_n^*$ then either $y \in A_{n-1}$ or $y \in X \setminus A_n$. It follows that either $x \in A_n$ or $x \in X \setminus A_n$. Thus

$$\mu(x - y) \geq \min\{\mu(x), \mu(y)\}.$$

If $x \notin X \setminus A_n^*$ and $y \in A_n^*$ then by similar process we have

$$\mu(x - y) \geq \min\{\mu(x), \mu(y)\}.$$

(ii) If $a, b \in X$ and $x \in A_n$ then $ax - a(b - x) \in A_n$. Then

$$\mu(ax - a(b - x)) \geq \min\{\mu(a), \mu(b)\}.$$

If $a, b \in X$ and $x \notin A_n$ then, we have

$$\mu(ax - a(b - x)) \geq \alpha_n = \mu(x).$$

(iii) Similarly, for $x, y \in X$, we have

$$\mu(xy) \geq \mu(y).$$

Hence μ is a fuzzy ideal of X . ■

Theorem 4.3: Let $\{A_n | n \in \mathbb{N}\}$ be a family of ideals of X which is nested, that is, $X = A_1 \supseteq A_2 \supseteq \dots$. Let μ be a fuzzy subset in X defined by

$$\mu(x) = \begin{cases} \frac{n}{n+1} & \text{if } x \in A_n \setminus A_{n+1}, n = 1, 2, 3, \dots, \\ 1 & \text{if } x \in \bigcap_{n=1}^{\infty} A_n. \end{cases}$$

for all $x \in X$. Then μ is a fuzzy ideal of X .

Proof: Let $x, y \in X$.

(i) Suppose that $x \in A_k \setminus A_{k+1}$ and $y \in A_r \setminus A_{r+1}$ for $k = 1, 2, \dots; r = 1, 2, \dots$. Without loss of generality, we may assume that $k \leq r$. Then $x - y \in A_k$ and so

$$\mu(x - y) \geq \frac{k}{k+1} = \min\{\mu(x), \mu(y)\}$$

If $x, y \in \bigcap_{n=1}^{\infty} A_n$ then $x - y \in \bigcap_{n=1}^{\infty} A_n$ and thus

$$\mu(x - y) = 1 = \min\{\mu(x), \mu(y)\}$$

■ If $x \in \bigcap_{n=1}^{\infty} A_n$ and $y \notin \bigcap_{n=1}^{\infty} A_n$, then there exists $i \in \mathbb{N}$ such that $y \in A_i \setminus A_{i+1}$. It follows that $x - y \in A_i$ so that

$$\mu(x - y) \geq \frac{i}{i+1} = \min\{\mu(x), \mu(y)\}$$

Similarly, we can prove that

$$\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$$

for all $x \notin \bigcap_{n=1}^{\infty} A_n$ then $y \in \bigcap_{n=1}^{\infty} A_n$.

(ii) Now, let $a, b \in X$. If $x \in A_r \setminus A_{r+1}$ for some $k = 1, 2, \dots$, then $ax - a(b - x) \in A_k$. Thus

$$\mu(ax - a(b - x)) \geq \frac{k}{k+1} = \mu(x)$$

If $x \in \bigcap_{n=1}^{\infty} A_n$ then $ax - a(b - x) \in \bigcap_{n=1}^{\infty} A_n$ for all $a, b \in X$. Thus

$$\mu(ax - a(b - x)) = 1 = \mu(x).$$

Assume that $a \in A_r \setminus A_{r+1}$ for some $r = 1, 2, 3, \dots$, and

$b \in \bigcap_{n=1}^{\infty} A_n$ (or, $a \in \bigcap_{n=1}^{\infty} A_n$ and $b \in A_r \setminus A_{r+1}$ for some $r = 1, 2, 3, \dots$). Then $x \in A_r$ and so

$$\mu(ax - a(b - x)) \geq \frac{r}{r+1} = \mu(x)$$

(iii) Now, if $x, y \in A_k \setminus A_{k+1}$ for some $r = 1, 2, 3, \dots$, then $y \in A_r$ as A_r is an ideal of X . Thus

$$\mu(xy) \geq \frac{r}{r+1} = \mu(y).$$

If $x, y \in \bigcap_{n=1}^{\infty} A_n$ then $y \in \bigcap_{n=1}^{\infty} A_n$ and so

$$\mu(xy) = 1 = \mu(y).$$

Hence, μ is a fuzzy ideal of X . ■

Let $\mu : X \rightarrow [0, 1]$ be a fuzzy subset of X . The smallest fuzzy ideal containing μ is called the fuzzy ideal generated by μ , and μ is said to be n -valued if $\mu(X)$ is a finite set of n elements. When no specific n is intended, we call μ a finite-valued fuzzy subset.

Theorem 4.4: A fuzzy ideal ν of X is finite valued if and only if a finite-valued fuzzy subset μ of X is generated by ν .

Proof: If $\nu : X \rightarrow [0, 1]$ is a finite-valued fuzzy ideal of X , then one may choose $\mu = \nu$. Consequently, assume that $\mu : X \rightarrow [0, 1]$ is a n -valued fuzzy subset with n distinct values t_1, t_2, \dots, t_n , where $t_1 > t_2 > \dots > t_n$. Let G^i be the inverse image of t_i under μ , that is, $G^i = \mu^{-1}(t_i)$. Obviously, $\bigcup_{i=1}^j G^i \subseteq \bigcup_{i=1}^r G^i$ when $j < r$. We denote by A^j the ideal of X generated by the set $\bigcup_{i=1}^j G^i$. Then we have the following chain of ideals:

$$A^1 \subseteq A^2 \subseteq \dots \subseteq A^n = X.$$

Define a fuzzy $\nu : X \rightarrow [0, 1]$ by

$$\nu(x) = \begin{cases} t_n & \text{if } x \in A^n, \\ t_j & \text{if } x \in A^j \setminus A^{j-1}; j = 1, 2, \dots, n-1. \end{cases}$$

We claim that ν is a fuzzy ideal of X and μ is generated by ν . Let $x, y \in X$ and let i and j be the smallest integer such that $x \in A^i$ and $y \in A^j$. We may assume that $i > j$ without loss of generality. Then $x - y \in A^i$ and $xy \in A^i$ and so

$$\nu(x - y) \geq t_j = \min\{t_i, t_j\} = \min\{\nu(x), \nu(y)\}$$

and

$$\nu(xy) \geq t_j = \nu(y).$$

Now, let $a, b \in X$. If $x \in A^j$ for some $i < j$, then $x \in A^i$ as A^i is an ideal of X . Thus

$$\nu(ax - a(b - x)) \geq t_j = \nu(x).$$

Hence, μ is a fuzzy ideal of X .

If $x \in X$ and $\mu(x) = t_j$, then $x \in G^j$ and so $x \in A^j$. But we get $\nu(x) \geq t_j = \mu(x)$. Consequently, $\mu \subseteq \nu$. Let γ be any fuzzy ideal of X which is a subset of μ . Then, $\bigcup_{i=1}^j G^i = U(\mu; t_j) \subseteq U(\gamma; t_j)$, and thus $A^j \subseteq U(\gamma; t_j)$. Hence, $\gamma \subseteq \mu$ and μ is generated by ν . Note that $|Im\mu| = n = |Im\nu|$. This completes the proof. ■

A near-subtraction semigroup X is said to be *Noetherian* (see [9]) if it satisfies the ascending chain condition on ideals of X .

Theorem 4.5: If X is a Noetherian near-subtraction semigroup, then every fuzzy ideal of X is finite valued.

Proof: Let $\mu : X \rightarrow [0, 1]$ be a fuzzy ideal of X which is not finite valued. Then, there exists a sequence of distinct numbers $\mu(0) = t_1 > t_2 > \dots > t_n$, where $t_i = \mu(x_i)$ for some $x_i \in X$. This sequence induces an infinite sequence of distinct ideals of X :

$$U(\mu; t_1) \subset U(\mu; t_2) \subset \dots \subset U(\mu; t_n) \subset \dots$$

This is a contradiction. ■

Combining Theorem 4.4 and Theorem 4.5, we have the following corollary.

Corollary 4.6: If X is a Noetherian near-subtraction semigroup, then every fuzzy ideal of X is generated by a finite fuzzy subset in X .

V. NORMAL FUZZY IDEALS

Definition 5.1: A fuzzy ideal μ of X is said to be *normal* if there exists $a \in X$ such that $\mu(a) = 1$.

We note that if μ is a normal fuzzy ideal μ of X is normal if and only if $\mu(1) = 1$. Let $\mathbb{F}_N(X)$ denote the set of all normal fuzzy ideal of X .

Theorem 5.2: Let μ be a fuzzy ideal of X and let μ^+ be a fuzzy set in X given by $\mu^+(x) = \mu(x) + 1 - \mu(1)$, for all $x \in X$. Then $\mu^+ \in \mathbb{F}_N(X)$ and $\mu \subseteq \mu^+$.

Proof: For any $x, y, z \in X$ we have $\mu^+(1) = \mu(1) + 1 - \mu(1) = 1 \geq \mu^+(x)$ and

(i) For all $x, y \in X$, we have

$$\begin{aligned} \mu^+(x - y) &= \mu(x - y) + 1 - \mu(1) \\ &\geq \min\{\mu(x), \mu(y)\} + 1 - \mu(1) \\ &= \min\{\mu(x) + 1 - \mu(1), \mu(y) + 1 - \mu(1)\} \\ &= \min\{\mu^+(x), \mu^+(y)\}. \end{aligned}$$

(ii) For all $x, a, b \in X$, we have

$$\begin{aligned} \mu^+(ax - a(b - x)) &= \mu(ax - b(x - a)) + 1 - \mu(1) \\ &\geq \mu(x) + 1 - \mu(1) \\ &= \mu^+(x). \end{aligned}$$

(iii) For all $x, y \in X$, we have

$$\begin{aligned} \mu^+(xy) &= \mu(xy) + 1 - \mu(1) \\ &\geq \mu(y) + 1 - \mu(1) \\ &= \mu^+(y). \end{aligned}$$

Hence $\mu^+ \in \mathbb{F}_N(X)$. Obviously, $\mu \subseteq \mu^+$. ■

Corollary 5.3: If μ be a fuzzy ideal of X satisfying $\mu^+(a) = 0$ for some $a \in X$, then $\mu(a) = 0$.

It is clear that fuzzy ideal μ of X is normal if and only if $\mu^+ = \mu$, and for any fuzzy ideal μ of X we have $(\mu^+)^+ = \mu^+$. Hence if μ is a normal fuzzy ideal of X , then $(\mu^+)^+ = \mu$.

Theorem 5.4: Let μ be a fuzzy ideal of X and let $\phi : [0, \mu(0)] \rightarrow [0, 1]$ be an increasing function. Let μ_ϕ be a fuzzy set in X defined by $\mu_\phi(x) = \phi(\mu(x))$ for all $x \in X$. Then μ_ϕ is a fuzzy ideal of X . Moreover, if $\phi(\mu(0)) = 1$ then $\mu_\phi \in \mathbb{F}_N(X)$, and if $\phi(t) \geq t$ for all $t \in [0, 1]$ then $\mu \subseteq \mu_\phi$.

Proof: (i) Let $x, y \in X$. Then

$$\begin{aligned} \mu_\phi(x - y) &= \phi(\mu(x - y)) \\ &\geq \phi(\min\{\mu(x), \mu(y)\}) \\ &= \min\{\phi(\mu(x)), \phi(\mu(y))\} \\ &= \min\{\mu_\phi(x), \mu_\phi(y)\}. \end{aligned}$$

(ii) Let $a, b, x \in X$. Then

$$\begin{aligned}\mu_\phi(ax - a(b - x)) &= \phi(\mu(ax - a(b - x))) \\ &\geq \phi(\mu(x)) \\ &= \mu_\phi(x).\end{aligned}$$

(iii) Let $x, y \in X$. Then

$$\begin{aligned}\mu_\phi(xy) &= \phi(\mu(xy)) \\ &\geq \phi(\mu(y)) \\ &= \mu_\phi(y).\end{aligned}$$

Hence μ_ϕ is a fuzzy ideal of X . If $\phi(\mu(0)) = 1$ then obviously μ_ϕ is normal, and so $\mu_\phi \in \mathbb{F}_N(X)$. Assume that $\phi(t) \geq t$ for all $t \in [0, \mu(0)]$. Then $\mu_\phi(x) = \phi(\mu(x)) \geq \mu(x)$ for all $x \in X$, which proves that $\mu \subseteq \mu_\phi$. ■

Theorem 5.5: Let $\mu \in \mathbb{F}_N(X)$ be a non-constant maximal element of the poset $(\mathbb{F}_N(X), \subseteq)$. Then μ takes only the values 0 and 1.

Proof: Since μ is normal, we have $\mu(0) = 1$. Let $\mu(x) \neq 1$ for some $x \in X$. We claim that $\mu(x) = 0$. If not, then there exists $x_0 \in X$ such that $0 < \mu(x_0) < 1$. Define on X a fuzzy set ν putting $\nu(x) = \frac{\mu(x) + \mu(x_0)}{2}$ for all $x \in X$. Then, clearly ν is well-defined.

(i) For all $x, y \in X$, we have

$$\begin{aligned}\nu(x - y) &= \frac{\mu(x - y) + \mu(x_0)}{2} \\ &\geq \frac{\min\{\mu(x), \mu(y)\} + \mu(x_0)}{2} \\ &= \frac{\min\{\mu(x) + \mu(x_0), \mu(y) + \mu(x_0)\}}{2} \\ &= \min\left\{\frac{\mu(x) + \mu(x_0)}{2}, \frac{\mu(y) + \mu(x_0)}{2}\right\} \\ &= \min\{\nu(x), \nu(y)\}.\end{aligned}$$

(ii) For all $a, b, x \in X$, we have

$$\begin{aligned}\nu(ax - a(b - x)) &= \frac{\mu(ax - a(b - x)) + \mu(x_0)}{2} \\ &\geq \frac{\mu(x) + \mu(x_0)}{2} \\ &= \nu(x).\end{aligned}$$

(iii) For all $x, y \in X$, we have

$$\begin{aligned}\nu(xy) &= \frac{\mu(xy) + \mu(x_0)}{2} \\ &\geq \frac{\mu(y) + \mu(x_0)}{2} \\ &= \nu(y).\end{aligned}$$

Thus ν is a fuzzy ideal of X . By Theorem 5.2, ν^+ is a maximal fuzzy ideal of X . Note that

$$\begin{aligned}\nu^+(x_0) &= \nu(x_0) + 1 - \nu(0) \\ &= \frac{\mu(x_0) + \mu(x_0)}{2} + 1 - \frac{\mu(0) + \mu(x_0)}{2} \\ &= \frac{\mu(x_0) + 1}{2}.\end{aligned}$$

and $\nu^+(x_0) < 1 = \frac{\mu(0) + 1}{2} = \nu^+(0)$. Hence ν^+ is non-constant, and μ is not a maximal element of $\mathbb{F}_N(X)$. This is a contradiction. ■

Definition 5.6: A fuzzy ideal μ of X is said to be *maximal* if it satisfies:

(M1) μ is non-constant, and

(M2) μ^+ is a maximal element of $(\mathbb{F}_N(X), \subseteq)$.

Theorem 5.7: If a fuzzy ideal of X is maximal, then

(i) μ is normal,

(ii) μ takes only the values 0 and 1,

(iii) $\chi_{\mu^0} = \mu$, where $\mu^0 = \{x \in X \mid \mu(0) = 1\}$,

(iv) μ^0 is a maximal ideal of X .

Proof: Let μ be a maximal fuzzy ideal of X . Then μ^+ is a non-constant maximal element of the poset $(\mathbb{F}_N(X), \subseteq)$. It follows from Theorem 5.5 that μ^+ takes only two values 0 and 1. Note that $\mu^+(x) = 1$ if and only if $\mu(x) = \mu(0)$, and $\mu^+(0) = 0$ if and only if $\mu(x) = \mu(0) - 1$. By corollary 5.3, we have $\mu(x) = 0$ and so $\mu(0) = 1$. Hence μ is normal and $\mu^+ = \mu$. This proves (i) and (ii).

(iii) Obvious.

(iv) It is clear that μ^0 is a proper ideal of X . Obviously $\mu^0 \neq X$ because μ takes two values. Let A be an ideal containing μ^0 . Then $\mu_{\mu^0} \subseteq \mu_A$, and consequently, $\mu = \mu_{\mu^0} \subseteq \mu_A$. Since μ is normal, μ_A also is normal and takes only two values 0 and 1. But, by the assumption, μ is maximal, so $\mu = \mu_A$ or $\mu = \phi$, where $\phi(x) = 1$ for all $x \in X$. In the last case $\mu^0 = X$, which is impossible. So, $\mu = \mu_A$ i.e. $\mu_A = \chi_A$. Hence $\mu^0 = A$. ■

Definition 5.8: A fuzzy ideal μ of X is said to be *complete* if it is normal and there exists $z \in X$ such that $\mu(z) = 0$.

Theorem 5.9: Let μ be a fuzzy ideal of X and let w be a fixed element of X such that $\mu(1) = \mu(w)$. Define a fuzzy set μ^* in X by $\mu^*(x) = \frac{\mu(x) - \mu(w)}{\mu(1) - \mu(w)}$ for all $x \in X$. Then μ^* is a complete fuzzy ideal of X .

Proof: (i) For any $x, y \in X$, we have

$$\begin{aligned}\mu^*(x - y) &= \frac{\mu(x - y) - \mu(w)}{\mu(1) - \mu(w)} \\ &\geq \frac{\min\{\mu(x), \mu(y)\} - \mu(w)}{\mu(1) - \mu(w)} \\ &= \min\left\{\frac{\mu(x) - \mu(w)}{\mu(1) - \mu(w)}, \frac{\mu(y) - \mu(w)}{\mu(1) - \mu(w)}\right\} \\ &= \min\{\mu^*(x), \mu^*(y)\}.\end{aligned}$$

(ii) For any $x, y \in X$, we have

$$\begin{aligned}\mu^*(ax - a(b - x)) &= \frac{\mu(ax - a(b - x)) - \mu(w)}{\mu(1) - \mu(w)} \\ &\geq \frac{\mu(x) - \mu(w)}{\mu(1) - \mu(w)} \\ &= \mu^*(x).\end{aligned}$$

(iii) For any $x, y \in X$, we have

$$\begin{aligned}\mu^*(xy) &= \frac{\mu(xy) - \mu(w)}{\mu(1) - \mu(w)} \\ &\geq \frac{\mu(y) - \mu(w)}{\mu(1) - \mu(w)} \\ &= \mu^*(y).\end{aligned}$$

Hence $\mu^* \in \mathbb{F}_N(S)$. Since $\mu^*(w) = 0$, thus μ^* is a complete fuzzy ideal of X . ■

Theorem 5.10: Every maximal fuzzy ideal of X is completely normal.

Proof: Let μ be a maximal fuzzy ideal of X . Then by Theorem 5.7, μ is a normal and $\mu = \mu^+$ takes only two values 0 and 1. Since μ is non-constant, it follows that $\mu(0) = 1$ and $\mu(x) = 0$ for some $x \in X$. Hence μ is completely normal. ■

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