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Existence of solutions for a nonlinear fractional differential equation with integral boundary condition

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Abstract—This paper deals with a nonlinear fractional differential equation with integral boundary condition of the following form:

$$\begin{cases} D_t^\alpha x(t) = f(t, x(t), D_t^\beta x(t)),\ t \in (0,1), \\ x(0) = 0, \quad x(1) = \int_0^1 g(s)x(s)ds, \end{cases}$$

where $1 < \alpha \leq 2, 0 < \beta < 1$. Our results are based on the Schauder fixed point theorem and the Banach contraction principle.

Keywords—Fractional differential equation; Integral boundary condition; Schauder fixed point theorem; Banach contraction principle.

I. Introduction

In the last few decades, fractional-order models are found to be more adequate than integerorder models for some real world problems. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional differential equations in comparison with classical integer-order models. Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, and so forth, involves derivatives of fractional order. In consequence, the subject of fractional differential equations is gaining much importance and attention. For examples and details, see [1-11] and the references therein. However, the theory of boundary value problems for nonlinear fractional differential equations is still in the initial stages and many aspects of this theory need to be explored.

Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multipoint, and nonlocal boundary value problems as special cases. For boundary value problems with integral boundary conditions and comments on their importance, we refer the reader to the papers [12-15] and the references therein.

In this paper, we consider the following boundary value problem for a nonlinear fractional differential equation with integral boundary conditions

$$\begin{cases} D_t^\alpha x(t) = f(t, x(t), D_t^\beta x(t)),\ t \in (0,1), \\ x(0) = 0, \quad x(1) = \int_0^1 g(s)x(s)ds. \end{cases} \tag{1}$$

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where $1 < \alpha \leq 2, 0 < \beta < 1$ and D_t^α represents the standard Riemannn-liouville fractional derivative, $f : [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is assumed to satisfy certain conditions, which will be specified later, $g \in L^1[0,1]$ satisfies $1 - \int_0^1 g(s)s^{\alpha-1}ds > 0$.

This paper is organized as follows. In next section, we present some basic definitions and preliminary lemmas. Section 3 is devoted to the existence results for (1) based on Schauder fixed point theorem and Banach contraction principle. In the last section, two examples are given to illustrate our main results.

II. Preliminaries

In this section, we shall first recall some basic definitions, lemmas which are used in what follows (see [8-11]).

Definition 2.1 The αth fractional order integral of the function $u : (0,\infty) \mapsto \mathbb{R}$ is defined by

$$I_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)ds,$$

where $\alpha > 0$, Γ is the gamma function, provided the right side is pointwise defined on $(0,\infty)$.

Definition 2.2 The αth fractional order derivative of a continuous function $u : (0,\infty) \mapsto \mathbb{R}$ is defined by

$$D_t^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t (t-s)^{n-\alpha-1} u(s)ds,$$

where $\alpha > 0$, $n = [\alpha] + 1$, provided that the right side is pointwise defined on $(0,\infty)$.

Lemma 2.1 Let $\alpha > 0$. then the fractional differential equation

$$D_t^\alpha u(t) = 0$$

has a solution

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n},$$

and $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, $n = [\alpha] + 1$.

Lemma 2.2 Let $\alpha > 0$. Then

$$I_t^\alpha D_t^\alpha u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n},$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, and $n = [\alpha] + 1$.

Lemma 2.3 Let $h \in C([0,1])$, then for $1 < \alpha \leq 2, 0 < \beta < 1$, the linear problem

$$\begin{cases} D_t^\alpha x(t) = h(t),\ t \in (0,1), \\ x(0) = 0, \quad x(1) = \int_0^1 g(s)x(s)ds. \end{cases} \tag{2}$$

has a unique solution

$$x(t) = \int_0^1 G(t,s)h(s)ds, \quad (3)$$

where

$$G(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\Theta t^{\alpha-1}}{\Gamma(\alpha)}\big[\int_s^1 g(r)(r-s)^{\alpha-1}dr \\ -(1-s)^{\alpha-1}\big],\ 0 \le s \le t \le 1, \\ \frac{\Theta t^{\alpha-1}}{\Gamma(\alpha)}\big[\int_s^1 g(r)(r-s)^{\alpha-1}dr \\ -(1-s)^{\alpha-1}\big],\ 0 \le t \le s \le 1, \end{cases} \quad (4)$$

here, $\Theta = \big[1 - \int_0^1 g(s)s^{\alpha-1}ds\big]^{-1}$.

Proof: In view of Lemma 2.2 and equation (2), we have

$$x(t) = I_t^\alpha h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \text{ for } c_1, c_2 \in \mathbb{R}.$$

Hence, the general solution of equation (2) is

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}h(s)}{\Gamma(\alpha)}ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2},$$

for $c_1, c_2 \in \mathbb{R}$. The boundary condition $x(0) = 0$ implies that $c_2 = 0$. And it is follows from

$$x(1) = \int_0^1 g(s)x(s)ds$$

that

$$\begin{aligned} c_1 &= \Theta\Big[\int_0^1 g(s)\int_0^s \frac{(s-r)^{\alpha-1}h(r)}{\Gamma(\alpha)}drds \\ &\quad -\int_0^1 \frac{(1-s)^{\alpha-1}h(s)}{\Gamma(\alpha)}ds\Big] \\ &= \frac{\Theta}{\Gamma(\alpha)}\Big[\int_0^1\int_0^s g(s)(s-r)^{\alpha-1}h(r)drds \\ &\quad -\int_0^1 (1-s)^{\alpha-1}h(s)ds\Big] \\ &= \frac{\Theta}{\Gamma(\alpha)}\Big[\int_0^1\int_s^1 g(r)(r-s)^{\alpha-1}h(s)drds \\ &\quad -\int_0^1 (1-s)^{\alpha-1}h(s)ds\Big] \\ &= \frac{\Theta}{\Gamma(\alpha)}\int_0^1\Big[\int_s^1 g(r)(r-s)^{\alpha-1}dr \\ &\quad -(1-s)^{\alpha-1}\Big]h(s)ds \end{aligned}$$

where, $\Theta = \big[1 - \int_0^1 g(s)s^{\alpha-1}ds\big]^{-1}$.

So

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{\alpha-1}h(s)}{\Gamma(\alpha)}ds \\ &\quad +\frac{\Theta t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1\Big[\int_s^1 g(r)(r-s)^{\alpha-1}dr \\ &\quad -(1-s)^{\alpha-1}\Big]h(s)ds \\ &= \int_0^1 G(t,s)h(s)ds. \end{aligned}$$

This completes the proof. ■

III. Main Results

Let $C([0,1],\mathbb{R})$ be the space of continuous functions defined on $[0,1]$. The space

$$\mathbb{B} = \{x : x \in C([0,1],\mathbb{R}), D_t^\alpha x \in C([0,1],\mathbb{R})\}$$

equipped with the norm $\|x\|_{\mathbb{B}} = \max_{t\in[0,1]}|x(t)| + \max_{t\in[0,1]}|D_t^\alpha x(t)|$ is a Banach space.

For the forthcoming analysis, we impose some growth conditions on the function f as follows:

(H_1) $f : [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous;

(H_2) There exists a nonnegative function $\phi \in L([0,1])$ such that $|f(t,x,y)| \le \phi(t) + c_1|x|^{\sigma_1} + c_1|y|^{\sigma_2}$, where $c_1, c_2 \in \mathbb{R}$ are nonnegative constants and $0 < \sigma_1, \sigma_2 < 1$;

(H_3) There exists a nonnegative function $\phi \in L([0,1])$ such that $|f(t,x,y)| \le \phi(t) + c_1|x|^{\sigma_1} + c_1|y|^{\sigma_2}$, where $c_1, c_2 \in \mathbb{R}$ are nonnegative constants and $\sigma_1, \sigma_2 > 1$;

(H_4) There exists a constant $k > 0$ such that

$$|f(t,x,y) - f(t,\bar{x},\bar{y})| \le k(|x-\bar{x}| + |y-\bar{y}|)$$

for each $t \in [0,1]$ and all $x, y, \bar{x}, \bar{y} \in \mathbb{R}$.

For convenience, we define the following constants:

$$\begin{aligned} p &= \max_{t\in[0,1]}\int_0^1 |G(t,s)\phi(s)|ds, \\ q &= \frac{1+\Theta}{\Gamma(\alpha+1)} + \frac{\Theta}{\Gamma(\alpha)}\int_0^1\int_s^1 |g(r)|(r-s)^{\alpha-1}drds, \\ \rho &= \frac{1+\Theta}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-\beta+1)} + \frac{\Theta}{\alpha\Gamma(\alpha-\beta)} \\ &\quad +\Big(\frac{\Theta}{\Gamma(\alpha-\beta)} + \frac{\Theta}{\Gamma(\alpha)}\Big)\int_0^1\int_s^1 |g(r)|(r-s)^{\alpha-1}drds. \end{aligned}$$

Lemma 3.1 Assume that (H_1) holds, then $x \in \mathbb{B}$ is a solution of BVP (1) if and only if $x \in \mathbb{B}$ is a solution of the integral equation

$$x(t) = \int_0^1 G(t,s)f(s,x(s),D_s^\beta x(s))ds. \quad (5)$$

Proof: Let $x \in \mathbb{B}$ is a solution of BVP (1), by the method used to prove Lemma 2.3, we can prove that $x \in \mathbb{B}$ is a solution of the integral equation (5).

Conversely, let $x \in \mathbb{B}$ be a solution of the integral equation (5). For

$$\begin{aligned} x(t) &= \int_0^1 G(t,s)f(s,x(s),D_s^\beta x(s))ds \\ &= \int_0^t \frac{(t-s)^{\alpha-1}f(s,x(s),D_s^\beta x(s))}{\Gamma(\alpha)}ds \\ &\quad +\Theta\Big[\int_0^1 g(s)\int_0^s \frac{(s-r)^{\alpha-1}f(r,x(r),D_r^\beta x(r))}{\Gamma(\alpha)}drds \\ &\quad -\int_0^1 \frac{(1-s)^{\alpha-1}f(s,x(s),D_s^\beta x(s))}{\Gamma(\alpha)}ds\Big]t^{\alpha-1} \\ &= I_t^\alpha f(t,x(t),D_t^\beta x(t)) \\ &\quad +\Theta\Big[\int_0^1 g(s)I_s^\alpha f(s,x(s),D_s^\beta x(s))ds \\ &\quad -I_1^\alpha f(1,x(1),D_1^\beta x(1))\Big]t^{\alpha-1} \end{aligned}$$

together with the relations

$$D_t^\alpha I_t^\alpha f(t) = f(t) \text{ and } D_t^\alpha t^{\alpha-1} = 0,$$

then

$$D_t^\alpha x(t) = f(t, x(t), D_t^\beta x(t)).$$

On the other hand, it is easy to show that $x(0) = 0$ and $x(1) = \int_0^1 g(s)x(s)ds$, which implies that $x \in \mathbb{B}$ is a solution of BVP (1). This completes the proof. ■

Theorem 3.1 Assume that (H_1) and (H_2) hold, then BVP (1) has a solution.

Proof: Define an operator $\Phi : \mathbb{B} \to \mathbb{B}$ by

$$(\Phi x)(t) = \int_0^1 G(t,s) f(s, x(s), D_s^\beta x(s))ds. \tag{6}$$

In view of the continuity of f and G, the operator Φ is continuous.

Let

$$\mathbb{M} = \{x \in \mathbb{B} : \|x\|_{\mathbb{B}} \leq R, t \in [0,1]\}$$

where

$$R \geq \max\{3p, (3c_1 q)^{\frac{1}{1-\sigma_1}}, (3c_2 q)^{\frac{1}{1-\sigma_2}}\}.$$

Firstly, we prove that $\Phi : \mathbb{M} \to \mathbb{M}$. In fact, for each $x \in \mathbb{M}$, we have

$$\begin{aligned}
|(\Phi x)(t)| &= \int_0^1 |G(t,s)||f(s, x(s), D_s^\beta x(s))|ds \\
&\leq \int_0^1 |G(t,s)\phi(s)|ds \\
&\quad +(c_1 R^{\sigma_1} + c_1 R^{\sigma_2}) \int_0^1 |G(t,s)|ds \\
&\leq \int_0^1 |G(t,s)\phi(s)|ds \\
&\quad +(c_1 R^{\sigma_1} + c_1 R^{\sigma_2}) \Big[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
&\quad + \int_0^1 \frac{\Theta t^{\alpha-1}}{\Gamma(\alpha)} \Big| \int_s^1 g(r)(r-s)^{\alpha-1} dr \\
&\quad -(1-s)^{\alpha-1} \Big| ds \Big] \\
&\leq \int_0^1 |G(t,s)\phi(s)|ds \\
&\quad +(c_1 R^{\sigma_1} + c_1 R^{\sigma_2}) \Big[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
&\quad + \frac{\Theta}{\Gamma(\alpha)} \int_0^1 \int_s^1 |g(r)|(r-s)^{\alpha-1} drds \\
&\quad + \frac{\Theta}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} ds \Big] \\
&\leq \int_0^1 |G(t,s)\phi(s)|ds \\
&\quad +(c_1 R^{\sigma_1} + c_1 R^{\sigma_2}) \Big[\frac{1+\Theta}{\Gamma(\alpha+1)} \\
&\quad + \frac{\Theta}{\Gamma(\alpha)} \int_0^1 \int_s^1 |g(r)|(r-s)^{\alpha-1} drds \Big] \\
&\leq p + (c_1 R^{\sigma_1} + c_1 R^{\sigma_2}) q \\
&\leq \frac{R}{3} + \frac{R}{3} + \frac{R}{3} = R.
\end{aligned}$$

Therefore, $\|(\Phi x)(t)\|_{\mathbb{B}} \leq R$. Thus, $\Phi : \mathbb{M} \to \mathbb{M}$.

Next, we show that Φ is completely continuous. In fact, Let $N = \max_{t \in [0,1]} |f(t, x(t), D_t^\beta x(t))| + 1$. For each $x \in \mathbb{M}$, and $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, then

$$\begin{aligned}
&|(\Phi x)(t_2) - (\Phi x)(t_1)| \\
&= \int_0^1 |G(t_2,s) - G(t_1,s)||f(s, x(s), D_s^\beta x(s))|ds \\
&\leq N|I_{t_2}^\alpha(1) - I_{t_1}^\alpha(1)| \\
&\quad + \frac{\Theta N}{\Gamma(\alpha)} \Big[\int_0^1 \int_s^1 |g(r)|(r-s)^{\alpha-1} drds \\
&\quad + \int_0^1 (1-s)^{\alpha-1} ds \Big] (t_2^{\alpha-1} - t_1^{\alpha-1}) \\
&\leq \frac{N}{\Gamma(\alpha+1)} ({t_2}^\alpha - {t_1}^\alpha) \\
&\quad + \frac{\Theta N}{\Gamma(\alpha)} \Big[\int_0^1 \int_s^1 |g(r)|(r-s)^{\alpha-1} drds \\
&\quad + \frac{1}{\alpha} \Big] (t_2^{\alpha-1} - t_1^{\alpha-1}).
\end{aligned}$$

By using $D_t^a t^b = \frac{\Gamma(b+1)}{\Gamma(b-a+1)} t^{b-a}$, then

$$\begin{aligned}
&|(D_{t_2}^\beta \Phi x)(t_2) - (D_{t_1}^\beta \Phi x)(t_1)| \\
&\leq |I_{t_2}^{\alpha-\beta} f(t_2, x(t_2), D_{t_2}^\beta x(t_2)) \\
&\quad - I_{t_1}^{\alpha-\beta} f(t_1, x(t_1), D_{t_1}^\beta x(t_1))| \\
&\quad + \frac{\Theta N}{\Gamma(\alpha-\beta)} \Big[\int_0^1 \int_s^1 |g(r)|(r-s)^{\alpha-1} drds + \frac{1}{\alpha} \Big] \\
&\quad (t_2^{\alpha-\beta-1} - t_1^{\alpha-\beta-1}) \\
&\leq N|I_{t_2}^{\alpha-\beta}(1) - I_{t_1}^{\alpha-\beta}(1)| \\
&\quad + \frac{\Theta N}{\Gamma(\alpha-\beta)} \Big[\int_0^1 \int_s^1 |g(r)|(r-s)^{\alpha-1} drds + \frac{1}{\alpha} \Big] \\
&\quad (t_2^{\alpha-\beta-1} - t_1^{\alpha-\beta-1}) \\
&\leq \frac{N}{\Gamma(\alpha-\beta+1)} ({t_2}^{\alpha-\beta} - {t_1}^{\alpha-\beta}) \\
&\quad + \frac{\Theta N}{\Gamma(\alpha-\beta)} \Big[\int_0^1 \int_s^1 |g(r)|(r-s)^{\alpha-1} drds + \frac{1}{\alpha} \Big] \\
&\quad (t_2^{\alpha-\beta-1} - t_1^{\alpha-\beta-1}).
\end{aligned}$$

Now, we conclude that $\Phi\mathbb{M}$ is equicontinuous, since the functions $t_2^{\alpha-1} - t_1^{\alpha-1}, t_2^\alpha - t_1^\alpha, t_2^{\alpha-\beta-1} - t_1^{\alpha-\beta-1}, {t_2}^{\alpha-\beta} - {t_1}^{\alpha-\beta}$ are uniformly continuous on $[0,1]$. Also, $\Phi\mathbb{M}$ is a uniformly bounded set. So, $\Phi\mathbb{M} \subset \mathbb{M}$. By the Arzela-Ascoli theorem, $\Phi : \mathbb{M} \to \mathbb{M}$ is completely continuous. Hence the Schauder fixed point theorem implies the existence of a solution in $\mathbb{M}$ for BVP (1). This completes the proof. ■

Theorem 3.2 Assume that (H_1) and (H_3) hold, then BVP (1) has a solution.

Proof: The proof is similar to that of Theorem 3.1, so we omit it here. ■

Theorem 3.3 Assume that (H_1) and (H_4) hold. If $k\rho < 1$, then BVP (1) has a unique solution.

Proof: For any $x, y \in \mathbb{B}$, by (H_4), we have

$$\begin{aligned} & |(\Phi x)(t) - (\Phi y)(t)| \\ = & \int_0^1 |G(t,s)||f(s,x(s),D_s^\beta x(s)) \\ & -f(s,y(s),D_s^\beta y(s))|ds \\ \leq & k\left[\frac{1+\Theta}{\Gamma(\alpha+1)} + \frac{\Theta}{\Gamma(\alpha)}\int_0^1\int_s^1 |g(r)|(r-s)^{\alpha-1}drds\right] \\ & \|x-y\|, \end{aligned}$$

and

$$\begin{aligned} & |(D_t^\beta \Phi x)(t) - (D_t^\beta \Phi y)(t)| \\ \leq & I_t^{\alpha-\beta}|f(t,x(t),D_t^\beta x(t_2)) - f(t,y(t),D_t^\beta y(t))| \\ & +\frac{\Theta}{\Gamma(\alpha-\beta)}\left[\int_0^1\int_s^1 |g(r)|(r-s)^{\alpha-1}drds + \frac{1}{\alpha}\right] \\ & t^{\alpha-\beta-1}|f(t,x(t),D_t^\beta x(t_2)) - f(t,y(t),D_t^\beta y(t))| \\ \leq & \left\{I_t^{\alpha-\beta}(1) + \frac{\Theta}{\Gamma(\alpha-\beta)}\left[\int_0^1\int_s^1 |g(r)|(r-s)^{\alpha-1}drds\right.\right. \\ & \left.\left.+\frac{1}{\alpha}\right]t^{\alpha-\beta-1}\right\}|f(t,x(t),D_t^\beta x(t_2)) \\ & -f(t,y(t),D_t^\beta y(t))| \\ \leq & k\left\{\frac{1}{\Gamma(\alpha-\beta+1)}\right. \\ & +\frac{\Theta}{\Gamma(\alpha-\beta)}\left[\int_0^1\int_s^1 |g(r)|(r-s)^{\alpha-1}drds\right. \\ & \left.\left.+\frac{1}{\alpha}\right]\right\}\|x-y\|. \end{aligned}$$

Then

$$\begin{aligned} & \|(\Phi x)(t) - (\Phi y)(t)\| \\ \leq & k\left[\frac{1+\Theta}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-\beta+1)} + \frac{\Theta}{\alpha\Gamma(\alpha-\beta)}\right. \\ & \left.+\left(\frac{\Theta}{\Gamma(\alpha-\beta)} + \frac{\Theta}{\Gamma(\alpha)}\right)\int_0^1\int_s^1 |g(r)|(r-s)^{\alpha-1}drds\right] \\ & \|x-y\| \\ = & k\rho\|x-y\| \\ < & \|x-y\|. \end{aligned}$$

By the contraction mapping principle, BVP (1) has a unique solution. This completes the proof. ■

IV. Examples

Consider the following boundary value problem

$$\begin{cases} D_t^{\frac{3}{2}}x(t) = f(t,x(t),D_t^{\frac{1}{2}}x(t)),\ t\in(0,1), \\ x(0)=0,\quad x(1)=\int_0^1 sx(s)ds. \end{cases} \tag{7}$$

Then $\Theta = \left[1-\int_0^1 s^{\frac{3}{2}}ds\right]^{-1} = \frac{5}{3} > 0$.

Example 1. $f(t,x(t),D_t^{\frac{1}{2}}x(t)) = \frac{(t-\frac{1}{4})^2e^t}{1+t^3} + \frac{\sin\pi t}{\sqrt{\pi}}|x(t)|^{\sigma_1} + \frac{e^{-t}}{5+|D_t^{\frac{1}{2}}x(t)|}|D_t^{\frac{1}{2}}x(t)|^{\sigma_2}$. Let $\phi(t) = \frac{(t-\frac{1}{4})^2e^t}{1+t^3}, c_1 = \frac{1}{\sqrt{\pi}}, c_2 = \frac{1}{5}$, then $|f(t,x(t),D_t^{\frac{1}{2}}x(t))| < \phi(t) + c_1|x(t)|^{\sigma_1} + c_2|D_t^{\frac{1}{2}}x(t)|^{\sigma_2}$. For $0<\sigma_1,\sigma_2<1$, the assumption (H_2) holds and for $\sigma_1,\sigma_2>1$, the assumption (H_3) holds. Therefore, by Theorem 3.1 and Theorem 3.2, BVP (7) has a solution.

Example 2. $f(t,x(t),D_t^{\frac{1}{2}}x(t)) = \frac{(t-\frac{1}{2})^2(x(t)+D_t^{\frac{1}{2}}x(t))}{(4+e^{2t})(1+x(t)+D_t^{\frac{1}{2}}x(t))^2}$. Let $k=\frac{1}{20}$, then the assumption (H_4) holds. By a direct calculation, we can get $\rho = 5.4324$ and $k\rho = 0.2716 < 1$. Therefore, by Theorem 3.3, BVP (7) has a unique solution.

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