International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:4, No:7, 2010

Existence of Solution for Four-Point Boundary Value Problems of Second-Order Impulsive Differential Equations (I)

Li Ge

Abstract—In this paper, we study the existence of solution of the four-point boundary value problem for second-order differential equations with impulses by using leray-Schauder theory:

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), \quad t \in [0,1],\ t \neq t_k,\ k = 1, 2, \dots, m \\ \Delta x(t_k) = I_k(x(t_k)), \quad k = 1, 2, \dots, m \\ \Delta x'(t_k) = \overline{I}_k(x(t_k), x'(t_k)), \quad k = 1, 2, \dots, m \\ x(0) = \alpha x(\xi),\ x(1) = \beta x(\eta), \end{cases} \tag{E}$$

where $0 < \xi \leq \eta < 0$, $\alpha\xi(1-\beta) + (1-\alpha)(1-\beta\eta) \neq 0$, $f \in C[J \times R^2, R]$, $I_k \in C[R, R]$, $\overline{I}_k \in C[R^2, R]$, $J = [0, 1]$. We also give a corresponding example to demonstrate our results.

Keywords—impulsive differential equations, impulsive integral-differential equation, boundary value problems

I. Introduction

The theory of impulsive differential equations is emerging as an important area of investigation since it is much richer than the corresponding theory of concerning equations without impulses. Recently, some existence results concerning the boundary value problems of impulsive differential equations have been obtained ([1-3]). However, there are few papers about multi-point boundary value problems of differential equations with impulses. Recently, Sun [4] proved the existence of solutions for the three-point boundary value problem for second-order differential equations with impulses:

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), \quad t \in [0,1],\ t \neq t_k,\ k = 1, 2, \dots, m \\ \Delta x(t_k) = I_k(x(t_k)), \quad k = 1, 2, \dots, m \\ \Delta x'(t_k) = \overline{I}_k(x(t_k), x'(t_k)), \quad k = 1, 2, \dots, m \\ x(0) = 0,\ x(1) = \alpha x(\eta). \end{cases}$$

Our work was motivated by the work of Sun [4]. In the paper, we study the existence of solution for BVP(E).

Consider the following second order impulsive differential equations

$$\begin{cases} x''(t) = f(t, x(t), x'(t)),\ t \in [0,1],\ t \neq t_k,\ k = 1, 2, \dots, m \\ \Delta x(t_k) = I_k(x(t_k)),\ k = 1, 2, \dots, m \\ \Delta x'(t_k) = \overline{I}_k(x(t_k), x'(t_k)),\ k = 1, 2, \dots, m \\ x(0) = \alpha x(\xi),\ x(1) = \beta x(\eta), \end{cases} \tag{E}$$

where $f \in C[J \times R^2, R]$, $J = [0, 1]$, $0 < t_1 < t_2 < \cdots < t_m < 1$, $I_k \in C[R, R]$, $\overline{I}_k \in C[R \times R, R]$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-) = x(t_k + 0) - x(t_k - 0)$, $\Delta x'(t_k) = x'(t_k + 0) - x'(t_k - 0)$.

Li Ge is with the Department of Mathematics, Fuyang Nomal College, Fuyang, 230062 Anhui, P.R.China, Email: gl3381874@sina.com

Let $PC[J, R] = \{x : J \to R | x(t)$ is continuous at $t \neq t_k$ and left continuous at $t = t_k$, and $x(t_k^+)$ exists for $k = 1, 2, \dots, m\}$, and $PC^1[J, R] = \{x \in PC[J, R] | x'(t)$ is continuous at $t \neq t_k$ and $x'(t_k^+), x'(t_k^-)$ exists for $k = 1, 2, \dots, m\}$. It is easy to prove that $PC[J, R]$ is a Banach space with norm $\|x\|_{PC} = \sup_{t \in J} |x(t)|$, $PC^1[J, R]$ is also a Banach space with norm $\|x\|_{PC^1} = \max\{\|x\|_{PC}, \|x'\|_{PC}\}$. We also use the space $L^1[0, 1]$, and denote the norm in $L^1[0, 1]$ by $\| \cdot \|_1$.

For $x \in PC^1[J, R]$, by virtue of the mean value theorem ([5]), we know that the left derivation $x'_-(t_k)$ exists and $x'_-(t_k) = x'(t_k^-)$. In (E) and what follows, it is understood that $x'(t_k) = x'(t_k^-)$. So, for $x \in PC^1[J, R]$, we have $x' \in PC[J, R]$.

Let $J_0 = [0, t_1]$, $J_1 = (t_1, t_2], \dots$, $J_{m-1} = (t_{m-1}, t_m]$, $J_m = (t_m, 1]$, $J' = J \setminus \{t_1, t_2, \dots, t_m\}$, A map $x \in PC^1[J, R] \cap C^2[J', R]$ is called a solution of BVP(E) if it satisfies all equations of (E).

Throughout this paper, we assume that $0 < \xi \leq \eta < 1$. Furthermore, for convenience sake, we set $\Lambda = \alpha\xi(1-\beta) + (1-\alpha)(1-\beta\eta)$.

II. Preliminary Lemmas

Lemma 2.1 ([3]) $H \subset PC^1[J, R]$ is a relatively compact set if and only if both $x(t)$ and $x'(t)$ are uniformly bounded on J and equicontinuous on every $J_k (k = 1, 2, \dots, m)$ for any $x \in H$.

Lemma 2.2 ([3]) If $x \in PC^1[J, R] \cap C^2[J', R]$ satisfies $x'' = f(t, x(t), x'(t))$, $t \neq t_k$, $k = 1, 2, \dots, m$, then

$$\begin{aligned} x'(t) &= x'(0) + \int_0^t f(s, x(s), x'(s))ds \\ &\quad + \sum_{0<t_k<t} (x'(t_k^+) - x'(t_k)), \quad \forall t \in J, \end{aligned} \tag{1}$$

$$\begin{aligned} x(t) &= x(0) + x'(0)t + \int_0^t (t-s) f(s, x(s), x'(s))ds \\ &\quad + \sum_{0<t_k<t} (x(t_k^+) - x(t_k)) \\ &\quad + \sum_{0<t_k<t} (x'(t_k^+) - x'(t_k))(t - t_k), \quad \forall t \in J. \end{aligned} \tag{2}$$

Lemma 2.3 let $x \in PC^1[J, R] \cap C^2[J', R]$ is a solution of BVP(E), if and only if $x \in PC^1[J, R]$ is a solution of the following impulsive integral-differential equations:

$$\begin{aligned} x'(t) &= x'(0)+\int_0^t f(s,x(s),x'(s))ds \\ &\quad + \sum_{0<t_k<t} \overline{I}_k(x(t_k),x'(t_k)). \end{aligned} \quad (3)$$

$$\begin{aligned} x(t) &= x(0)+x'(0)t+\int_0^t (t-s)f(s,x(s),x'(s))ds \\ &\quad + \sum_{0<t_k<t} \overline{I}_k(x(t_k),x'(t_k))(t-t_k) \\ &\quad + \sum_{0<t_k<t} I_k(x(t_k)), \end{aligned} \quad (4)$$

where

$$\begin{aligned} x(0) &= \frac{1}{\Lambda}\Bigg[\alpha(1-\beta\eta)\int_0^\xi (\xi-s)f(s,x(s),x'(s))ds \\ &\quad +\alpha\beta\xi\int_0^\eta (\eta-s)f(s,x(s),x'(s))ds \\ &\quad -\alpha\xi\int_0^1 (1-s)f(s,x(s),x'(s))ds \\ &\quad +\alpha(1-\beta\eta)\sum_{0<t_k<\xi} I_k(x(t_k)) \\ &\quad +\alpha\beta\xi\sum_{0<t_k<\eta} I_k(x(t_k)) - \alpha\xi\sum_{k=1}^m I_k(x(t_k)) \\ &\quad +\alpha(1-\beta\eta)\sum_{0<t_k<\xi}(\xi-t_k)\overline{I}_k(x(t_k),x'(t_k)) \\ &\quad +\alpha\beta\xi\sum_{0<t_k<\eta}(\eta-t_k)\overline{I}_k(x(t_k),x'(t_k)) \\ &\quad -\alpha\xi\sum_{k=1}^m (1-t_k)\overline{I}_k(x(t_k),x'(t_k))\Bigg]. \end{aligned} \quad (5)$$

and

$$\begin{aligned} x'(0) &= \frac{1}{\Lambda}\Bigg[\alpha(\beta-1)\int_0^\xi (\xi-s)f(s,x(s),x'(s))ds \\ &\quad +\beta(1-\alpha)\int_0^\eta (\eta-s)f(s,x(s),x'(s))ds \\ &\quad -(1-\alpha)\int_0^1 (1-s)f(s,x(s),x'(s))ds \\ &\quad -\alpha(1-\beta)\sum_{0<t_k<\xi} I_k(x(t_k)) \\ &\quad +\beta(1-\alpha)\sum_{0<t_k<\eta} I_k(x(t_k)) \\ &\quad -(1-\alpha)\sum_{k=1}^m I_k(x(t_k)) \\ &\quad -\alpha(1-\beta)\sum_{0<t_k<\xi}(\xi-t_k)\overline{I}_k(x(t_k),x'(t_k)) \\ &\quad +\beta(1-\alpha)\sum_{0<t_k<\eta}(\eta-t_k)\overline{I}_k(x(t_k),x'(t_k)) \\ &\quad -(1-\alpha)\sum_{k=1}^m (1-t_k)\overline{I}_k(x(t_k),x'(t_k))\Bigg]. \end{aligned} \quad (6)$$

Proof If $x(t)$ is a solution of BVP(E), then

$$\begin{aligned} x(t) &= x(0)+x'(0)t+\int_0^t (t-s)f(s,x(s),x'(s))ds \\ &\quad + \sum_{0<t_k<t}(x(t_k^+)-x(t_k)) \\ &\quad + \sum_{0<t_k<t}(x'(t_k^+)-x'(t_k))(t-t_k), \ \ \forall t\in J. \end{aligned}$$

In view of $x(0)=\alpha x(\xi),\ x(1)=\beta x(\eta)$, we easily obtain (5) and (6). The combination of (1), (2), (5) and (6), yields (3) and (4).

On the other hand, assume that $x\in PC^1[J,R]$ is a solution of Eqs (3) and (4). It is clear that $x(0)=\alpha x(\xi),\ x(1)=\beta x(\eta),\ \Delta x(t_k)=I_k(x(t_k))$. By performing differentiation of (4) twice, we get

$$\begin{aligned} x'(t) &= x'(0)+\int_0^t f(s,x(s),x'(s))ds \\ &\quad + \sum_{0<t_k<t} \overline{I}_k(x(t_k),x'(t_k)), \ \ t\neq t_k, \end{aligned}$$

and

$$x''(t)=f(t,x(t),x'(t)), \ \ t\neq t_k,$$

which imply $x\in C^2[J',R]$ and $\Delta x'(t_k)=\overline{I}_k(x(t_k),x'(t_k))$. Therefore $x\in PC^1[J,R]\cap C^2[J',R]$ and x is a solution of BVP(E).

Operator $A: PC^1[J,R]\to PC^1[J,R]$ is defined as follows:

$$\begin{aligned} (Ax)(t) &= x(0)+x'(0)t+\int_0^t (t-s)f(s,x(s),x'(s))ds \\ &\quad + \sum_{0<t_k<t} I_k(x(t_k)) \\ &\quad + \sum_{0<t_k<t} \overline{I}_k(x(t_k),x'(t_k))(t-t_k), \forall t\in J. \end{aligned} \quad (7)$$

where $x(0)$ and $x'(0)$ are defined by (5) and (6) respectively.

Lemma 2.4 Operator A is a completely continuous one mapping $PC^1[J,R]$ into $PC^1[J,R]$.

Proof By (3), we get

$$\begin{aligned} (Ax)'(t) &= x'(0)t+\int_0^t f(s,x(s),x'(s))ds \\ &\quad + \sum_{0<t_k<t} \overline{I}_k(x(t_k)).) \end{aligned} \quad (8)$$

where $x'(0)$ is defined by (6).

From (7) and (8), it is easy to see that A is continuous operator from $PC^1[J,R]$ into $PC^1[J,R]$. Let S be a bounded set of $PC^1[J,R]$, then $A(S)\subset PC^1[J,R]$ is bounded and the elements of $A(S)$ and their derivatives are all uniformly bounded on J and equicontinuous on each $J_k (k=1,2,\dots,m)$. Therefore, $A(S)$ is a relatively compact set of $PC^1[J,R]$ by Lemma 2.1. So, operator A is completely continuous.

Lemma 2.5 ([6]) Let X be a real normed linear space and $T: X\to X$ be a compact operator. Suppose that $\Omega=\bigcup_{\lambda\in(0,1)}\Omega_\lambda$ is a bounded set, where $\Omega_\lambda=\{x\in X|x=\lambda Tx\}$, then the equation $x=\lambda Tx$ has at least a solution when $\lambda=1$.

III. Existence results for BVP(E)

In this section, we will prove existence results for BVP(E) in following cases:

(i) $\alpha \geq 1,\ \beta \geq 1,\ \beta\eta > 1$, and $\Lambda > 0$.
(ii) $\alpha \geq 1,\ \beta \geq 1,\ \beta\eta > 1$, and $\Lambda < 0$.
(iii) $\alpha \geq 1,\ \beta \geq 1,\ \beta\eta < 1$, then $\Lambda < 0$.
(iv) $0 \leq \alpha < 1,\ \beta \geq 1,\ \beta\eta > 1$, then $\Lambda < 0$.
(v) $0 \leq \alpha < 1,\ \beta \geq 1,\ \beta\eta < 1$, and $\Lambda < 0$.
(vi) $0 \leq \alpha < 1,\ \beta \geq 1,\ \beta\eta < 1$, and $\Lambda > 0$.
(vii) $0 \leq \alpha < 1,\ 0 \leq \beta < 1$, then $\Lambda > 0$.

Theorem 3.1 Let $f : [0,1] \times R^2 \to R$ be a continuous function, $I_k \in C[R,R],\ \overline{I}_k \in C[R \times R, R]$. Assume that
(A_1) There exist functions $p,\ q,\ r$ in $L^1[0,1]$, such that for all $(x,y) \in R^2,\ t \in [0,1]$

$$|f(t,x,y)| \leq p(t)|x| + q(t)|y| + r(t). \qquad (9)$$

(A_2) There exist constants $0 \leq \beta_k < 1,\ M_k \geq 0$ satisfying $|I_k(x)| \leq M_k$ for any $\forall x \in R$, and

$$\overline{\lim}_{|x|+|y|\to\infty} \frac{|\overline{I}_k(x,y)|}{|x|+|y|} = \beta_k, \quad k = 1,2,\dots,m. \qquad (10)$$

(A_3) There exist constants $\alpha,\ \beta,\ \xi,\ \eta$ satisfying (i).
Then BVP(E) has at least one solution in $PC^1[J,R] \cap C^2[J',R]$ provided that

$$\|p\|_1 + \|q\|_1 + \sum_{k=1}^{m} 2\beta_k < \frac{\Lambda}{2\beta\eta(\alpha\xi + 2\alpha - 2)}. \qquad (11)$$

Proof We will verify that the set of all possible solution of the family of equations:

$$\begin{cases} x''(t) = \lambda f(t,x(t),x'(t)), & t \neq t_k, k = 1,2,\dots,m \\ \Delta x(t_k) = \lambda I_k(x(t_k)), & k = 1,2,\dots,m \\ \Delta x'(t_k) = \lambda \overline{I}_k(x(t_k),x'(t_k)), & k = 1,2,\dots,m \\ x(0) = \alpha x(\xi),\ x(1) = \beta x(\eta), \end{cases} \qquad (E_\lambda)$$

is priori bounded in $PC^1[J,R] \cap C^2[J',R]$ by a constant independent of $\lambda \in (0,1)$.

If $x \in PC^1[J,R] \cap C^2[J',R]$, with $x(0) = \alpha x(\xi), x(1) = \beta x(\eta)$, from $x(t) = x(0) + \int_0^t x'(s)ds + \sum_{0<t_k<t} I_k(x(t_k))$, we have

$$|x(t)| \leq |x(0)| + \int_0^1 |x'(s)|ds + \sum_{k=1}^{m} M_k. \qquad (12)$$

From (3) and (6), we get

$$x'(t) = x'(0) + \int_0^t f(s,x(s),x'(s))ds + \sum_{0<t_k<t} \overline{I}_k(x(t_k),x'(t_k))$$

That is

$$\begin{aligned} |x'(t)| \leq\ & \frac{1}{\Lambda}\Bigg[\alpha(\beta-1)\int_0^\xi (\xi - s)|f(s,x(s),x'(s))|ds \\ & +\beta(\alpha-1)\int_0^\eta (\eta - s)|f(s,x(s),x'(s))|ds \\ & +(\alpha-1)\int_0^1 (1-s)|f(s,x(s),x'(s))|ds \\ & +\alpha(\beta-1)\sum_{0<t_k<\xi} |I_k(x(t_k))| \\ & +\beta(\alpha-1)\sum_{0<t_k<\eta} |I_k(x(t_k))| \\ & +(\alpha-1)\sum_{k=1}^{m} |I_k(x(t_k))| \\ & +\alpha(\beta-1)\sum_{0<t_k<\xi} (\xi - t_k)|\overline{I}_k(x(t_k),x'(t_k))| \\ & +\beta(\alpha-1)\sum_{0<t_k<\eta} (\eta - t_k)|\overline{I}_k(x(t_k),x'(t_k))| \\ & +(\alpha-1)\sum_{k=1}^{m} (1-t_k)|\overline{I}_k(x(t_k),x'(t_k))|\Bigg] \\ & +\int_0^t |f(s,x(s),x'(s))|ds + \sum_{0<t_k<t} |\overline{I}_k(x(t_k),x'(t_k))| \\ \leq\ & \frac{1}{\Lambda}\Bigg[\alpha\xi(\beta-1)\int_0^\xi |f(s,x(s),x'(s))|ds \\ & +\beta\eta(\alpha-1)\int_0^\eta |f(s,x(s),x'(s))|ds \\ & +(\alpha-1)\int_0^1 |f(s,x(s),x'(s))|ds \\ & +\alpha(\beta-1)\sum_{0<t_k<\xi} |I_k(x(t_k))| \\ & +\beta(\alpha-1)\sum_{0<t_k<\eta} |I_k(x(t_k))| \\ & +(\alpha-1)\sum_{k=1}^{m} |I_k(x(t_k))| \\ & +\alpha\xi(\beta-1)\sum_{0<t_k<\xi} |\overline{I}_k(x(t_k),x'(t_k))| \\ & +\beta\eta(\alpha-1)\sum_{0<t_k<\eta} |\overline{I}_k(x(t_k),x'(t_k))| \\ & +(\alpha-1)\sum_{k=1}^{m} |\overline{I}_k(x(t_k),x'(t_k))| \\ & +\Lambda\int_0^t |f(s,x(s),x'(s))|ds \\ & +\Lambda\sum_{0<t_k<t} |\overline{I}_k(x(t_k),x'(t_k))|\Bigg] \\ \leq\ & \frac{1}{\Lambda}\Bigg[2\beta\eta(\alpha-1)\int_0^1 |f(s,x(s),x'(s))|ds \end{aligned}$$

$$+(2\alpha\beta-\beta-1)\sum_{k=1}^{m}|I_k(x(t_k))|$$
$$+2\beta\eta(\alpha-1)\sum_{k=1}^{m}|\overline{I}_k(x(t_k),x'(t_k))|\Bigg]$$
$$\le \frac{2\beta\eta(\alpha-1)}{\Lambda}\Bigg[\int_0^1|f(s,x(s),x'(s))|ds + \sum_{k=1}^{m}|\overline{I}_k(x(t_k),x'(t_k))|\Bigg] + \frac{2\alpha\beta-\beta-1}{\Lambda}\sum_{k=1}^{m}M_k. \quad (13)$$

By (9), set $\|p\|_1+\|q\|_1+\sum_{k=1}^{m}2\beta_k=M_0$, then exists $\varepsilon_0>0$, such that

$$m\varepsilon<\frac{1}{4}[\frac{\Lambda}{2\beta\eta(\alpha\xi+2\alpha-2)}-M_0],\quad \forall\varepsilon<\varepsilon_0.$$

From (10), we know exists an $M(\varepsilon)$ for any ε defined above such that $|x|+|x'|\ge M(\varepsilon)$, $|\overline{I}_k(x(t_k),x'(t_k))|\le(\beta_k+\varepsilon)(|x|+|x'|)$. Now, we assume that $|x'(t)|$ is unbounded, that is there exists some $\lambda\in(0,1)$ such that $|x'|>max\{M(\varepsilon),\overline{M}\}$, where

$$M_{41} = \frac{2\beta\eta(\alpha-1)}{\Lambda}\|r\|_1+\frac{2\alpha\beta-\beta-1}{\Lambda}\sum_{k=1}^{m}M_k.$$
$$M_{42} = [1-\frac{2\beta\eta(\alpha-1)}{\Lambda}(\|q\|_1+\sum_{k=1}^{m}(\beta_k+\varepsilon))]^{-1}M_{41}.$$
$$M_{43} = \frac{\frac{2\alpha\beta\eta+2\alpha\xi+2\alpha\beta-2\alpha-\beta\eta-\beta}{\Lambda}\sum_{k=1}^{m}M_k}{1-\frac{2\beta\eta(\alpha\xi+\alpha-1)}{\Lambda}[\|p\|_1+\sum_{k=1}^{m}(\beta_k+\varepsilon)]}.$$
$$M_{44} = \frac{\frac{2\beta\eta(\alpha\xi+\alpha-1)}{\Lambda}[\|q\|_1+\sum_{k=1}^{m}(\beta_k+\varepsilon)]M_{42}}{1-\frac{2\beta\eta(\alpha\xi+\alpha-1)}{\Lambda}[\|p\|_1+\sum_{k=1}^{m}(\beta_k+\varepsilon)]}.$$

$\overline{M}=\frac{a}{b}$, where,

$$a = [1-\frac{2\beta\eta(\alpha\xi+\alpha-1)}{\Lambda}(\|p\|_1+\sum_{k=1}^{m}(\beta_k+\varepsilon))][1-\frac{2\beta\eta(\alpha-1)}{\Lambda}(\|q\|_1+\sum_{k=1}^{m}(\beta_k+\varepsilon))](M_{43}+M_{44}),$$

$$b = 1-\frac{2\beta\eta(\alpha\xi+\alpha-1)}{\Lambda}[\|p\|_1+\sum_{k=1}^{m}(\beta_k+\varepsilon)]-\frac{2\beta\eta(\alpha-1)}{\Lambda}[\|q\|_1+\sum_{k=1}^{m}(\beta_k+\varepsilon)].$$

Hence

$$|x'(t)| \le \frac{2\beta\eta(\alpha-1)}{\Lambda}\Bigg[\|p\|_1|x|+\|q\|_1|x'|+\|r\|_1+\sum_{k=1}^{m}(\beta_k+\varepsilon)(|x|+|x'|)\Bigg]+\frac{2\alpha\beta-\beta-1}{\Lambda}\sum_{k=1}^{m}M_k$$
$$= \frac{2\beta\eta(\alpha-1)}{\Lambda}\Bigg[\|q\|_1+\sum_{k=1}^{m}(\beta_k+\varepsilon)\Bigg]|x'|+\frac{2\beta\eta(\alpha-1)}{\Lambda}\Bigg[\|p\|_1+\sum_{k=1}^{m}(\beta_k+\varepsilon)\Bigg]|x|+M_{41}$$

That is

$$\|x'\|_{PC} \le \frac{\frac{2\beta\eta(\alpha-1)}{\Lambda}[\|p\|_1+\sum_{k=1}^{m}(\beta_k+\varepsilon)]}{1-\frac{2\beta\eta(\alpha-1)}{\Lambda}[\|q\|_1+\sum_{k=1}^{m}(\beta_k+\varepsilon)]}\|x\|_{PC}+M_{42}. \quad (14)$$

By (5), (12) and (13), we get

$$|x(t)| \le \frac{1}{\Lambda}\Bigg[\alpha\xi(\beta\eta-1)\int_0^\xi|f(s,x(s),x'(s))|ds$$
$$+\alpha\beta\xi\eta\int_0^\eta|f(s,x(s),x'(s))|ds$$
$$+\alpha\xi\int_0^1|f(s,x(s),x'(s))|ds$$
$$+\alpha(\beta\eta-1)\sum_{0<t_k<\xi}|I_k(x(t_k))|$$
$$+\alpha\beta\xi\sum_{0<t_k<\eta}|I_k(x(t_k))|$$
$$+\alpha\xi\sum_{k=1}^{m}|I_k(x(t_k))|$$
$$+\alpha\xi(\beta\eta-1)\sum_{0<t_k<\xi}|\overline{I}_k(x(t_k),x'(t_k))|$$
$$+\alpha\beta\xi\eta\sum_{0<t_k<\eta}|\overline{I}_k(x(t_k),x'(t_k))|$$
$$+\alpha\xi\sum_{k=1}^{m}|\overline{I}_k(x(t_k),x'(t_k))|\Bigg]$$
$$+\frac{2\beta\eta(\alpha-1)}{\Lambda}\Bigg[\int_0^1|f(s,x(s),x'(s))|ds+\sum_{k=1}^{m}|\overline{I}_k(x(t_k),x'(t_k))|\Bigg]$$
$$+\frac{2\alpha\beta-\beta-1}{\Lambda}\sum_{k=1}^{m}M_k+\sum_{k=1}^{m}M_k$$
$$\le \frac{2\alpha\beta\xi\eta+2\alpha\beta\eta-2\beta\eta}{\Lambda}\Bigg[\int_0^1|f(s,x(s),x'(s))|ds+\sum_{k=1}^{m}|\overline{I}_k(x(t_k),x'(t_k))|\Bigg]$$

$$
\begin{aligned}
&+\frac{2\alpha\beta\eta+2\alpha\xi+2\alpha\beta-2\alpha-\beta\eta-\beta}{\Lambda}\sum_{k=1}^{m}M_k\\
\leq\ & \frac{2\beta\eta(\alpha\xi+\alpha-1)}{\Lambda}\left[\|p\|_1+\sum_{k=1}^{m}(\beta_k+\varepsilon)\right]|x|\\
&+\frac{2\beta\eta(\alpha\xi+\alpha-1)}{\Lambda}\left[\|q\|_1+\sum_{k=1}^{m}(\beta_k+\varepsilon)\right]|x'|\\
&+\frac{2\alpha\beta\eta+2\alpha\xi+2\alpha\beta-2\alpha-\beta\eta-\beta}{\Lambda}\cdot\\
&\sum_{k=1}^{m}M_k. \qquad (15)
\end{aligned}
$$

That is

$$
\begin{aligned}
\|x\|_{PC}\ \leq\ & \frac{\frac{2\beta\eta(\alpha\xi+\alpha-1)}{\Lambda}[\|q\|_1+\sum_{k=1}^{m}(\beta_k+\varepsilon)]}{1-\frac{2\beta\eta(\alpha\xi+\alpha-1)}{\Lambda}[\|p\|_1+\sum_{k=1}^{m}(\beta_k+\varepsilon)]}\cdot\\
&\|x'\|_{PC}+M_{43}. \qquad (16)
\end{aligned}
$$

The combination of (14) and (16), yields

$$
\begin{aligned}
\|x\|_{PC}\ \leq\ & k\cdot\frac{1-\frac{2\beta\eta(\alpha\xi+\alpha-1)}{\Lambda}[\|q\|_1+\sum_{k=1}^{m}(\beta_k+\varepsilon)]}{1-\frac{2\beta\eta(\alpha\xi+\alpha-1)}{\Lambda}[\|p\|_1+\sum_{k=1}^{m}(\beta_k+\varepsilon)]}\\
&+M_{43},
\end{aligned}
$$

where $k=\left[\frac{\frac{2\beta\eta(\alpha-1)}{\Lambda}[\|p\|_1+\sum_{k=1}^{m}(\beta_k+\varepsilon)]}{1-\frac{2\beta\eta(\alpha-1)}{\Lambda}[\|q\|_1+\sum_{k=1}^{m}(\beta_k+\varepsilon)]}\|x\|_{PC}+M_{42}\right]$

Hence

$$\|x\|_{PC}\leq\frac{c}{d}(M_{43}+M_{44}).$$

$$
\begin{aligned}
c\ =\ & [1-\frac{2\beta\eta(\alpha\xi+\alpha-1)}{\Lambda}(p_b+\sum_{k=1}^{m}(\beta_k+\varepsilon))]\\
&[1-\frac{2\beta\eta(\alpha-1)}{\Lambda}(q_b+\sum_{k=1}^{m}(\beta_k+\varepsilon))],
\end{aligned}
$$

$$
\begin{aligned}
d\ =\ & 1-\frac{2\beta\eta(\alpha\xi+\alpha-1)}{\Lambda}[p_b+\sum_{k=1}^{m}(\beta_k+\varepsilon)]\\
&-\frac{2\beta\eta(\alpha-1)}{\Lambda}[q_b+\sum_{k=1}^{m}(\beta_k+\varepsilon)].
\end{aligned}
$$

It lead to $\|x\|_{PC}\leq\overline{M}$ for any $\lambda\in(0,1)$, a contradiction. It is now immediate from (14), that $\|x'\|_{PC}$ is also bounded, so is $\|x'\|_{PC^1}$. This completes the proof.

By using the same method as the proof of Theorem 3.1, we can show that the following Theorem 3.2 - Theorem 3.7 hold.

Theorem 3.2 Let $f:[0,1]\times R^2\to R$ be a continuous function, $I_k\in C[R,R],\ \overline{I}_k\in C[R\times R,R]$. Assume that the conditions (A_1) and (A_2) of Theorem 3.1 are satisfied and (A_4) There exist constants $\alpha,\ \beta,\ \xi,\ \eta$ satisfying (ii). Then BVP(E) has at least one solution in $PC^1[J,R]\cap C^2[J',R]$ provided that

$$\|p\|_1+\|q\|_1+\sum_{k=1}^{m}2\beta_k<\frac{-\Lambda}{2(\alpha\beta\xi\eta+\alpha\beta\xi+\alpha-\alpha\xi-1)}.$$

Theorem 3.3 Let $f:[0,1]\times R^2\to R$ be a continuous function, $I_k\in C[R,R],\ \overline{I}_k\in C[R\times R,R]$. Assume that the conditions (A_1) and (A_2) of Theorem 3.1 are satisfied and
(A_5) There exist constants $\alpha,\ \beta,\ \xi,\ \eta$ satisfying (iii).
Then BVP(E) has at least one solution in $PC^1[J,R]\cap C^2[J',R]$ provided that

$$\|p\|_1+\|q\|_1+\sum_{k=1}^{m}2\beta_k<\frac{-\Lambda}{2(\alpha\beta\xi+\alpha-1)}.$$

Theorem 3.4 Let $f:[0,1]\times R^2\to R$ be a continuous function, $I_k\in C[R,R],\ \overline{I}_k\in C[R\times R,R]$. Assume that the conditions (A_1) and (A_2) of Theorem 3.1 are satisfied and
(A_6) There exist constants $\alpha,\ \beta,\ \xi,\ \eta$ satisfying (iv).
Then BVP(E) has at least one solution in $PC^1[J,R]\cap C^2[J',R]$ provided that

$$\|p\|_1+\|q\|_1+\sum_{k=1}^{m}2\beta_k<\frac{-\Lambda}{2(\alpha\beta\xi\eta+\beta\eta+\alpha\beta\xi-\alpha\beta\eta-\alpha\xi)}.$$

Theorem 3.5 Let $f:[0,1]\times R^2\to R$ be a continuous function, $I_k\in C[R,R],\ \overline{I}_k\in C[R\times R,R]$. Assume that the conditions (A_1) and (A_2) of Theorem 3.1 are satisfied and
(A_7) There exist constants $\alpha,\ \beta,\ \xi,\ \eta$ satisfying (v).
Then BVP(E) has at least one solution in $PC^1[J,R]\cap C^2[J',R]$ provided that

$$\|p\|_1+\|q\|_1+\sum_{k=1}^{m}2\beta_k<\frac{-\Lambda}{2(\beta\eta+\alpha\beta\xi-\alpha\beta\eta)}.$$

Theorem 3.6 Let $f:[0,1]\times R^2\to R$ be a continuous function, $I_k\in C[R,R],\ \overline{I}_k\in C[R\times R,R]$. Assume that the conditions (A_1) and (A_2) of Theorem 3.1 are satisfied and
(A_8) There exist constants $\alpha,\ \beta,\ \xi,\ \eta$ satisfying (vi).
Then BVP(E) has at least one solution in $PC^1[J,R]\cap C^2[J',R]$ provided that

$$\|p\|_1+\|q\|_1+\sum_{k=1}^{m}2\beta_k<\frac{\Lambda}{2(\alpha\xi+1-\alpha)}.$$

Theorem 3.7 Let $f:[0,1]\times R^2\to R$ be a continuous function, $I_k\in C[R,R],\ \overline{I}_k\in C[R\times R,R]$. Assume that the conditions (A_1) and (A_2) of Theorem 3.1 are satisfied and
(A_9) There exist constants $\alpha,\ \beta,\ \xi,\ \eta$ satisfying (vii).
Then BVP(E) has at least one solution in $PC^1[J,R]\cap C^2[J',R]$ provided that

$$\|p\|_1+\|q\|_1+\sum_{k=1}^{m}2\beta_k<\frac{\Lambda}{2(2\alpha\xi+1-\alpha\beta\xi-\beta)}.$$

Remark In cases (iv) and (vi), the special cases: $\alpha=0$ have discussed by Sun [4].

Example Consider the BVP:

$$
\begin{cases}
x''(t)=\frac{1}{240}x+\frac{1}{228}x'+2\ln(1+t^2),\ t\neq\frac{1}{2};\\
\Delta x(\frac{1}{2})=\cos^2 x(\frac{1}{2});\\
\Delta x'(\frac{1}{2})=\frac{1}{228}[x(\frac{1}{2})-x'(\frac{1}{2})];\\
x(0)=3x(\frac{1}{4}),\ x(1)=2x(\frac{3}{4})
\end{cases}
\qquad (E')
$$

Where $f\in C[J\times R^2,R],\ I_1\in C[R,R],\ \overline{I}_1\in C[R^2,R]$.

Note that $m = 1,\ t_1 = \frac{1}{2},\ \alpha = 3$.
Furthermore

$$|f(t,x,y)| \leq \frac{1}{240}|x| + \frac{1}{228}|y| + 2\ln(1+t^2),$$

$$|I_1| = |\cos^2 x| \leq 1,$$

$$|\overline{I}_1(x,y)| \leq \frac{1}{228}(|x|+|y|),\ \forall t \in J,\ x,y \in R.$$

Therefore

$$\frac{(a-1)(\beta-1)}{2\beta\eta(\alpha\xi+2\alpha-2)} = \frac{1}{57},\ \beta_1 = \frac{1}{228},$$

$$\|p\|_1 + \|q\|_1 + 2\beta_1 = \frac{79}{4560} < \frac{1}{57}.$$

Hence from Theorem 3.1, there exises a solution $x \in PC^1[J,R] \cap C^2[J',R]$ to (E'), where $(J' = [0,\frac{1}{2}) \bigcup (\frac{1}{2},1)])$.

REFERENCES

[1] Z. M. He and J. S. Yu, Periodic boundary value problem for first order impulsive functional differential equations. Appl. Math. Comput., **138**(2002)205-217.

[2] D. J. Guo, Multiple positive solutions of boundary value problems for first order nonlinear impulsive intergro-differential equations in a Banach space. Appl. Math. Comput., **143**(2003)233-249.

[3] D. J. Guo, J. X. Sun, Z. L. Liu, *Functional methods of nonlinear ordinary differential equation.* Jinan: shangdong Sci and Tech Press, 2005.

[4] Y. Sun and D. M. Zhu, Existence theorems for a second order three-point boundary value problem with impulses, Appl. Math. J. Chinese Univ. Ser. B., 20(2005)165-174.

[5] D. J. Guo, J. X. Sun, *Ordinary differential equation in abstract space.* Jinan: shangdong Sci and Tech Press, 2005.

[6] Y. C. Zhao, *Nonlinear analysis with applications.* Beijing: Higher Education Press, 1989.