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Fixed Point Equations Related to Motion Integrals in Renormalization Hopf Algebra

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Abstract—In this paper we consider quantum motion integrals depended on the algebraic reconstruction of BPHZ method for perturbative renormalization in two different procedures. Then based on Bogoliubov character and Baker-Campbell-Hausdorff (BCH) formula, we show that how motion integral condition on components of Birkhoff factorization of a Feynman rules character on Connes-Kreimer Hopf algebra of rooted trees can determine a family of fixed point equations.

Keywords—Birkhoff Factorization, Connes-Kreimer Hopf Algebra of Rooted Trees, Integral Renormalization, Lax Pair Equation, Rota-Baxter Algebras.

I. Introduction

THE study of perturbative Quantum Field Theory (QFT) underlying the scheme minimal subtraction in dimensional regularization with respect to the Connes-Kreimer Hopf algebraic approach determines important physical information of a given theory for instance counterterms, renormalization group and $\beta-$function in a combinatorial algebro-geometric setting. In other words, firstly with attention to the Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) method in perturbative renormalization and with the help of a decorated version of the Connes-Kreimer Hopf algebra of rooted trees, a hidden Hopf algebra from the process of renormalization can be described [14], [15]. Secondly the Riemann-Hilbert correspondence introduces a new algebraic reinterpretation from physical theory such that it is formulated by the Birkhoff decomposition of loops with values in an infinite dimensional Lie group (induced with the Hopf algebra) [4], [5], [6], [9], [14], [16], [22]. And finally in a more general configuration, one can associate a neutral Tannakian category to each theory such that its objects (i.e. equisingular flat connections) enable to store a completely geometric description from counterterms. Connes and Marcolli developed this story until they introduced a universal treatment in the study of renormalizable QFTs. [7]

The value of this point of view to QFT will be cleared more, when we consider its application in the analyzing of gauge theories and non-perturbative QFT [17], [25]. Kreimer in [1] characterizes non-perturbative situations based on a new version of Dyson-Schwinger equations namely, combinatorial level. It is necessary to know that these equations can be studied with Hochschild one cocycles of the Hopf algebra of renormalization.

Relation between the perturbative renormalization and the Riemann-Hilbert correspondence is capsulated in the Birkhoff factorization and moreover the existence of this unique decomposition is strongly related with an algebraic condition (namely, Rota-Baxter property) of the couple regularization scheme and renormalization map. It shows that Rota-Baxter algebras play the role of a bridge between the study of ill-defined divergent Feynman integrals in QFT (by renormalization) and the extraction of finite values based on the Riemann-Hilbert problem in the study of a special class of differential systems. [9], [10], [11], [13], [16]

It is also interesting to know that this group of algebras determines (modified) classical Yang-Baxter equations ([9], [10], [11], [18]) such that their solutions apply to consider quantum integrable systems [8], [19], [20]. On the other hand, with working at the level of Lie bialgebras in [2], the authors provide the semisimplicity of the Lie algebra of infinitesimal characters and then they apply Connes-Kreimer Birkhoff decomposition to a Feynman rules character to identify its corresponding Lax pair equation.

Therefore it does make sense to say that Rota-Baxter type algebras enable to obvious an interesting notion for the study of integrable systems in QFTs underlying the Connes-Kreimer theory.

With attention to this foundation, we are going to apply the Rota-Baxter property of the scheme dimensional regularization in minimal subtraction to introduce its related integrals of motion with respect to two different strategies namely, Rosenberg's approach and noncommutative differential forms. After that based on Bogoliubov character and Baker-Campbell-Hausdorff (BCH) formula and with using integral renormalization theorems, we characterize a family of fixed point equations related to Feynman rules characters and connected with Nijenhuis type flows. Specially in this process a class of equations associated to the renormalization group flow will be determined.

II. From Classical Yang-Baxter Equations to the Concept of Flow in Connes-Kreimer Theory

For each renormalizable theory Φ, the reconstruction of its associated Hopf algebra of Feynman diagrams $H_F(\Phi)$ is available with a decorated version of the Connes-Kreimer Hopf algebra H_{rt} of rooted trees such that primitive (sub-)divergences of Feynman graphs are reserved in these labels. It is a free (as an algebra) connected graded commutative non-cocommutative Hopf algebra on the set of all non-planar rooted trees such that its coproduct structure is given by

$$\begin{aligned}\Delta B^{+}(t_1...t_n) &= t\otimes\mathbb{I}+(id\otimes B^{+})\Delta(t_1...t_n)\\ &= t\otimes\mathbb{I}+\mathbb{I}\otimes t+\sum_{c}P_c(t)\otimes R_c(t)\end{aligned}\qquad(1)$$

where $B^{+}:H_{rt}\longrightarrow H_{rt}$ is an isomorphism of graded vector spaces and it maps a forest to a rooted tree by connecting

the roots of rooted trees in the forest to a new root. And also the sum is over all possible non-trivial admissible cuts c on t. With induction, one can induce the antipode of this Hopf algebra given by

$$S(t) = -t - \sum_{c} S(P_c(t))R_c(t). \quad (2)$$

This Hopf algebra contains a connected graded commutative cocommutative Hopf subalgebra H_{lrt} of ladder trees. [4], [9], [10]

It is observed that Hopf algebra H_{rt} has universal property with respect to the Hochschid cohomology theory and therefore it is reasonable to apply this Hopf algebra, as kind of a simplified model, to consider perturbative renormalization. [4], [14], [16]

BPHZ method in renormalization can be reconstructed in a algebraic procedure such that regularization is described with the unital commutative algebra $A_{dr} = \mathbb{C}[[z, z^{-1}]$ of Laurent series with finite pole part and renormalization scheme R_{ms} is determined with the projection of a Laurent series onto its pole part. It means that

$$R_{ms}(\sum_{i\geq -m}^{\infty} c_i z^i) := \sum_{i\geq -m}^{-1} c_i z^i. \quad (3)$$

[7], [9], [13]

Definition II.1. For a given unital associative algebra A over a field $\mathbb{K}$ of characteristic zero together with a $\mathbb{K}-$linear map $R : A \longrightarrow A$, the pair (A, R) is called *Rota-Baxter algebra*, if for all $x, y \in A$, it satisfies

$$R(x)R(y) + R(xy) = R(R(x)y + xR(y)).$$

Remark II.2. For a given Rota-Baxter algebra (A, R),

(i) The map $\widetilde{R} := Id_A - R$ has the Rota-Baxter property,

(ii) Its related Lie algebra $(A, [.,.])$ (such that $[.,.]$ is the commutator with respect to the product of A) obeys from the equation

$$[R(x), R(y)] + R([x, y]) = R([R(x), y] + [x, R(y)]).$$

(iii) The pair (A_{dr}, R_{ms}) is an idempotent Rota-Baxter algebra.

The consideration of different Poisson Lie brackets on a given Lie algebra is an important part of the study of integrable systems such that in this process for instance one can find a closed relation between classical r-matrices and factorization in Lie bialgebras [20], [21]. There is also another procedure closely related to the Rota-Baxter theory namely, deformation of the initial associative product of a given algebra such that at the Lie algebra level, a general version of the (modified) classical Yang-Baxter equation can be investigated [10], [11], [18], [20]. This technique provides a new direction to proceed the study of integrable systems based on a particular group of deformed products such that they are determined by an algebraic formula. It can be seen that this condition has a mutual source with the existence of Birkhoff factorization and therefore one can apply this process to investigate integrable systems with respect to the Connes-Kreimer perturbative renormalization.

For a given associative algebra A and a linear map $N : A \longrightarrow A$, consider a new product on this algebra defined by

$$(x, y) \longmapsto x \circ_N y := N(x)y + xN(y) - N(xy). \quad (4)$$

The associativity of this new product makes clear one important and interesting equation.

Theorem II.3. *The product (4) is associative iff for each $x, y \in A$,*

$$N(x \circ_N y) - N(x)N(y) = 0.$$

The pair (A, N) together with this condition is called Nijenhuis algebra. [3]

One essentially note is that a Nijenhuis algebra (A, N) provides the relation

$$[N(x), N(y)] = N([N(x), y]) + N([x, N(y)]) - N^2([x, y]) \quad (5)$$

on the Lie algebra $(A, [.,.])$ such that it applies to induce a new Lie bracket

$$[x, y]_N := [N(x), y] + [x, N(y)] - N([x, y]). \quad (6)$$

The compatibility of this bracket is derived from the given condition in II.3 and moreover it is observed that

$$[x, y]_N = x \circ_N y - y \circ_N x \quad (7)$$

[3], [10], [11].

An interesting family of Nijenhuis algebras is introduced with Rota-Baxter maps. For a given Rota-Baxter algebra (A, R) with the idempotent map R and each $\lambda \in \mathbb{K}$, one can show that the operator $R_\lambda := R - \lambda\widetilde{R}$ has Nijenhuis property. The pair (A_{dr}, R_{ms}) is the most important example for this class of Rota-Baxter algebras in physics.

Let $L(H_{rt}, A_{dr})$ be the set of all linear maps on H_{rt} with values in A_{dr}. The convolution product $*$ determines a complete filtered noncommutative associative unital Rota-Baxter algebra such that its idempotent Rota-Baxter map $\mathcal{R}$ is defined by

$$\mathcal{R} : L(H_{rt}, A_{dr}) \longrightarrow L(H_{rt}, A_{dr}), \quad \mathcal{R}(f) := R_{ms} \circ f \quad (8)$$

[9]. It is clear that for each $\lambda \in \mathbb{K}$, $\mathcal{R}_\lambda$ has Nijenhuis property and in addition theorem II.3 shows that each of these maps defines a new associative product $\circ_\lambda$ and a new compatible Lie bracket $[.,.]_\lambda$ on $L(H_{rt}, A_{dr})$. It means that one can have different deformations from the main algebra (based on $\mathcal{R}_\lambda$s) such that they will be applied to introduce new Poisson brackets.

Proposition II.4. *Consider the associative algebra $C^x_\lambda := (L(H_x, A_{dr}), \circ_\lambda)$ with the center $Z^x(\lambda)$ such that $x = lrt, rt$. There is a differential graded algebra $\Omega^\bullet_{Der}(C^x_\lambda)$ based on the space of all derivations of C^x_λ.*

Proof: With help of the results in [12], [23], [24], define $\Omega^\bullet_{Der}(C^x_\lambda) = \bigoplus_n \Omega^n_{Der}(C^x_\lambda)$ such that

- $\Omega^0_{Der}(C^x_\lambda) = C^x_\lambda$,

- $\Omega^n_{Der}(C^x_\lambda)$ is the set of all $Z^x(\lambda)-$multilinear antisymmetric maps from $Der(C^x_\lambda)^n$ into C^x_λ,

- For each $\omega \in \Omega^n_{Der}(C^x_\lambda)$ and $\theta_i \in Der(C^x_\lambda)$, the anti-derivation differential operator d_λ is defined by

$$(d_\lambda\omega)(\theta_0,...,\theta_n) := \sum_{k=0}^{n}(-1)^k\theta_k \circ_\lambda \omega(\theta_0,...,\widehat{\theta_k},...,\theta_n)$$
$$+ \sum_{0\leq r<s\leq n}(-1)^{r+s}\omega([\theta_r,\theta_s]_\lambda,\theta_0,...,\widehat{\theta_r},...,\widehat{\theta_s},...,\theta_n).$$

■

The Lie brackets $[.,.]_\lambda$ make available two procedures to study integrable systems at this level namely, identifying integral curves from a Lax pair equation or inducing motion integrals depended on symplectic structures.

First Approach. Consider the loop algebra of the semisimple trivial Lie bialgebra $\mathcal{C}(x,\lambda) = \mathcal{C}^x_\lambda \oplus \mathcal{C}^{x*}_\lambda$ such that $\mathcal{C}^x_\lambda := (C^x_\lambda, [.,.]_\lambda)$. One can show that $\mathcal{C}(x,\lambda)$ is the associated Lie algebra of the Lie group $\widetilde{\mathcal{C}}(x,\lambda) := \mathcal{C}^x_\lambda \rtimes_\sigma \mathcal{C}^{x*}_\lambda$ such that

$$\sigma : \mathcal{C}^x_\lambda \times \mathcal{C}^{x*}_\lambda \longrightarrow \mathcal{C}^{x*}_\lambda, \quad (f,X) \longmapsto Ad^*(f)(X). \tag{9}$$

[2]. The loop algebra of this Lie bialgebra is given by the set

$$L\mathcal{C}(x,\lambda) := \{F(c) = \sum_{j=-\infty}^{\infty} c^j F_j, F_j \in \mathcal{C}(x,\lambda)\} \tag{10}$$

and the Lie bracket

$$[\sum c^i F_i, \sum c^j G_j] := \sum_k c^k \sum_{i+j=k}[F_i,G_j]_\lambda. \tag{11}$$

Decompose this set of formal power series into two parts

$$L\mathcal{C}_+(x,\lambda) = \{\sum_{j=0}^{\infty} c^j F_j\}, \quad L\mathcal{C}_-(x,\lambda) = \{\sum_{j=-\infty}^{-1} c^j F_j\} \tag{12}$$

and let $P_\pm$ are the natural projections on these components where $P := P_+ - P_-$. It is proved that for a given Casimir function α on $L\mathcal{C}(x,\lambda)$, integral curve $G(t)$ of the Lax pair equation $\frac{dG}{dt} = [M,G]$ where $M = \frac{1}{2}P(I(d\alpha(F(c)))) \in L\mathcal{C}(x,\lambda)$ is determined by

$$G(t) = Ad^*_{L\widetilde{\mathcal{C}}(x,\lambda)}\gamma_\pm(t).G(0) \tag{13}$$

such that smooth curves $\gamma_\pm$ solve the Birkhoff factorization problem

$$exp(-tX) = \gamma_-^{-1}(t)\gamma_+(t) \tag{14}$$

where $X = I(d\alpha(F(c))) \in L\mathcal{C}(x,\lambda)$. It is important to know that one can project the above Lax pair equation to an equation on loop algebra of the original Lie algebra $\mathcal{C}^x_\lambda$. [2], [21]

Second Approach. The Lie bracket $[.,.]_\lambda$ is naturally a Poisson bracket such that for each element $f \in C^x_\lambda$, its associated *Hamiltonian vector field* is defined by

$$ham(f) : g \longmapsto [f,g]_\lambda. \tag{15}$$

This class of derivations can give us a symplectic structure. On the other hand, with restriction to the $Z^x(\lambda)-$module $Der_{Ham}(C^x_\lambda)$ (generated by all Hamiltonian derivations), one can induce a symplectic structure ω_λ in $\Omega^2_{Der_{Ham}}(C^x_\lambda)$ given by

$$\omega_\lambda(\theta_1,\theta_2) := \sum_{i,j} u^1_i \circ_\lambda u^2_j \circ_\lambda [f_i,g_j]_\lambda \tag{16}$$

such that $\theta_1 = \sum_i u^1_i \circ_\lambda ham(f_i), \quad \theta_2 = \sum_j u^2_j \circ_\lambda ham(g_j)$.

The Hamiltonian vector field θ^λ_f is the unique solution of the equation

$$i_{\theta^\lambda_f}\omega_\lambda = d_\lambda f. \tag{17}$$

and in fact it can be a description from the correspondence between Hamiltonian derivations (related to $\circ_\lambda$) and closed noncommutative deRham one forms on the algebra C^x_λ [12], [23], [24]. Moreover based on these Hamiltonian derivations, one can obtain a new Poisson bracket on C^x_λ (associated with the symplectic structure) such that for each $f,g \in C^x_\lambda$, it is defined by

$$\{f,g\}_\lambda := i_{\theta^\lambda_f}(d_\lambda g) = i_{\theta^\lambda_f}i_{\theta^\lambda_g}\omega_\lambda. \tag{18}$$

Generally Poisson brackets $[.,.]_\lambda$ are degenerate and it means that all derivations of C^x_λ are not Hamiltonian (i.e. $Der(C^x_\lambda) \neq Der_{Ham}(C^x_\lambda)$). It provides this fact that for each λ, the bracket $\{.,.\}_\lambda$ (defined by the symplectic structure ω_λ) may not coincide with $[.,.]_\lambda$.

We know that for each Hamiltonian derivation (vector field) θ^λ_g on the algebra C^x_λ, the one parameter group $\{exp(t\theta^\lambda_g)\}_t$ are integral curves generated by this infinitesimal automorphism ([12]) and therefore constant elements along these integral curves will identify the associated *integrals of motion*. It means that integral of motion f of θ^λ_g is determined by the equation

$$\{f,g\}_\lambda = 0. \tag{19}$$

In the next section, by applying integral renormalization, we will lift the equation (19) to the level of fixed point equations.

III. Integral Renormalization and Motion Integrals

Physical information of a given renormalizable QFT Φ are stored in Feynman diagrams equipped with related Feynman rules. Kreimer shows that one can find these rules in very specific characters of the Hopf algebra $H_F(\Phi)$. In better words, components of the Birkhoff factorization of a Feynman rules character are another characters such that they determine renormalized values, counterterms, renormalization group and $\beta-$function [4], [5], [7], [13], [22]. This fact shows that these Birkhoff components have the ability of saving physical meanings and it can be interested to find situations for these characters to play the role of integral of motion for the given Feynman rules character. Working on this problem clarifies more hidden physical nature in these characters and moreover it helps to study the compatibility of the condition (19) with the Connes-Kreimer renormalization group flow. In this section we consider this question underlying the context of integral renormalization and in this process some reformulations from the equation (19) with respect to Bogoliubov character and BCH series are obtained such that they apply to induce a family of fixed point equations related to integrals of motion.

The mathematical description of the BPHZ method in renormalization is designed basically by the Atkinson's theorem. It provides inductive formulaes (i.e. integral renormalization theorems) for components of the Birkhoff factorization of characters on rooted trees such that at this level one can find

the notion of a decomposition of determined Lie algebras with the Connes-Kreimer theory.

Let $char_{A_{dr}}\ H_x$ be the infinite dimensional Lie group of all characters with corresponding Lie algebra $\partial\ char_{A_{dr}}\ H_x$ (i.e. the set of all infinitesimal characters (or derivations)). This Lie algebra is generated by derivations Z^t indexed by ladder tree (rooted tree) t and defined by the natural paring $< Z^t, s >= \delta_{t,s}$. There is a natural bijection between $char_{A_{dr}}\ H_x$ and $\partial\ char_{A_{dr}}\ H_x$ such that for each character g, its corresponding derivation Z_g given by the exponential map

$$g = exp^*(Z_g) = \sum_{n=0}^{\infty} \frac{Z_g^{*n}}{n!}. \tag{20}$$

One can show that the Lie algebra $\partial\ char_{A_{dr}}\ H_x$ together with the map $\mathcal{R}$ define a Lie Rota-Baxter algebra such that each derivation Z of this Lie algebra has a unique decomposition $Z = \mathcal{R}(Z) + \widetilde{\mathcal{R}}(Z)$. On the other hand, idempotent property of R provides a decomposition $A_{dr} = A_{dr}^+ + A_{dr}^-$ such that it can be extended to $\partial\ char_{A_{dr}}\ H_x$ and it means that

$$\partial\ char_{A_{dr}}\ H_x = (\partial\ char_{A_{dr}}\ H_x)_+ + (\partial\ char_{A_{dr}}\ H_x)_- \tag{21}$$

where

$$(\partial\ char_{A_{dr}}\ H_x)_+ := \widetilde{\mathcal{R}}(\partial\ char_{A_{dr}}\ H_x),$$

$$(\partial\ char_{A_{dr}}\ H_x)_- := \mathcal{R}(\partial\ char_{A_{dr}}\ H_x) \tag{22}$$

[9], [10], [11].

For a fixed character $g \in C_\lambda^x$, let f be its integral of motion. Product $\circ_\lambda$ and equations (17), (18), (19) provide that

$$\{f, g\}_\lambda = i_{\theta_g^\lambda} i_{\theta_f^\lambda} \omega_\lambda = \omega_\lambda(\theta_f^\lambda, \theta_g^\lambda) = [f, g]_\lambda = 0. \tag{23}$$

Relation (6) and equation (23) show that

$$[\mathcal{R}_\lambda(f), g] + [f, \mathcal{R}_\lambda(g)] - \mathcal{R}_\lambda([f, g]) = 0 \iff$$
$$\mathcal{R}_\lambda(f) * g - g * \mathcal{R}_\lambda(f) + f * \mathcal{R}_\lambda(g)$$
$$-\mathcal{R}_\lambda(g) * f - \mathcal{R}_\lambda(f * g) + \mathcal{R}_\lambda(g * f) = 0. \tag{24}$$

Lemma III.1. *For a given character $g \in C_\lambda^x$ with the Birkhoff factorization (g_-, g_+),*

(i) $g = g_-^{-1} * g_+, \quad R_{ms} \circ g_- = g_-, \quad R_{ms} \circ g_+ = 0.$

(ii) The negative component g_- is an integral of motion for g iff

$$g_+ - g * g_- + (1+\lambda) g_- * \mathcal{R}(g) - (1+\lambda)\mathcal{R}(g) * g_-$$
$$+(1+\lambda)\mathcal{R}(g * g_-) = 0.$$

(iii) The positive component g_+ is an integral of motion for g iff

$$-\lambda g_+ * g + \lambda g * g_+ + (1+\lambda) g_+ * \mathcal{R}(g) - (1+\lambda)\mathcal{R}(g) * g_+$$
$$-(1+\lambda)\mathcal{R}(g_+ * g) + (1+\lambda)\mathcal{R}(g * g_+) = 0.$$

With the help of ladder tree version of the integral renormalization (based on the Atkinson's theorem and the abelianess of $\partial\ char_{A_{dr}}\ H_{lrt}$) given in [10], one can obtain a unique Birkhoff factorization for each character $\phi \in char_{A_{dr}}\ H_{lrt}$ given by

$$\phi = exp^*(\mathcal{R}(Z_\phi) + \widetilde{\mathcal{R}}(Z_\phi)) = exp^*(\mathcal{R}(Z_\phi)) * exp^*(\widetilde{\mathcal{R}}(Z_\phi)) \tag{25}$$

such that components of the decomposition are determined with

$$\phi_- = exp^*(-\mathcal{R}(Z_\phi)), \quad \phi_+ = exp^*(\widetilde{\mathcal{R}}(Z_\phi)). \tag{26}$$

Moreover in general, components of the character $\psi \in char_{A_{dr}}\ H_{rt}$ such that

$$\psi = exp^*(Z_\psi) = exp^*(\mathcal{R}(\chi(Z_\psi))) * exp^*(\widetilde{\mathcal{R}}(\chi(Z_\psi))) \tag{27}$$

(where infinitesimal character χ is characterized with the BCH series) are given by

$$\psi_- = exp^*(-\mathcal{R}(\chi(Z_\psi))), \quad \psi_+ = exp^*(\widetilde{\mathcal{R}}(\chi(Z_\psi))). \tag{28}$$

Therefore by applying (26) and (28) in the lemma III.1, one can receive equations at the level of Lie algebra for while these components are motion integrals of a given character in the algebra C_λ^x.

In addition, there is another representation from components based on the double Rota-Baxter structures such that it can be applied to characterize a class of fixed point equations related to the condition (19). With help of the Rota-Baxter map $\mathcal{R}$, one can deform the convolution product $*$ to obtain a well known associative product on the set $L(H_x, A_{dr})$ given by

$$f *_{\mathcal{R}} g := f * \mathcal{R}(g) + \mathcal{R}(f) * g - f * g. \tag{29}$$

It is easy to show that $C_{\mathcal{R}}^x := (L(H_x, A_{dr}), *_{\mathcal{R}}, \mathcal{R})$ is a Rota-Baxter algebra with the corresponding $\mathcal{R}-$bracket

$$[f, g]_{\mathcal{R}} = [f, \mathcal{R}(g)] + [\mathcal{R}(f), g] - [f, g]. \tag{30}$$

For each infinitesimal character Z, it can be seen that

$$exp^*(\mathcal{R}(Z)) = \mathcal{R}(exp^{*\mathcal{R}}(Z)),\ exp^*(\widetilde{\mathcal{R}}(Z)) = -\widetilde{\mathcal{R}}(exp^{*\mathcal{R}}(-Z)). \tag{31}$$

From equations (23), (29) and (30), one can prove that

Lemma III.2. *For a given character $g \in C_{\mathcal{R}}^x$ with the Birkhoff factorization (g_-, g_+), the components are integrals of motion for g iff*

(i) $[g_-, \mathcal{R}(g)] = 0$,

(ii) $[g_+, \mathcal{R}(g)] - [g_+, g] = 0$, *respectively.*

Now from (28), (31) and lemma III.2, the following equations at the level of infinitesimal characters are introduced.

$$(\mathbf{i})\ \mathcal{R}(exp^{*\mathcal{R}}(-\chi(Z_\psi))) * \mathcal{R}(exp^*(Z_\psi))$$
$$-\mathcal{R}(exp^*(Z_\psi)) * \mathcal{R}(exp^{*\mathcal{R}}(-\chi(Z_\psi))) = 0, \tag{32}$$

$$(\mathbf{ii}) - \widetilde{\mathcal{R}}(exp^{*\mathcal{R}}(-\chi(Z_\psi))) * \mathcal{R}(exp^*(Z_\psi))$$
$$+\mathcal{R}(exp^*(Z_\psi)) * \widetilde{\mathcal{R}}(exp^{*\mathcal{R}}(-\chi(Z_\psi)))$$
$$+\widetilde{\mathcal{R}}(exp^{*\mathcal{R}}(-\chi(Z_\psi))) * exp^*(Z_\psi)$$
$$-exp^*(Z_\psi) * \widetilde{\mathcal{R}}(exp^{*\mathcal{R}}(-\chi(Z_\psi))) = 0. \tag{33}$$

On the other hand, one can approximate the *Bogoliubov character* $b[\psi] := exp^{*\mathcal{R}}(-\chi(Z_\psi))$ by the formula

$$\mathcal{R}(b[\psi]) = -R_{ms} \circ \{exp^*(Z_\psi) + \alpha_\psi\} \tag{34}$$

such that

$$\alpha_\psi := \sum_{n\geq 0} \frac{1}{n!} \sum_{j=1}^{n-1} \frac{n!}{j!(n-j)!} \mathcal{R}(-\chi(Z_\psi))^{*(n-j)} * Z_\psi^{*j} \tag{35}$$

[9], [11]. Based on this estimation, it is possible to rewrite the equations (32) and (33). We have

$$(\mathbf{i})'$$

$$-\mathcal{R}(\psi+\alpha_\psi)*\mathcal{R}(exp^*(Z_\psi))+\mathcal{R}(exp^*(Z_\psi))*\mathcal{R}(\psi+\alpha_\psi) = 0 \tag{36}$$

and

$$(\mathbf{ii})' \quad \widetilde{\mathcal{R}}(\psi+\alpha_\psi)*\mathcal{R}(exp^*(Z_\psi)) - \mathcal{R}(exp^*(Z_\psi))*\widetilde{\mathcal{R}}(\psi+\alpha_\psi)$$

$$-\widetilde{\mathcal{R}}(\psi+\alpha_\psi)*exp^*(Z_\psi)+exp^*(Z_\psi)*\widetilde{\mathcal{R}}(\psi+\alpha_\psi) = 0. \tag{37}$$

Moreover we know that the Birkhoff factorization of characters of the Connes-Kreimer Hopf algebra are characterized with the special infinitesimal character χ such that for each infinitesimal character $Z \in \partial char_{A_{dr}}\ H_{rt}$, $\chi(Z) = Z + \sum_{k=1}^{\infty} \chi_Z^{(k)}$. The sum is a finite linear combination of infinitesimal characters such that $\chi_Z^{(k)}$s are determined by unique solution of the fixed point equation

$$E: \quad \chi(Z) = Z - \sum_{k=1}^{\infty} c_k K^{(k)}(\mathcal{R}(\chi(Z)), \widetilde{\mathcal{R}}(\chi(Z))) \tag{38}$$

where terms $K^{(k)}$s are identified by BCH series. By putting the equation E in the Bogoliubov character and with notice to the relations (36) and (37), one can reformulate the motion integral condition (19) for components of a given character underlying the fixed point equation E.

In this procedure it should be important to consider the behavior of $\beta-$function and renormalization group. These physical information are based on the grading operator Y (that providing the scaling evolution of the coupling constant). This element is defined with the extension of the Lie algebra $\partial\ char_{A_{dr}} H_{rt}$ by an element Z^0 such that for each rooted tree t, we have $[Z^0, Z^t] = Y(Z^t) = |t|Z^t$. For each character $\psi \in char_{A_{dr}} H_{rt}$, its related $\beta-$function is given by

$$\beta(\psi) = \psi_- * [Z^0, \psi_-^{-1}] = \psi_- * Z^0 * \psi_-^{-1} - Z^0. \tag{39}$$

With applying the exponential map, its related renormalization group is determined by $F_t = exp^*(t\beta)$. For each $t \in \mathbb{R}$, F_t is a character given by a polynomial of the variable t and therefore $R_{ms} \circ F_t = 0$ ([7], [10], [22]). Equations (24) and (30) show that each element of the renormalization group plays the role of an integral of motion for ψ in the algebras C_λ^x and $C_{\mathcal{R}}^x$ if and only if

$$-\lambda[F_t, \psi] + [F_t, \mathcal{R}_\lambda(\psi)] - \mathcal{R}_\lambda([\psi, F_t]) = 0, \tag{40}$$

$$[F_t, \mathcal{R}(\psi)] - [F_t, \psi] = 0, \tag{41}$$

respectively. The condition (41) is corresponded with a fixed point equation related to the Feynman rules character ψ and its related $\beta-$function.

Corollary III.3. *For a given Feynman rules character $\psi \in char_{A_{dr}}(H_{rt})$, an element F_t of the related renormalization group plays the role of integral of motion for ψ iff the $\beta-$function satisfies in the equation*

$$[exp^*(t\beta), \mathcal{R}\{exp^*(\mathcal{R}(E)) * exp^*(\widetilde{\mathcal{R}}(E))\}]$$

$$-[exp^*(t\beta), exp^*(\mathcal{R}(E)) * exp^*(\widetilde{\mathcal{R}}(E))] = 0.$$

At last relation between the renormalization group flow and the determined Nijenhuis type flows should be emphasized. We know that the renormalization group $\{F_t\}_t$ is a $1-$parameter subgroup of characters and it means that $F_t * F_s = F_{t+s}$. Therefore it is easy to show that in the cases C_0^x and $C_{\mathcal{R}}^x$, each F_t is an integral of motion for a fixed element F_{t_0} of the renormalization group.

IV. Conclusion

Connes-Kreimer treatment to perturbative renormalization provides a very practical instruction to study QFTs based on a combinatorial Hopf algebra such that rooted trees can give us a toy model to consider this interesting Hopf algebra. This approach determines important physical parameters in an algebro-geometrical constructive procedure. In this short article we applied the Rota-Baxter nature of the scheme BPHZ in renormalization (i.e. its algebraic reformulation) to characterize quantum motion integrals with respect to two different techniques namely, Rosenberg's strategy in identifying Lax pair equations and noncommutative differential forms. Then with using the representation of Birkhoff components based on integral renormalization, we considered the possibility of saving motion integrals in Birkhoff components of Feynman rules characters on Connes-Kreimer Hopf algebra of rooted trees such that it was formulated with specific family of fixed point equations. The very essential fact is that this type of motion integrals can address Connes-Kreimer renormalization flow. Finally, one can not ignore the role of Birkhoff factorization in both indicated techniques such that it would be an important signal about the study of integrable systems based on the Riemann-Hilbert correspondence.

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