Free Flapping Vibration of Rotating Inclined Euler Beams

Chih-Ling Huang, Wen-Yi Lin, and Kuo-Mo Hsiao

Abstract—A method based on the power series solution is proposed to solve the natural frequency of flapping vibration for the rotating inclined Euler beam with constant angular velocity. The vibration of the rotating beam is measured from the position of the corresponding steady state axial deformation. In this paper the governing equations for linear vibration of a rotating Euler beam are derived by the d'Alembert principle, the virtual work principle and the consistent linearization of the fully geometrically nonlinear beam theory in a rotating coordinate system. The governing equation for flapping vibration of the rotating inclined Euler beam is linear ordinary differential equation with variable coefficients and is solved by a power series with four independent coefficients. Substituting the power series solution into the corresponding boundary conditions at two end nodes of the rotating beam, a set of homogeneous equations can be obtained. The natural frequencies may be determined by solving the homogeneous equations using the bisection method. Numerical examples are studied to investigate the effect of inclination angle on the natural frequency of flapping vibration for rotating inclined Euler beams with different angular velocity and slenderness ratio.

Keywords—Flapping vibration, Inclination angle, Natural frequency, Rotating beam.

I. INTRODUCTION

ROTATING beams are often used as a simple model for propellers, turbine blades, and satellite booms. Rotating beam differs from a non-rotating beam in having additional centrifugal force and Coriolis effects on its dynamics. The free vibration frequencies of rotating beams have been extensively studied [1-12]. However, the vibration analysis of rotating inclined beam is rather rare in the literature [8, 11, 12]. To the authors' knowledge, the natural frequency for very slender rotating inclined beam at high angular velocity is not reported in the literature. The objective of this paper is to derive the correct governing equations for linear flapping vibration of a rotating inclined Euler beam, and investigate the effects of inclination angle and slenderness ratio on the natural frequency of rotating Euler beams. The equations of motion for rotating

C. L. Huang is with Department of Mechanical Engineering, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu, 300, Taiwan (phone: 886-930816808; e-mail: clhuang@aaroc.edu.tw).

W. Y. Lin is with Department of Mechanical Engineering, De Lin Institute of Technology, 1 Alley 380, Ching Yun Road, Tucheng, 236, Taiwan (e-mail: wylin@dlit.edu.tw).

K. M. Hsiao is with Department of Mechanical Engineering, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu, 300, Taiwan (e-mail: kmhsiao@mail.nctu.edu.tw).

Euler beam are derived by the d'Alembert principle and the virtual work principle. In order to capture all inertia effect and coupling between extensional and flexural deformation, the consistent linearization [13, 14] of the fully geometrically non-linear beam theory [14, 15] is used in the derivation. A method based on the power series solution is proposed to solve the natural frequency. Numerical examples are studied to investigate the effect of inclination angle and slenderness ratio on the natural frequency of flapping vibration for rotating inclined Euler beams with different angular velocity.

II. FORMULATION

A. Description of Problem

Consider an inclined uniform Euler beam of length L rigidly mounted with an inclination angle α on the periphery of rigid hub with radius R rotating about its axis fixed in space at a constant angular velocity Ω as shown in Fig. 1. The deformation displacements of the beam are defined in a rotating rectangular Cartesian coordinate system which is rigidly tied to the hub. The origin of this coordinate system is chosen to be the intersection of the periphery of the hub and the centroid axis of the undeformed beam. The X_1 axis is chosen to coincide with the centroid axis of the undeformed beam, and the X_2 and X_3 axes are chosen to be the principal directions of the beam cross section at the undeformed state.

The direction of the X_3 axis is coincident with the axis of the rotating hub. Thus, the angular velocity of the hub may be given by

$$\Omega = \{0 \quad 0 \quad \Omega\} \tag{1}$$

where the symbol { } denotes a column matrix, which is used through the paper.

Here it is assumed that the beam is only deformed in the $X_1 - X_2$ plane. Thus only axial and flapping vibrations are considered. Note that the axial and flapping vibrations are not coupled and can be analyzed independently. It is well known that the beam sustains a steady state deformations (time-independent displacement) induced by constant rotation [16]. In this study, the vibration (time-dependent displacement) of the beam is measured from the position of the steady state axial deformation, and only infinitesimal free vibration is

considered. Here the engineering strain and stress are used for the measure of the strain and stress. It is assumed that the strains are small and the stress-strain relationships are linear.
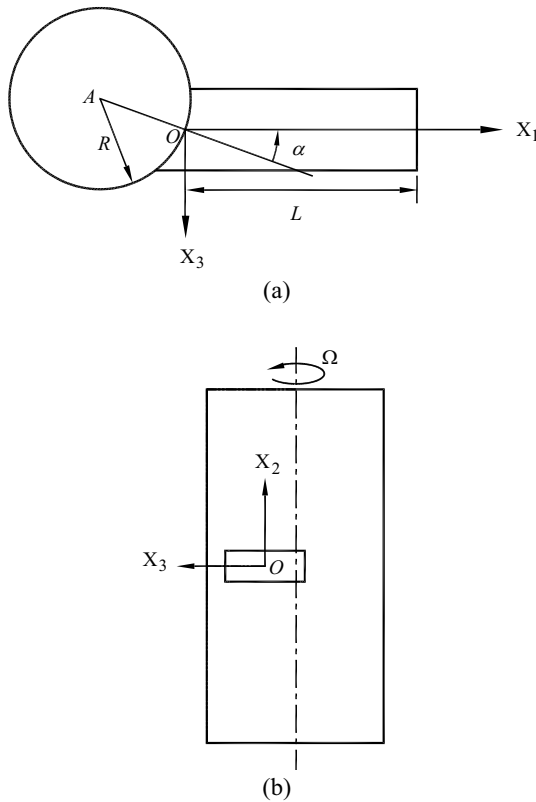

(a)

(b)

Fig. 1 A rotating inclined beam (a) Top view, (b) Side view

B. Kinematics of Euler Beam

Let P (see Fig. 2) be an arbitrary point in the rotating beam, and Q be the point corresponding to the beam cross-section of P on the centroid axis. The position vector of point P in the undeformed and deformed configurations may be expressed as

$$\mathbf{r}_0 = \{R + x \quad y \quad z\} \tag{2}$$

$$\mathbf{r} = \{x + \bar{u}(x,t) - y\sin\theta \quad v(x,t) + y\cos\theta \quad z\} = r_i \mathbf{e}_i \tag{3}$$

$$\bar{u}(x,t) = u_s(x) + u(x,t) \tag{4}$$

where t is time, $u_s(x)$ is the steady-state axial deformations induced by constant rotation, $u(x,t)$ and $v(x,t)$ are the infinitesimal displacements of point Q in the X_1 and X_2 directions, respectively, caused by the free vibration, $\theta = \theta(x,t)$ is the infinitesimal angle of rotation of the cross section passing through point Q about the X_3 axis, caused by the free vibration, $\mathbf{e}_i$ ($i = 1, 2, 3$) are unit vectors in the X_i directions.
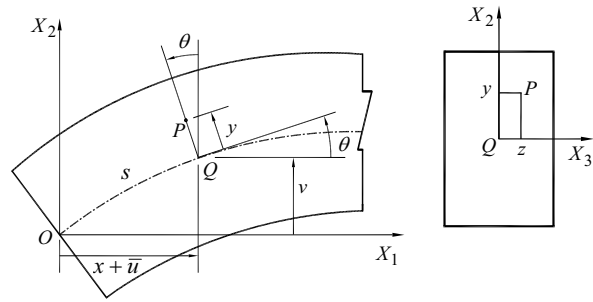

Fig. 2 Kinematics of deformed beam

From (3) and the definition of engineering strain, making use of the assumption of small strain, and using the approximation $\sin\theta \approx \theta \approx v_{,x}$ and $\cos\theta \approx 1$, the engineering strain in the Euler beam may be approximated by

$$\varepsilon = \bar{u}_{,x} + \frac{1}{2}v_{,x}^2 - yv_{,xx} \tag{5}$$

C. Equations of Motion

The equations of motion for rotating inclined Euler beam are derived by the d'Alembert principle, the virtual work principle and the consistent first order linearization of the fully geometrically non-linear beam theory [14]. Fig. 3 shows a portion of the deformed centerline of the beam. Here the generalized displacements are chosen to be $\bar{u}$, v, and θ defined in (3). The corresponding generalized forces are F_1, F_2, and M, the forces in X_1, X_2 directions, and moment about X_3 axis. F_{1j}, F_{2j}, and M_j ($j = a, b$) in Fig. 3 denote the values of F_1, F_2, and M at section j.
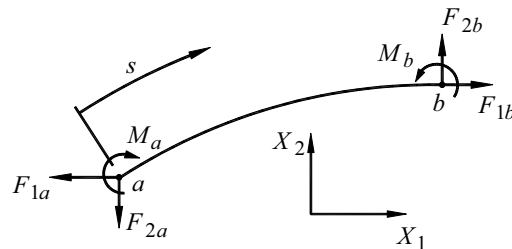

Fig. 3 Free body of a portion of deformed beam

For linear elastic material, the virtual work principle may be written as

$$\delta W_{ext} = \delta W_{int} \tag{6}$$

$$\delta W_{ext} = (F_1 \delta \bar{u} + F_2 \delta v + M \delta \theta)\Big|_a^b$$

$$= \int_a^b \frac{d}{dx}(F_1 \delta \bar{u} + F_2 \delta v + M \delta \theta)\, dx \tag{7}$$

$$\delta W_{int} = E\int_{V_{ab}} \delta\varepsilon^t \varepsilon\, dV + \rho\int_{V_{ab}} \ddot{\mathbf{r}}^t \delta\mathbf{r}\, dV \qquad (8)$$

where δW_{ext} and δW_{int} are the virtual work of the external forces and the internal stresses, respectively, $(\ )\big|_a^b$ is the value of $(\ )$ at section b minus the value of $(\ )$ at section a, $\delta\bar{u}$, δv and $\delta\theta$ are the virtual displacements, $\delta\varepsilon$ is the variation of ε given in (5), E is Young's modulus, V_{ab} is the volume of the undeformed beam between section a and section b. The differential volume dV may be expressed as $dV = dA\,dx$, where dA is the differential cross section area of the beam, ρ is the density, $\delta\mathbf{r}$ is the variation of $\mathbf{r}$ given in (3), and $\ddot{\mathbf{r}} = d^2\mathbf{r}/dt^2$ is the absolute acceleration. The symbol $(\dot{\ })$ denotes differentiation with respect to time t.

The exact expression of δW_{int} may be very complicated. However, due to the assumption of infinitesimal vibration, the quantities u, v, and θ defined in (3) and (4), and their derivatives with respect to x and t are all infinitesimal quantities. For linear vibration analysis only the terms up to the first order of infinitesimal quantities are required. All terms up to the first order of infinitesimal quantities in δW_{int} are retained. Note that the steady state axial deformations $u_s(x)$ in (4) and its derivatives with respect to x are small finite quantities, not infinitesimal quantities, and are all retained as zeroth order terms of infinitesimal quantities.

From (3) and (5), using the approximation $\theta \approx v_{,x}$, and retaining all terms up to the first order of infinitesimal quantities, $\delta\mathbf{r}$ and $\delta\varepsilon$ may be approximated by

$$\delta\mathbf{r} = \{\delta\bar{u} - y\delta v_{,x} \quad \delta v - y\delta v_{,x} \quad 0\} \qquad (9)$$

$$\delta\varepsilon = \delta\bar{u}_{,x} + v_{,x}\delta v_{,x} - y\delta v_{,xx} \qquad (10)$$

The second time derivative of $\mathbf{r}$ in (3) may be expressed as

$$\ddot{\mathbf{r}} = \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}_{AO}) + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}) + 2\boldsymbol{\Omega}\times(\dot{r}_i\mathbf{e}_i) + \ddot{r}_i\mathbf{e}_i \qquad (11)$$

$$\mathbf{r}_{AO} = \{R\cos\alpha \quad -R\sin\alpha \quad 0\} \qquad (12)$$

where $i = 1, 2, 3$ and $\boldsymbol{\Omega}$ is given in (1).

From (1), (3), (11) and (12), using the approximation $\theta \approx v_{,x}$, $\ddot{\mathbf{r}}$ in (11) may be approximated by

$$\ddot{\mathbf{r}} = \begin{Bmatrix} \ddot{u} - y\ddot{v}_{,x} - \Omega^2(R\cos\alpha + x + \bar{u} - yv_{,x}) \\ \ddot{v} \\ -2\Omega(\dot{u} - y\dot{v}_{,x}) - z\Omega^2 - R\Omega^2\sin\alpha \end{Bmatrix} \qquad (13)$$

Substituting (5), (7)-(10) and (13) into (6), using $\int_A y\,dA = 0$, and retaining all terms up to the first order of infinitesimal quantities, and then equating the terms in both sides of (6)

corresponding to the same generalized virtual displacements, one may obtain

$$F_{1,x} = \rho A[\ddot{u} - \Omega^2(R\cos\alpha + x + \bar{u})] \qquad (14)$$

$$F_{2,x} = \rho A\ddot{v} \qquad (15)$$

$$F_2 + M_{,x} = EA\bar{u}_{,x}v_{,x} + \rho I(\ddot{v}_{,x} - \Omega^2 v_{,x}) \qquad (16)$$

$$M = EIv_{,xx} \qquad (17)$$

$$F_1 = EA\bar{u}_{,x} \qquad (18)$$

where $I = \int_A y^2\, dA$ is the moment of inertia of the cross-section. (14)-(16) are equations of motion and (17) and (18) are constitutive equations.

Substituting (18) into (14), and substituting (15) and (17) into (16), one may obtain

$$EA\bar{u}_{,xx} = \rho A[\ddot{u} - \Omega^2(R\cos\alpha + x + \bar{u})] \qquad (19)$$

$$EIv_{,xxxx} = EA(\bar{u}_{,x}v_{,x})_{,x} + \rho I(\ddot{v}_{,xx} - \Omega^2 v_{,xx}) - \rho A\ddot{v} \qquad (20)$$

The boundary conditions for a rotating Euler beam with fixed end at $x = 0$ and free end at $x = L$ are given by

$$u_s(0) = u(0,t) = 0, \quad v(0,t) = 0, \quad v_{,x}(0,t) = 0 \qquad (21)$$

$$F_1(L,t) = 0, \quad M(L,t) = 0, \quad F_2(L,t) = 0$$

D. Steady-State Axial Deformation

For the steady-state axial deformations, $\bar{u}(x,t) = u_s(x)$, $u(x,t) = v(x,t) = 0$. Thus (19) and (21) can be reduced to

$$Eu_{s,xx} = -\rho\Omega^2(R\cos\alpha + x + u_s) \qquad (22)$$

$$u_s(0) = 0, \quad u_{s,x}(L) = 0 \qquad (23)$$

Let

$$k = \Omega L\sqrt{\rho/E} \qquad (24)$$

where k is a dimensionless angular velocity. If $k \ll 1$, the steady-state axial deformation $u_s(x)$, which satisfies (22) and (23), may be may be approximated by [9]

$$u_s(x) = \frac{k^2}{L^2}\left[\frac{-x^3}{6} - \frac{R\cos\alpha x^2}{2} + (RL\cos\alpha + \frac{L^2}{2})x\right] \qquad (25)$$

The maximum value of the steady state axial strain corresponding to the axial deformation given in (25) occurs at the root of the beam and may be expressed as

$$\varepsilon_{max} = u_{s,x}(0) = k^2(r\cos\alpha + 0.5) \qquad (26)$$

$$r = R/L \tag{27}$$

where R is the radius of the rotating hub.

E. Free Vibration

The vibration of the rotating beam is measured from the position of the steady-state axial deformation. From (4), (19), (20) and (22), the governing equations for free vibration may be expressed as

$$u_{,xx} + \frac{\rho\Omega^2}{E}u - \frac{\rho}{E}\ddot{u} = 0 \tag{28}$$

$$v_{,xxxx} = \frac{A}{I}(u_{s,x}v_{,x})_{,x} + \frac{\rho}{E}(\ddot{v}_{,xx} - \Omega^2 v_{,xx}) - \frac{\rho A}{EI}\ddot{v} \tag{29}$$

It can be seen from (28) and (29) that the axial vibration and the flapping vibration are not coupled and can be solved independently.

Let ω denote the natural frequency of rotating beam and

$$K = \omega L \sqrt{\rho/E} \tag{30}$$

denote a nondimensional natural frequency.

From (28) (30)and boundary conditions in (21), the natural frequency and vibration mode of rotating beam corresponding to the axial vibration may be expressed as

$$K = (a^2 - k^2)^{1/2} \tag{31}$$

$$u_R = \sin(ax/L) \tag{32}$$

where $a = (2n+1)\pi/2$, u_R is the axial vibration mode.

For convenience, the following nondimensional variables are used:

$$\xi = x/L - 0.5 , \; U_s = U_s(\xi) = u_s/L\frac{u_s}{L}, \; V = V(\xi) = v/L \tag{33}$$

$$\eta = \sqrt{AL^2/I} \tag{34}$$

where $0 \le x \le L$, η is the slenderness ratio of the beam.

From (29), (33) and (34), the dimensionless governing equations of free vibration may be expressed as

$$V_{,\xi\xi\xi\xi} - (\eta^2 U_{s,\xi} - k^2)V_{,\xi\xi} - \eta^2 U_{s,\xi\xi}V_{,\xi}$$
$$-\frac{\rho L^2}{E}\ddot{V}_{,\xi\xi} + \frac{\eta^2 \rho L^2}{E}\ddot{V} = 0 \tag{35}$$

$$U_{s,\xi} = -k^2[0.5\xi^2 + (r\cos\alpha + 0.5)\xi - 0.5r\cos\alpha - 0.375] \tag{36}$$

We shall seek a solution of (35) in the form

$$V(\xi,t) = V_R(\xi)e^{i\omega t} \tag{37}$$

where $i = \sqrt{-1}$, and ω is the natural frequency to be determined

Introducing (37) into (35), one may obtain

$$V_{R,\xi\xi\xi\xi} + (b\xi^2 + c\xi + d)V_{R,\xi\xi} + (2b\xi + c)V_{R,\xi} + eV_R = 0 \tag{38}$$

$$b = 0.5\eta^2 k^2 \tag{39}$$

$$c = \eta^2 k^2 (r\cos\alpha + 0.5)$$

$$d = K^2 + k^2 - \eta^2 k^2 (0.5r\cos\alpha + 0.375)$$

$$e = -\eta^2 K^2$$

F. Power Series Solution

The solution of (38) can be expressed as a power series in the independent variable ξ:

$$V_R(\xi) = \sum_{n=0}^{\infty} C_n \xi^n \tag{40}$$

where C_n are undetermined coefficients.

Substituting (40) into (38) and equating coefficients of like power of ξ, we obtain the recurrence formula

$$C_n = \sum_{j=1}^{4} A_n^j C_{n-j} , \; n \ge 4 \tag{41}$$

$$A_n^1 = 0 , \qquad A_n^2 = \frac{-d}{n(n-1)} \tag{42}$$

$$A_n^3 = \frac{-c(n-3)}{n(n-1)(n-2)}$$

$$A_n^4 = \frac{-[e + b(n-3)(n-4)]}{n(n-1)(n-2)(n-3)}$$

From (41), It can be seen that only C_0, C_1, C_2 and C_3 are independent constants in (40), and C_n ($n \ge 4$) can be rewritten as

$$C_n = \sum_{i=0}^{3} Y_i^n C_i , \; n \ge 4 \tag{43}$$

$$Y_i^n = \sum_{j=1}^{4} A_n^j Y_i^{n-j} , i = 0,1,2,3 , n \ge 4 \tag{44}$$

$$Y_i^j = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \ne j \end{cases} , \quad i, j = 0, 1, 2, 3 \tag{45}$$

Substituting (43) into (40), one may obtain

$$V_R(\xi) = \mathbf{E}^t(\xi)\mathbf{C} \tag{46}$$

$$\mathbf{E}(\xi) = \{E_0 \quad E_1 \quad E_2 \quad E_3\} \tag{47}$$

$$E_0 = 1 + \sum_{n=4}^{\infty} Y_0^n \xi^n \;,\quad E_1 = \xi + \sum_{n=4}^{\infty} Y_1^n \xi^n$$

$$E_2 = \xi^2 + \sum_{n=4}^{\infty} Y_2^n \xi^n \;,\quad E_3 = \xi^3 + \sum_{n=4}^{\infty} Y_3^n \xi^n$$

$$\mathbf{C} = \{C_0 \quad C_1 \quad C_2 \quad C_3\} \tag{48}$$

From boundary conditions given in (21), and (16), (17), (25), (36), (37) and (46), one may obtain a set of homogeneous equations expressed by

$$\mathbf{K}(K)\mathbf{C} = \mathbf{0} \tag{49}$$

$$\mathbf{K}(K) = \begin{bmatrix} \mathbf{E}^t(\xi_1) \\ \mathbf{E}'^t(\xi_1) \\ \dfrac{EI}{L^2}[(\eta^2 U_{s,\xi}(\xi_2) - K^2 - k^2)\mathbf{E}'^t(\xi_2) - \mathbf{E}'''^t(\xi_2)] \\ \dfrac{EI}{L}\mathbf{E}''^t(\xi_2) \end{bmatrix} \tag{50}$$

where $\mathbf{K}$ is a 4×4 matrix, $\mathbf{K}(K)$ denotes $\mathbf{K}$ is a function of K, $\mathbf{E}' = \mathbf{E}_{,\xi}$, $\mathbf{E}'' = \mathbf{E}_{,\xi\xi}$, $\mathbf{E}''' = \mathbf{E}_{,\xi\xi\xi}$, and $\xi_1 = -0.5$ and $\xi_2 = 0.5$ denote the values of the nondimensional coordinates ξ at two end nodes for the rotating beam.

For a nontrivial $\mathbf{C}$, the determinant of the matrix $\mathbf{K}$ must be equal to zero. The values of K which make the determinant vanish are called eigenvalues of matrix $\mathbf{K}$ and give the natural frequencies of the rotating Euler beam through K in Eq. (30). The bisection method is used here to find the eigenvalues.

III. Numerical Examples

To demonstrate the accuracy of the proposed method and to investigate the effect of inclination angle on the natural frequency of rotating inclined Euler beams with different angular velocity and slenderness ratio, several numerical examples are studied. Here the following cases are considered: $r = 0.5$, 1.0, $k = 0, 0.03, 0.06$, and $\alpha(°) = 0$, 15, 30, 45, 60, 75, 90.

From (26), one may obtain $\varepsilon_{\max} = 0.0054$, the maximum steady axial strain occurred at the root of the beam for all cases studied.

The axial and lateral vibration modes are not coupled here. For convenience, let K_i and K_i^a denote the ith dimensionless natural frequencies of lateral and axial vibration, respectively. From (31), it is noted that K_i^a are functions of the dimensionless angular velocity k only, and from (35) and (36), it is noted that K_i are functions of slenderness ratio η, the dimensionless angular velocity k, the dimensionless radius of

the rotating hub r, and the inclination angle α.

From (31), the first three dimensionless natural frequency of axial vibration corresponding to the dimensionless angular velocity $k = 0$, 0.03, 0.06 may be obtained and given as follows: $K_1^a = 1.57080$, 1.57051, 1.56965, $K_2^a = 4.71239$, 4.71229, 4.71201, and $K_3^a = 7.85398$, 7.85392, 7.85375. As expected, the value of K_i^a decreases slightly with the increase of k.

Tables 1-6 present dimensionless natural frequencies K_i ($i = 1$-5) for the rotating inclined beam with different slenderness ratio. It can be seen from Tables 1-6 that the values of K_i corresponding to the same η increases with increase of k and r, but decreases with increase of α. However, when $\alpha = 90°$, the values of K_i corresponding to the same η and k are identical for different r. These results may be explained by the centrifugal stiffening effect and (36). As can be seen from (36) that the centrifugal force increases with increase of k and r, but decreases with increase of α. However, $r\cos 90° = 0$ for all values of r.

It can be seen from Tables 1-3 and 4-6 that the centrifugal force, which is proportional to k^2, has stronger effect on the lower K_i. For $\eta = 50$, the effect of the centrifugal force on K_i ($i \geq 4$) is negligible. However, for $\eta = 500$, the effect of the centrifugal force on K_5 is still remarkable. It indicates that the effect of the centrifugal force on K_i increases with increase of η.

IV. Conclusions

In this paper, the correct governing equations for linear vibration of a rotating inclined Euler beam are derived. The vibration of the beam is measured from the position of the steady-state axial deformation, and only infinitesimal free vibration is considered. The equations of motion for rotating Euler beam are derived by the d'Alembert principle, the virtual work principle and the consistent linearization of the fully geometrically non-linear beam theory. The governing equation for linear flapping vibration of rotating beam is solved by a power series with four independent coefficients. Substituting the power series solution into the corresponding boundary conditions at two end nodes of the rotating beam, a set of homogeneous equations can be obtained. The natural frequencies may be determined by solving the homogeneous equations using the bisection method.

The results of numerical examples show that the effect of the centrifugal force on the natural frequencies corresponding to lateral vibration mode decreases with increase of the inclination angle, but increases with increase of the slenderness ratio for the inclined rotating Euler beam.

TABLE 1
DIMENSIONLESS FREQUENCIES FOR ROTATING BEAM
($r = 0.5$, $\eta = 50$)

k	α	K_1	K_2	K_3	K_4	K_5
0	0°	0.07026	0.43786	1.21530	2.35176	3.82644
0.03	0°	0.08191	0.44868	1.22613	2.36290	3.83770
	15°	0.08176	0.44854	1.22598	2.36274	3.83754
	30°	0.08133	0.44811	1.22553	2.36228	3.83707
	45°	0.08064	0.44742	1.22483	2.36154	3.83632
	60°	0.07973	0.44653	1.22391	2.36058	3.83534
	75°	0.07866	0.44550	1.22284	2.35947	3.83420
	90°	0.07749	0.44438	1.22168	2.35826	3.83298
0.06	0°	0.10946	0.47966	1.25799	2.39597	3.87125
	15°	0.10902	0.47911	1.25740	2.39535	3.87061
	30°	0.10773	0.47751	1.25568	2.39353	3.86874
	45°	0.10565	0.47495	1.25293	2.39063	3.86577
	60°	0.10288	0.47160	1.24934	2.38684	3.86190
	75°	0.09954	0.46765	1.24514	2.38242	3.85738
	90°	0.09583	0.46338	1.24062	2.37766	3.85252

TABLE 4
DIMENSIONLESS FREQUENCIES FOR ROTATING BEAM
($r = 1.0$, $\eta = 50$)

k	α	K_1	K_2	K_3	K_4	K_5
0	0°	0.07026	0.43786	1.21530	2.35176	3.82644
0.03	0°	0.08609	0.45294	1.23056	2.36753	3.84241
	15°	0.08582	0.45265	1.23025	2.36721	3.84209
	30°	0.08499	0.45180	1.22937	2.36629	3.84115
	45°	0.08367	0.45045	1.22796	2.36482	3.83965
	60°	0.08191	0.44868	1.22613	2.36290	3.83770
	75°	0.07981	0.44661	1.22399	2.36067	3.83542
	90°	0.07749	0.44438	1.22168	2.35826	3.83298
0.06	0°	0.12153	0.49536	1.27507	2.41411	3.88986
	15°	0.12075	0.49431	1.27391	2.41288	3.88860
	30°	0.11842	0.49121	1.27052	2.40927	3.88488
	45°	0.11462	0.48623	1.26510	2.40351	3.87897
	60°	0.10946	0.47966	1.25799	2.39597	3.87125
	75°	0.10312	0.47188	1.24965	2.38716	3.86223
	90°	0.09583	0.46338	1.24062	2.37766	3.85252

TABLE 2
DIMENSIONLESS FREQUENCIES FOR ROTATING BEAM
($r = 0.5$, $\eta = 100$)

k	α	K_1	K_2	K_3	K_4	K_5
0	0°	0.03515	0.21999	0.61460	1.20047	1.97618
0.03	0°	0.05477	0.24099	0.63621	1.22305	1.99933
	15°	0.05455	0.24072	0.63591	1.22274	1.99900
	30°	0.05391	0.23991	0.63504	1.22181	1.99804
	45°	0.05287	0.23863	0.63365	1.22033	1.99650
	60°	0.05148	0.23694	0.63183	1.21839	1.99450
	75°	0.04981	0.23497	0.62971	1.21614	1.99217
	90°	0.04796	0.23282	0.62742	1.21371	1.98966
0.06	0°	0.09036	0.29495	0.69649	1.28804	2.06696
	15°	0.08984	0.29407	0.69542	1.28684	2.06570
	30°	0.08830	0.29146	0.69228	1.28334	2.06199
	45°	0.08580	0.28725	0.68724	1.27774	2.05607
	60°	0.08241	0.28166	0.68060	1.27041	2.04833
	75°	0.07828	0.27500	0.67278	1.26179	2.03927
	90°	0.07358	0.26765	0.66425	1.25247	2.02949

TABLE 5
DIMENSIONLESS FREQUENCIES FOR ROTATING BEAM
($r = 1.0$, $\eta = 100$)

k	α	K_1	K_2	K_3	K_4	K_5
0	0°	0.03515	0.21999	0.61460	1.20047	1.97618
0.03	0°	0.06080	0.24888	0.64484	1.23231	2.00894
	15°	0.06041	0.24835	0.64426	1.23168	2.00829
	30°	0.05925	0.24679	0.64254	1.22984	2.00637
	45°	0.05735	0.24429	0.63980	1.22690	2.00332
	60°	0.05477	0.24099	0.63621	1.22305	1.99933
	75°	0.05160	0.23709	0.63199	1.21856	1.99467
	90°	0.04796	0.23282	0.62742	1.21371	1.98966
0.06	0°	0.10442	0.31976	0.72703	1.32243	2.10361
	15°	0.10352	0.31813	0.72500	1.32012	2.10114
	30°	0.10085	0.31332	0.71900	1.31333	2.09387
	45°	0.09644	0.30549	0.70933	1.30242	2.08224
	60°	0.09036	0.29495	0.69649	1.28804	2.06696
	75°	0.08271	0.28214	0.68117	1.27103	2.04899
	90°	0.07358	0.26765	0.66425	1.25247	2.02949

TABLE 3
DIMENSIONLESS FREQUENCIES FOR ROTATING BEAM
($r = 0.5$, $\eta = 500$)

k	α	K_1	K_2	K_3	K_4	K_5
0	0°	0.00703	0.04407	0.12338	0.24173	0.39954
0.03	0°	0.04141	0.10696	0.20338	0.33491	0.50166
	15°	0.04113	0.10637	0.20251	0.33382	0.50038
	30°	0.04031	0.10462	0.19996	0.33057	0.49661
	45°	0.03897	0.10178	0.19581	0.32532	0.49054
	60°	0.03714	0.09793	0.19023	0.31831	0.48248
	75°	0.03488	0.09323	0.18347	0.30988	0.47287
	90°	0.03228	0.08787	0.17586	0.30051	0.46227
0.06	0°	0.08087	0.19752	0.33807	0.51095	0.71403
	15°	0.08030	0.19629	0.33616	0.50834	0.71076
	30°	0.07863	0.19262	0.33047	0.50060	0.70105
	45°	0.07590	0.18663	0.32119	0.48797	0.68523
	60°	0.07217	0.17849	0.30857	0.47083	0.66386
	75°	0.06755	0.16847	0.29305	0.44980	0.63775
	90°	0.06219	0.15692	0.27518	0.42570	0.60804

TABLE 6
DIMENSIONLESS FREQUENCIES FOR ROTATING BEAM
($r = 1.0$, $\eta = 500$)

k	α	K_1	K_2	K_3	K_4	K_5
0	0°	0.00703	0.04407	0.12338	0.24173	0.39954
0.03	0°	0.04882	0.12287	0.22694	0.36528	0.53741
	15°	0.04835	0.12185	0.22543	0.36332	0.53507
	30°	0.04695	0.11883	0.22093	0.35747	0.52814
	45°	0.04463	0.11384	0.21353	0.34791	0.51686
	60°	0.04141	0.10696	0.20338	0.33491	0.50166
	75°	0.03730	0.09827	0.19072	0.31891	0.48317
	90°	0.03228	0.08787	0.17586	0.30051	0.46227
0.06	0°	0.09585	0.23062	0.38939	0.58102	0.80246
	15°	0.09491	0.22853	0.38614	0.57658	0.79684
	30°	0.09208	0.22227	0.37643	0.56331	0.78005
	45°	0.08740	0.21190	0.36037	0.54136	0.75232
	60°	0.08087	0.19752	0.33807	0.51095	0.71403
	75°	0.07249	0.17920	0.30967	0.47232	0.66572
	90°	0.06219	0.15692	0.27518	0.42570	0.60804

REFERENCES

[1] M. J. Schilhansl, "Bending frequency of a rotating cantilever beam," *ASME Journal of Applied Mechanics,* vol. 25, pp. 28-30, 1958.

[2] J. T. S. Wang, O. Mahrenholtz and J. Bohm, "Extended Galerkin's method for rotating beam vibrations using Legendre polynomials," *Solid Mechanics Archives*, vol. 1, pp. 341-365, 1976.

[3] A. Leissa, "Vibrational aspects of rotating turbomachinery blades," *ASME Applied Mechanics Reviews*, vol. 34, pp. 629-635, 1981.

[4] D. H. Hodges and M. J. Rutkowski, "Free-vibration analysis of rotating beams by a variable-order finite-element method," *AIAA Journal*, vol. 19, pp. 1459-1466, 1981.

[5] T. Yokoyama, "Free vibration characteristics of rotating Timoshenko beam," *International Journal of Mechanical Science*, vol. 30, pp. 743-755, 1988.

[6] H. H. Yoo and S. H. Shin, "Vibration analysis of rotating cantilever beams," *Journal of Sound and vibration*, vol. 212, pp. 807-828, 1988.

[7] S. Y. Lee and Y. H. Kuo, "Bending frequency of a rotating beam with an elastically restrained root," *ASME Journal of Applied Mechanics*, vol. 58, pp. 209-214, 1991.

[8] H. P. Lee, "Vibration on an inclined rotating cantilever beam with tip mass," *ASME Journal of Vibration and Acoustics*, vol. 115, pp. 241 – 245, 1993.

[9] S. C. Lin and K. M. Hsiao, "Vibration analysis of rotating Timoshenko beam," *Journal of Sound and Vibration*, vol. 240, pp. 303-322, 2001.

[10] A. A. Al-Qaisia, "Non-linear dynamics of a rotating beam clamped with an attachment angle and carrying an inertia element," *The Arabian Journal for Science and Engineering,* vol. 29, pp. 81-98, 2004.

[11] S. Y. Lee and J. J. Sheu, "Free vibrations of a rotating inclined beam," *ASME Journal of Applied Mechanics,* vol. 74, pp. 406-414, 2007.

[12] S. Y. Lee and J. J. Sheu, "Free vibration of an extensible rotating inclined Timoshenko beam," *Journal of Sound and Vibration*, vol. 304, pp. 606-624, 2007.

[13] J. C. Simo and K. Vu-Quac , "The role of non-linear theories in transient dynamic analysis of flexible structures," *Journal of Sound and Vibration*, vol. 119, pp. 487-508, 1987.

[14] K. M. Hsiao, "Corotational total Lagrangian formulation for three-dimensional beam element," *AIAA Journal*, vol. 30, pp. 797-804, 1992.

[15] K. M. Hsiao, R. T. Yang, and A. C. Lee, "A cosistent finite element formulation for nonlinear dynamic analysis of planar beam," *International Journal for Numerical Methods in Engineering*, vol. 37, pp. 75-89, 1994

[16] P. W. Likins, "Mathematical modeling of spinning elastic bodies for model analysis," *AIAA Journal*, vol. 11, pp. 1251-1258, 1973.