# Existence of Positive Solutions for Second-Order Difference Equation with Discrete Boundary Value Problem 

Thanin Sitthiwirattham, Jiraporn Reunsumrit


#### Abstract

We study the existence of positive solutions to the three points difference-summation boundary value problem. We show the existence of at least one positive solution if $f$ is either superlinear or sublinear by applying the fixed point theorem due to Krasnoselskii in cones.


Keywords-Positive solution, Boundary value problem, Fixed point theorem, Cone.

## I. Introduction

THE study of the existence of solutions of multipoint boundary value problems for linear second-order ordinary differential and difference equations was initiated by Ilin [1]. Then Gupta [2] studied three-point boundary value problems for nonlinear second-order ordinary differential equations. Since then, nonlinear second-order three-point boundary value problems have also been studied by many authors, one may see the text books [3-4] and the papers [6-11]. However, all these papers are concerned with problems with three-point boundary condition restrictions on the difference of the solutions and the solutions themselves, for example,

$$
\begin{aligned}
u(0)=0, & u(T+1)=0 \\
u(0)=0, & a u(s)=u(T+1), \\
u(0)=0, & u(T+1)-a u(s)=b . \\
u(0)-\alpha \Delta u(0)=0, & u(T+1)=\beta u(s) . \\
u(0)-\alpha \Delta u(0)=0, & \Delta u(T+1)=0 \\
u(0)=0, & u(T+1)=\alpha \sum_{s=1}^{\eta} u(s) \\
u(0)=\beta \sum_{s=1}^{\eta} u(s), & u(T+1)=\alpha \sum_{s=1}^{\eta} u(s)
\end{aligned}
$$

and so forth.
In [6], Leggett-Williams developed a fixed point theorem to prove the existence of three positive solutions for Hammerstein integral equations. Since then, this theorem has been reported to be a successful technique for dealing with the existence of three solutions for the two-point boundary value problems of differential and difference equations; see [7,8]. In [9], X. Lin and W. Liu using the properties of the associate Green's

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function and Leggett-Williams fixed point theorem, studied the existence of positive solutions of the problem.
G. Zhang and R. Medina [10], T. Sitthiwirattham and J.Tariboon [11], studied the existence of positive solutions for second order boundary value problems of difference equations by applying the Krasnoselskii's fixed point theorem. In [12], J. Henderson and H.B. Thompson used lower and upper solution methods.

In this paper, we consider the existence of positive solutions to the equation

$$
\begin{equation*}
\Delta^{2} u(t-1)+a(t) f(u)=0, \quad t \in\{1,2, \ldots, T\}, \tag{1}
\end{equation*}
$$

with difference-summation boundary condition

$$
\begin{equation*}
u(0)=\beta \Delta u(0), \quad u(T+1)=\alpha \sum_{s=1}^{\eta} u(s), \tag{2}
\end{equation*}
$$

where $f$ is continuous.
The aim of this paper is to give some results for existence of positive solutions to (1)-(2).
Let $\mathbb{N}$ be the nonnegative integer, we let $\mathbb{N}_{i, j}=\{k \in \mathbb{N} \mid i \leq$ $k \leq j\}$ and $\mathbb{N}_{p}=\mathbb{N}_{0, p}$. By the positive solution of (1)-(2) we mean that a function $u(t): \mathbb{N}_{T+1} \rightarrow[0, \infty)$ and satisfies the problem (1)-(2).

Throughout this paper, we suppose the following conditions hold:
(H1) $T \geq 3$ is a fixed positive integer, $\eta \in\{1,2, \ldots, T-1\}$, constant $\alpha, \beta>0$ such that $0<\alpha<\frac{2(T+1)}{\eta(\eta+1)}$ and $0<\beta<$ $2(T+1)-\alpha \eta(\eta+1)$.
$(H 2) \stackrel{2(\alpha \eta-1)}{f} C([0, \infty),[0, \infty)), f$ is either superlinear or sublinear. Set

$$
f_{0}=\lim _{u \rightarrow 0^{+}} \frac{f(u)}{u}, \quad f_{\infty}=\lim _{u \rightarrow \infty} \frac{f(u)}{u}
$$

Then $f_{0}=0$ and $f_{\infty}=\infty$ correspond to the superlinear case, and $f_{0}=\infty$ and $f_{\infty}=0$ correspond to the sublinear case. (H3) $a \in C\left(\mathbb{N}_{T+1},[0, \infty)\right)$ and there exists $t_{0} \in \mathbb{N}_{\eta, T+1}$ such that $a\left(t_{0}\right)>0$.

The proof of the main theorem is based upon an application of the following Krasnoselskii's fixed point theorem in a cone.

Theorem 1. ([5]). Let $E$ be a Banach space, and let $K \subset E$ be a cone. Assume $\Omega_{1}, \Omega_{2}$ are open subsets of $E$ with $0 \in \Omega_{1}$, $\bar{\Omega}_{1} \subset \Omega_{2}$, and let

$$
A: K \cap\left(\bar{\Omega}_{2} \backslash \Omega_{1}\right) \longrightarrow K
$$

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be a completely continuous operator such that
(i) $\|A u\| \leqslant\|u\|, \quad u \in K \cap \partial \Omega_{1}$, and $\|A u\| \geqslant\|u\|, \quad u \in$ $K \cap \partial \Omega_{2} ;$ or
(ii) $\|A u\| \geqslant\|u\|, \quad u \in K \cap \partial \Omega_{1}$, and $\|A u\| \leqslant\|u\|, \quad u \in$ $K \cap \partial \Omega_{2}$.
Then $A$ has a fixed point in $K \cap\left(\bar{\Omega}_{2} \backslash \Omega_{1}\right)$.

## II. Preliminaries

We now state and prove several lemmas before stating our main results.
Lemma 1. The problem

$$
\begin{gather*}
\Delta^{2} u(t-1)+y(t)=0, \quad t \in \mathbb{N}_{1, T},  \tag{3}\\
u(0)=\beta \Delta u(0), \quad u(T+1)=\alpha \sum_{s=1}^{\eta} u(s), \tag{4}
\end{gather*}
$$

has a unique solution

$$
\begin{aligned}
u(t)= & \frac{2(t+\beta)}{2(T+1)-\alpha \eta(\eta+1)-2 \beta(\alpha \eta-1)} \times \\
& \sum_{s=1}^{T}(T-s+1) y(s) \\
& -\frac{\alpha(t+\beta)}{2(T+1)-\alpha \eta(\eta+1)-2 \beta(\alpha \eta-1)} \times \\
& \sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1) y(s) \\
& -\sum_{s=1}^{t-1}(t-s) y(s), \quad t \in \mathbb{N}_{T+1} .
\end{aligned}
$$

Proof. From $\Delta^{2} u(t-1)=\Delta u(t)-\Delta u(t-1)$ and the first equation of (3), we get

$$
\begin{aligned}
\Delta u(t)-\Delta u(t-1) & =-y(t) \\
\Delta u(t-1)-\Delta u(t-2) & =-y(t-1)
\end{aligned}
$$

$$
\Delta u(1)-\Delta u(0)=-y(1) .
$$

We sum the above equations to obtain

$$
\begin{equation*}
\Delta u(t)=\Delta u(0)-\sum_{s=1}^{t} y(s), t \in \mathbb{N}_{T} \tag{5}
\end{equation*}
$$

We define $\sum_{s=p}^{q} y(s)=0$; if $p<q$. Similarly, we sum (5) from $t=0$ to $t=h$, and by using the boundary condition $u(0)=\beta \Delta u(0)$ in (4), we obtain
$u(h+1)=(h+1+\beta) \Delta u(0)-\sum_{s=1}^{h}(h+1-s) y(s), h \in \mathbb{N}_{T}$, by changing the variable from $h+1$ to $t$, we have

$$
\begin{equation*}
u(t)=(t+\beta) \Delta u(0)-\sum_{s=1}^{t-1}(t-s) y(s), t \in \mathbb{N}_{T+1} \tag{6}
\end{equation*}
$$

From (6),

$$
\begin{aligned}
\sum_{s=1}^{\eta} u(s)= & \left(\frac{1}{2} \eta(\eta+1)+\beta \eta\right) \Delta u(0)-\sum_{s=1}^{\eta-1} \sum_{l=1}^{\eta-s} l y(s) \\
= & \left(\frac{1}{2} \eta(\eta+1)+\beta \eta\right) \Delta u(0) \\
& -\frac{1}{2} \sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1) y(s)
\end{aligned}
$$

Again using the boundary condition $u(T+1)=\alpha \sum_{s=1}^{\eta} u(s)$ in (4), we obtain
$(T+1+\beta) \Delta u(0)-\sum_{s=1}^{T}(T-s+1) y(s)=$
$\alpha\left(\frac{1}{2} \eta(\eta+1)+\beta \eta\right) \Delta u(0)-\frac{\alpha}{2} \sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1) y(s)$ Thus,

$$
\begin{aligned}
\Delta u(0)= & \frac{2}{2(T+1)-\alpha \eta(\eta+1)-2 \beta(\alpha \eta-1)} \times \\
& \sum_{s=1}^{T}(T-s+1) y(s) \\
& -\frac{\alpha}{2(T+1)-\alpha \eta(\eta+1)-2 \beta(\alpha \eta-1)} \times \\
& \sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1) y(s) .
\end{aligned}
$$

Therefore, (3)-(4) has a unique solution

$$
\begin{aligned}
u(t)= & \frac{2(t+\beta)}{2(T+1)-\alpha \eta(\eta+1)-2 \beta(\alpha \eta-1)} \times \\
& \sum_{s=1}^{T}(T-s+1) y(s) \\
& -\frac{\alpha(t+\beta)}{2(T+1)-\alpha \eta(\eta+1)-2 \beta(\alpha \eta-1)} \times \\
& \sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1) y(s) \\
& -\sum_{s=1}^{t-1}(t-s) y(s), \quad t \in \mathbb{N}_{T+1} .
\end{aligned}
$$

Lemma 2. The function

$$
G(t, s)=\frac{1}{\Lambda}\left\{\begin{array}{c}
(s+\beta)[2(T+1)-\alpha \eta(\eta+1)+2 t(\alpha \eta-1)]  \tag{7}\\
\quad+\alpha s(t+\beta)(1-s), s \in \mathbb{N}_{1, t-1} \cap \mathbb{N}_{1, \eta-1} \\
2(s+\beta)(T+1-t)+\alpha \eta(t-s)(\eta+1+2 \beta), \\
s \in \mathbb{N}_{\eta, t-1} \\
(t+\beta)[2(T+1)-\alpha \eta(\eta+1)+2 s(\alpha \eta-1)+ \\
\alpha s(1-s)], s \in \mathbb{N}_{t, \eta-1} \\
2(T+\beta)(T+1-s), s \in \mathbb{N}_{t, T} \cap \mathbb{N}_{\eta, T}
\end{array}\right.
$$

where

$$
\Lambda=2(T+1)-\alpha \eta(\eta+1)-2 \beta(\alpha \eta-1)>0
$$

is the Green's function of the problem

$$
\begin{align*}
& -\Delta^{2} u(t-1)=0, \quad t \in \mathbb{N}_{1, T}, \\
& u(0)=\beta \Delta u(0), \quad u(T+1)=\alpha \sum_{s=1}^{\eta} u(s) . \tag{8}
\end{align*}
$$

Proof. Suppose $t<\eta$. The unique solution of problem (3)-(4) can be written

$$
\begin{aligned}
u(t)= & -\sum_{s=1}^{t-1}(t-s) y(s)+\frac{2(t+\beta)}{\Lambda}\left[\sum_{s=1}^{t-1}(T-s+1) y(s) \times\right. \\
& \left.+\sum_{s=t}^{\eta-1}(T-s+1) y(s)+\sum_{s=\eta}^{T}(T-s+1) y(s)\right] \\
& -\frac{\alpha(t+\beta)}{\Lambda}\left[\sum_{s=1}^{t-1}(\eta-s)(\eta-s+1) y(s)\right. \\
& \left.+\sum_{s=t}^{\eta-1}(\eta-s)(\eta-s+1) y(s)\right] \\
= & \frac{1}{\Lambda} \sum_{s=1}^{t-1}[(s+\beta)[2(T+1)-\alpha \eta(\eta+1)] \\
& +\alpha s(t+\beta)(1-s)] y(s) \\
& +\frac{1}{\Lambda} \sum_{s=t}^{\eta-1}[(t+\beta)[2(T+1)-\alpha \eta(\eta+1)+2 s(\alpha \eta-1) \\
& \left.\left.+\alpha s-\alpha s^{2}\right]\right] y(s) \\
& +\frac{1}{\Lambda} \sum_{s=\eta}^{T} 2(T+\beta)(T+1-s) y(s) \\
= & \sum_{s=1}^{T} G(t, s) y(s) .
\end{aligned}
$$

Suppose $t \geq \eta$. The unique solution of problem (3)-(4) can be written

$$
\begin{aligned}
u(t)= & -\sum_{s=1}^{\eta-1}(t-s) y(s)-\sum_{s=\eta}^{t-1}(t-s) y(s) \\
& +\frac{2(t+\beta)}{\Lambda}\left[\sum_{s=1}^{\eta-1}(T-s+1) y(s)+\sum_{s=\eta}^{t-1}(T-s+1) y(s)\right. \\
& \left.+\sum_{s=t}^{T}(T-s+1) y(s)\right] \\
& -\frac{\alpha(t+\beta)}{\Lambda} \sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1) y(s)
\end{aligned}
$$

$$
\begin{aligned}
u(t)= & \frac{1}{\Lambda} \sum_{s=1}^{\eta-1}[(s+\beta)[2(T+1)-\alpha \eta(\eta+1)] \\
& +\alpha s(t+\beta)(1-s)] y(s)+\frac{1}{\Lambda} \sum_{s=\eta}^{t-1}[2(s+\beta)(T+1-t) \\
& +\alpha \eta(t-s)(\eta+1+2 \beta)] y(s) \\
& +\frac{1}{\Lambda} \sum_{s=t}^{T} 2(T+\beta)(T+1-s) \\
= & \sum_{s=1}^{T} G(t, s) y(s) .
\end{aligned}
$$

Then the unique solution of problem (3)-(4) can be written as $u(t)=\sum_{s=1}^{T} G(t, s) y(s)$. The proof is complete.

We observe that the condition $0<\alpha<\frac{2(T+1)}{\eta(\eta+1)}$ and $0<\beta<$ $\frac{2(T+1)-\alpha \eta(\eta+1)}{2(\alpha \eta-1)}$. implies $G(t, s)$ is positive on $\mathbb{N}_{1, T} \times \mathbb{N}_{1, T}$, which mean that the finite set

$$
\left\{\frac{G(t, s)}{G(t, t)}: t \in \mathbb{N}_{T+1}, s \in \mathbb{N}_{1, T}\right\}
$$

take positive values. Then we let

$$
\begin{align*}
& M_{1}=\min \left\{\frac{G(t, s)}{G(t, t)}: t \in \mathbb{N}_{T+1}, s \in \mathbb{N}_{1, T}\right\}  \tag{9}\\
& M_{2}=\max \left\{\frac{G(t, s)}{G(t, t)}: t \in \mathbb{N}_{T+1}, s \in \mathbb{N}_{1, T}\right\} \tag{10}
\end{align*}
$$

Lemma 3. Let $(t, s) \in \mathbb{N}_{1, T} \times \mathbb{N}_{1, T}$. Then we have

$$
\begin{equation*}
G(t, s) \geq M_{1} G(t, t .) \tag{11}
\end{equation*}
$$

where $0<M_{1}<1$ is a constant given by

Proof. In order that (11) holds, it is sufficient that $M_{1}$ satisfies

$$
\begin{equation*}
M_{1} \leq \min _{(t, s) \in \mathbb{N}_{1, T} \times \mathbb{N}_{1, T}} \frac{G(t, s)}{G(t, t)} . \tag{13}
\end{equation*}
$$

Then we may choose
$M_{1} \leq \min \left\{\min _{(t, s) \in \mathbb{N}_{1, \eta-1} \times \mathbb{N}_{1, T}} \frac{G(t, s)}{G(t, t)}, \min _{(t, s) \in \mathbb{N}_{\eta, T} \times \mathbb{N}_{1, T}} \frac{G(t, s)}{G(t, t)}\right\}$.

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since

$$
\begin{aligned}
& (t, s) \in \mathbb{N}_{1, \eta-1} \times \mathbb{N}_{1, T} \frac{G(t, s)}{G(t, t)} \\
& \geq\left\{\begin{array}{l}
\min _{t \in \mathbb{N}_{1, \eta-1}} \\
\left\{\begin{array}{l}
\left\{\frac{(1+\beta)[2(T+1)-\alpha \eta(\eta+1)+2 t(\alpha \eta-1)]+\alpha(t+\beta)(2-t)}{(t+\beta)[2(T+1)-\alpha \eta(\eta+1)+2 t(\alpha \eta-1)+\alpha t(1-t)]},\right. \\
\frac{2(T+1)-\alpha \eta(\eta+1)-2(\eta-1)+\alpha(2-\eta)}{2(T+1)-\alpha \eta(\eta+1)+2 t(\alpha \eta-1)+\alpha t(1-t)}, \\
\frac{2}{2(T+1)-\alpha \eta(\eta+1)+2 t(\alpha \eta-1)+\alpha t(1-t)}
\end{array}\right\} ; \quad \text { if } \alpha>\frac{1}{\eta} \\
\min _{t \in \mathbb{N}_{1, \eta-1}} \\
\left\{\begin{array}{l}
\frac{(1+\beta)[2(T+1)-\alpha \eta(\eta+1)+2 t(\alpha \eta-1)]+\alpha(t+\beta)(2-t)}{(t+\beta)[2(T+1)-\alpha \eta(\eta+1)+2 t(\alpha \eta-1)+\alpha t(1-t)]}, \\
\frac{2(T+1)-\alpha \eta(\eta+1)-2(\eta-1)(\alpha \eta-1)+\alpha(2-\eta)}{2(T+1)-\alpha \eta(\eta+1)+2 t(\alpha \eta-1)+\alpha t(1-t)}, \\
\left.\frac{2}{2(T+1)-\alpha \eta(\eta+1)+2 t(\alpha \eta-1)+\alpha t(1-t)}\right\} ;
\end{array}\right.
\end{array}, \begin{array}{l}
\text { if } 0<\alpha<\frac{1}{\eta}
\end{array}\right.
\end{aligned}
$$

$$
\geq\left\{\begin{aligned}
\min & \left\{\frac{(1+\beta)[2 T-\alpha \eta(\eta+4)+3 \alpha]}{(\eta+\beta-1)[2(T+2)+\alpha \eta(\eta-3)-2 \eta]}, \frac{2(T+2+\alpha)-\alpha \eta(\eta+4)}{2(T+2)+\alpha \eta(\eta-3)-2 \eta},\right. \\
& \left.\frac{2}{2(T+2)+\alpha \eta(\eta-3)-2 \eta]}\right\} ; \text { if } \alpha>\frac{1}{\eta} \\
\min \quad & \left\{\frac{(1+\beta)[2(T+2)+\alpha \eta(\eta-4)-2 \eta+3 \alpha]}{(\eta+\beta-1)[2 T-\alpha \eta(\eta-1)]}, \frac{2(T+\eta+\alpha)-3 \alpha \eta^{2}}{2 T-\alpha \eta(\eta-1)},\right. \\
& \left.\frac{2}{2 T-\alpha \eta(\eta-1)}\right\} ; \text { if } 0<\alpha<\frac{1}{\eta}
\end{aligned}\right.
$$

Similarly, we get

$$
\begin{align*}
& \min _{(t, s) \in \mathbb{N}_{\eta, T} \times \mathbb{N}_{1, T}} \frac{G(t, s)}{G(t, t)} \\
& \geq\left\{\begin{aligned}
& \min \quad\left\{\frac{(1+\beta)[2(T+1-\eta)-\alpha \eta(3 \eta+1)]+\alpha(\eta+\beta)(2-\eta)}{2(T+\beta)(T+1-\eta)}\right. \\
& \frac{2(\eta+\beta)+\alpha \eta(\eta+1+2 \beta)}{2(T+\beta)(T+1-\eta)}, \\
&\left.\frac{1}{2(T+\beta)(T+1-\eta)}\right\} ; \text { if } \alpha>\frac{1}{\eta} \\
& \min \quad\left\{\frac{(1+\beta)[\alpha \eta(2 T-\eta-1)+2]+\alpha(\eta+\beta)(2-\eta)}{2(T+\beta)(T+1-\eta)}\right. \\
& \frac{2(\eta+\beta)+\alpha \eta(\eta+1+2 \beta)}{2(T+\beta)(T+1-\eta)}, \\
&\left.\frac{1}{2(T+\beta)(T+1-\eta)}\right\} ; \text { if } 0<\alpha<\frac{1}{\eta}
\end{aligned}\right. \tag{15}
\end{align*}
$$

The (12) is immediate from (15)-(16)

Lemma 4. Let $(t, s) \in \mathbb{N}_{T+1} \times \mathbb{N}_{1, T}$. Then we have

$$
\begin{equation*}
G(t, s) \leq M_{2} G(t, t .) \tag{16}
\end{equation*}
$$

where $M_{2} \geq 1$ is a constant given by
$M_{2}=\left\{\begin{aligned} \max & \left\{\frac{2(T+1-\eta)}{2(T+\alpha)-\alpha \eta^{2}}, \frac{(\eta-1+\beta)[\alpha \eta(2 T-\eta-1)+2]}{2(\eta+\beta)},\right. \\ & \left.\frac{2(T-1+\beta)+\alpha \eta(T-\eta)(\eta+1+2 \beta)}{2(\eta+\beta)}, 1\right\} ; \text { if } \alpha>\frac{1}{\eta} \\ \max & \left\{\frac{2(T+1-\eta)}{2(T+2-\eta-\alpha)+\alpha \eta(\eta-4)},\right. \\ & \frac{(\eta-1+\beta)[2(T+1-\eta)+\alpha \eta(\eta-1)}{2(\eta+\beta)}, \\ & \left.\frac{2(T-1+\beta)+\alpha \eta(T-\eta)(\eta+1+2 \beta)}{2(\eta+\beta)}, 1\right\} ; \text { if } 0<\alpha<\frac{1}{\eta}\end{aligned}\right.$

Proof. For $k=0$, from (7) we get

$$
G(0, s)=2 \beta(T+1-s)<2 \beta(T+1)=G(0,0) .
$$

Then we may choose $M_{2}=1$. For $k \in \mathbb{N}_{1, T}$, if (16) holds, it is sufficient that $M_{2}$ satisfies

$$
\begin{equation*}
M_{2} \geq \max _{(t, s) \in \mathbb{N}_{1, T} \times \mathbb{N}_{1, T}} \frac{G(t, s)}{G(t, t)} . \tag{18}
\end{equation*}
$$

Then we may choose
$M_{2} \leq \min \left\{\max _{(t, s) \in \mathbb{N}_{1, \eta-1} \times \mathbb{N}_{1, T}} \frac{G(t, s)}{G(t, t)}, \max _{(t, s) \in \mathbb{N}_{\eta, T} \times \mathbb{N}_{1, T}} \frac{G(t, s)}{G(t, t)}\right\}$.
since

Similarly, we get

$$
\max _{(t, s) \in \mathbb{N}_{\eta, T} \times \mathbb{N}_{1, T}} \frac{G(t, s)}{G(t, t)}
$$

$$
\geq\left\{\begin{aligned}
\max & \left\{\frac{(\eta-1+\beta)[\alpha \eta(2 T-\eta-1)+2]}{2(\eta+\beta)},\right. \\
& \left.\frac{2(T-1+\beta)+\alpha \eta(T-\eta)(\eta+1+2 \beta)}{2(\eta+\beta)}, 1\right\} ; \text { if } \alpha>\frac{1}{\eta} \\
\max & \left\{\frac{(\eta-1+\beta)[2(T+1-\eta)+\alpha \eta(\eta-1)}{2(\eta+\beta)},\right. \\
& \left.\frac{2(T-1+\beta)+\alpha \eta(T-\eta)(\eta+1+2 \beta)}{2(\eta+\beta)}, 1\right\} ; \quad \text { if } 0<\alpha<\frac{1}{\eta}
\end{aligned}\right.
$$

For $k=T+1 \mathrm{~m}$ from (7) we get,

$$
\begin{aligned}
G(T+1, s)= & \alpha \eta(s+\beta)[2(T+1)-(\eta+1)] \\
& +2 s(T+1+\beta)(1-s) \\
< & \alpha \eta(T+1+\beta)[2(T+1)-(\eta+1)] \\
& +2 T(T+1+\beta)(T+1) \\
= & G(T+1, T+1) .
\end{aligned}
$$

$$
\begin{align*}
& \max _{(t, s) \in \mathbb{N}_{1, \eta-1} \times \mathbb{N}_{1, T}} \frac{G(t, s)}{G(t, t)}  \tag{19}\\
& \geq\left\{\begin{array}{l}
\max _{t \in \mathbb{N}_{1, \eta-1}} \\
\left\{\frac{(t-1+\beta)[2(T+1)-\alpha \eta(\eta+1)+2 t(\alpha \eta-1)]}{(t+\beta)[2(T+1)-\alpha \eta(\eta+1)+2 t(\alpha \eta-1)+\alpha t(1-t)]},\right. \\
\frac{2(T+1)-\alpha \eta(\eta+1)+2(\eta-1)(\alpha \eta-1)+\alpha(\eta-1)(1-t)}{2(T+1)-\alpha \eta(\eta+1)+2 t(\alpha \eta-1)+\alpha t(1-t)}, \\
\left.\frac{2(T+1-\eta)}{2(T+1)-\alpha \eta(\eta+1)+2 t(\alpha \eta-1)+\alpha t(1-t)}\right\} ; \quad \text { if } \alpha>\frac{1}{\eta} \\
\max _{t \in \mathbb{N}_{1, \eta-1}} \\
\left\{\begin{array}{l}
\frac{(t-1+\beta)[2(T+1)-\alpha \eta(\eta+1)+2 t(\alpha \eta-1)]}{(t+\beta)[2(T+1)-\alpha \eta(\eta+1)+2 t(\alpha \eta-1)+\alpha t(1-t)]}, \\
\frac{2(T+1)-\alpha \eta(\eta+1)+2 t(\alpha \eta-1)+\alpha(\eta-1)(1-t)}{2(T+1)-\alpha \eta(\eta+1)+2 t(\alpha \eta-1)+\alpha t(1-t)}, \\
\left.\frac{2(T+1-\eta)}{2(T+1)-\alpha \eta(\eta+1)+2 t(\alpha \eta-1)+\alpha t(1-t)}\right\} ; \quad \text { if } 0<\alpha<\frac{1}{\eta}
\end{array}\right.
\end{array}\right. \\
& \left\{\operatorname { m a x } \left\{\frac{(\eta-2+\beta)[2(T+2-\eta)+\alpha \eta(\eta-3)}{(1+\beta)\left[2(T+\alpha)-\alpha \eta^{2}\right]}, \frac{2(T+2-\eta)+\alpha \eta(\eta-3)}{2(T+\alpha)-\alpha \eta^{2}},\right.\right. \\
& \left.\frac{2(T+1-\eta)}{2(T+\alpha)-\alpha \eta^{2}}\right\} ; \quad \text { if } \alpha>\frac{1}{\eta} \\
& \geq\left\{\begin{array}{l}
\quad \begin{array}{l}
2(T+\alpha)-\alpha \eta \\
\max
\end{array} \frac{(\eta-2+\beta)[2 T-\alpha \eta(\eta-1)}{(1+\beta)[2(T+2-\eta-\alpha)+\alpha \eta(\eta-4)]}, \frac{2 T-\alpha \eta(\eta-1)}{2(T+2-\eta-\alpha)+\alpha \eta(\eta-4)},
\end{array}\right. \\
& \left.\frac{2(T+1-\eta)}{2(T+2-\eta-\alpha)+\alpha \eta(\eta-4)}\right\} ; \quad \text { if } 0<\alpha<\frac{1}{\eta}
\end{align*}
$$

Then we choose $M_{2}=1$. So (18) is immediate from (21)-(22).

## III. Main Results

Now we are in the position to establish the main result.
Theorem 2. Assume $(H 1)$ - $(H 3)$ hold. Then the problem (1)-(2) has at least one positive solution.

Proof. In the following, we denote

$$
m=\min _{t \in \mathbb{N}_{\eta, T}} G(t, t), \quad M=\max _{t \in \mathbb{N}_{T+1}} G(t, t)
$$

Then $0<m<M$.
Let Ebe the Banach's space defined by $E=\left\{u: \mathbb{N}_{T+1} \rightarrow\right.$ $R\}$. Define
$K=\left\{u \in E: u \geqslant 0, t \in \mathbb{N}_{T+1}\right.$ and $\left.\min _{t \in \mathbb{N}_{1, T}} u(t) \geq \sigma\|u\|\right\}$.
where $\sigma=\frac{M_{1} m}{M_{2} M} \in(0,1),\|u\|=\max _{t \in \mathbb{N}_{T+1}}|u(t)|$. It is obvious that $K$ is a cone in $E$.

We define the operator $F: K \rightarrow E$ by

$$
(F u)(t)=\sum_{s=1}^{T} G(t, s) a(s) f(u(s)), t \in \mathbb{N}_{T+1}
$$

It is clear that problem (1)-(2) has a solution $u$ if and only if $u \in K$ is a fixed point of operator $F$. We shall now show that the operator $F$ maps $K$ to itself. For this, let $u \in K$, from $\left(H_{2}\right)-\left(H_{3}\right)$, we get

$$
\begin{equation*}
(F u)(t)=\sum_{s=1}^{T} G(t, s) a(s) f(u(s)) \geq 0, t \in \mathbb{N}_{T+1} \tag{20}
\end{equation*}
$$

from (10), we obtain

$$
\begin{aligned}
(F u)(t) & =\sum_{s=1}^{T} G(t, s) a(s) f(u(s)) \leq M_{2} \sum_{s=1}^{T} G(t, t) a(s) f(u(s)) \\
& \leq M_{2} M \sum_{s=1}^{T} a(s) f(u(s)), \quad t \in \mathbb{N}_{T+1}
\end{aligned}
$$

Therefore

$$
\begin{equation*}
\|F u\| \leq M_{2} M \sum_{s=1}^{T} a(s) f(u(s)) \tag{21}
\end{equation*}
$$

Now from $\left(H_{2}\right),\left(H_{3}\right)$, (2.7) and (3.2), for $t \in \mathbb{N}_{\eta, T}$, we have

$$
\begin{aligned}
(F u)(t) & \geq M_{1} \sum_{s=1}^{T} G(t, t) a(s) f(u(s)) \geq M_{1} m \sum_{s=1}^{T} a(s) f(u(s)) \\
& \geq \frac{M_{1} m}{M_{2} M}\|F u\|=\sigma\|u\| .
\end{aligned}
$$

Then

$$
\begin{equation*}
\min _{t \in \mathbb{N}_{\eta, T}}(F u)(t) \geq \sigma\|u\| \tag{22}
\end{equation*}
$$

From (20)-(21), we obtain $F u \in K$, Hence $F(K) \subseteq K$. So $F: k \rightarrow K$ is completely continuous.

Superlinear case. $f_{0}=0$ and $f_{\infty}=\infty$. Since $f_{0}=0$, we may choose $H_{1}>0$ so that $f(u) \leqslant \epsilon_{1} u$, for $0<u \leqslant H_{1}$, where $\epsilon_{1}>0$ satisfies

$$
\begin{equation*}
\epsilon_{1} M_{2} M \sum_{s=1}^{T} a(s) \leq 1 \tag{23}
\end{equation*}
$$

Thus, if we let

$$
\Omega_{1}=\left\{u \in E:\|u\|<H_{1}\right\}
$$

then for $u \in K \cap \partial \Omega_{1}$, we get

$$
\begin{aligned}
(F u)(t) & \leq M_{2} \sum_{s=1}^{T} G(t, t) a(s) f(u(s)) \leq \epsilon_{1} M_{2} M \sum_{s=1}^{T} a(s) u(s) \\
& \leq \epsilon_{1} M_{2} M \sum_{s=1}^{T} a(s)\|u\| \leq\|u\|
\end{aligned}
$$

Thus $\|F u\| \leq\|u\|, u \in K \cap \partial \Omega_{1}$.
Further, since $f_{\infty}=\infty$, there exists $\widehat{H}_{2}>0$ such that $f(u) \geq \epsilon_{2} u$, for $u \geq \widehat{H}_{2}$, where $\epsilon_{2}>0$ satisfies

$$
\begin{equation*}
\epsilon_{2} M_{1} \sigma \sum_{s=\eta}^{T} G(\eta, \eta) a(s) \geq 1 \tag{24}
\end{equation*}
$$

Let $H_{2}=\max \left\{2 H_{1}, \frac{\widehat{H}_{2}}{\sigma}\right\}$ and $\Omega_{2}=\left\{u \in E:\|u\|<H_{2}\right\}$. Then $u \in K \cap \partial \Omega_{2}$ implies

$$
\min _{t \in \mathbb{N}_{\eta, T}} u(t) \geq \sigma\|u\| \geq \widehat{H}_{2}
$$

Applying (9) and (24), we get

$$
\begin{aligned}
(F u)(\eta) & =M_{1} \sum_{s=1}^{T} G(\eta, s) a(s) f(u(s)) \geq M_{1} \sum_{s=\eta}^{T} G(\eta, \eta) a(s) f(u(s)) \\
& \geq \varepsilon_{2} M_{1} \sum_{s=\eta}^{T} G(\eta, \eta) a(s) y(s) \geq \varepsilon_{2} M_{1} \sigma \sum_{s=\eta}^{T} G(\eta, \eta) a(s)\|u\| \\
& \geq\|u\|
\end{aligned}
$$

Hence, $\|F u\| \geq\|u\|, u \in K \cap \partial \Omega_{2}$. By the first part of Theorem $1, F$ has a fixed point in $K \cap\left(\bar{\Omega}_{2} \backslash \Omega_{1}\right)$ such that $H_{1} \leqslant\|u\| \leqslant H_{2}$.
Sublinear case. $f_{0}=\infty$ and $f_{\infty}=0$. Since $f_{0}=\infty$, choose $H_{3}>0$ such that $f(u) \geqslant \epsilon_{3} u$ for $0<u \leqslant H_{3}$, where $\varepsilon_{3}>0$ satisfies

$$
\begin{equation*}
\epsilon_{3} M_{1} \sigma \sum_{s=\eta}^{T} G(\eta, \eta) a(s) \geqslant 1 \tag{25}
\end{equation*}
$$

Let

$$
\Omega_{3}=\left\{u \in E:\|u\|<H_{3}\right\}
$$

then for $u \in K \cap \partial \Omega_{3}$, we get

$$
\begin{aligned}
(F u)(\eta) & \geq M_{1} \sum_{s=\eta}^{T} G(\eta, \eta) a(s) f(u(s)) \geq \epsilon_{3} M_{1} \sum_{s=\eta}^{T} G(\eta, \eta) a(s) y(s) \\
& \geq \varepsilon_{3} M_{1} \sigma \sum_{s=\eta}^{T} G(\eta, \eta) a(s)\|u\| \geq\|u\|
\end{aligned}
$$

Thus, $\|F u\| \geqslant\|u\|, u \in K \cap \partial \Omega_{3}$.

Now, since $f_{\infty}=0$, there exists $\widehat{H}_{4}>0$ so that $f(u) \leqslant \epsilon_{4} u$ for $u \geqslant \widehat{H}_{4}$, where $\epsilon_{4}>0$ satisfies

$$
\begin{equation*}
\epsilon_{4} M_{2} M \sum_{s=\eta}^{T} a(s) \geqslant 1 . \tag{26}
\end{equation*}
$$

Subcase 1. Suppose $f$ is bounded, $f(u) \leq L$ for all $u \in[0, \infty)$ for some $L>0$. Let $H_{4}=$ $\max \left\{2 H_{3}, L M_{2} M \sum_{s=1}^{T} a(s)\right\}$.

Then for $u \in K$ and $\|u\|=H_{4}$, we get

$$
\begin{aligned}
(F u)(\eta) & \leq M_{2} \sum_{s=1}^{T} G(t, t) a(s) f(u(s)) \leq L M_{2} M \sum_{s=1}^{T} a(s) \\
& \leq H_{4}=\|u\|
\end{aligned}
$$

Thus $(F u)(t) \leq\|u\|$.
Subcase 2. Suppose $f$ is unbounded, there exist $H_{4}>$ $\max \left\{2 H_{3}, \frac{\widehat{H}_{4}}{\sigma}\right\}$ such that $f(u) \leq f\left(H_{4}\right)$ for all $0<u \leq H_{4}$. Then for $u \in K$ with $\|u\|=H_{4}$ from (10) and (26), we have

$$
\begin{aligned}
(F u)(t) & \leq M_{2} \sum_{s=1}^{T} G(t, t) a(s) f(u(s)) \leq M_{2} M \sum_{s=1}^{T} a(s) f\left(H_{4}\right) \\
& \leq \epsilon_{4} M_{2} M \sum_{s=1}^{T} a(s) H_{4} \leq H_{4}=\|u\| .
\end{aligned}
$$

Thus in both cases, we may put Omega $_{4}=\{u \in E$ : $\left.\|u\|<H_{4}\right\}$. Then

$$
\|F u\| \leqslant\|u\|, u \in K \cap \partial \Omega_{4} .
$$

By the second part of Theorem 1, $A$ has a fixed point $u$ in $K \cap\left(\bar{\Omega}_{4} \backslash \Omega_{3}\right)$, such that $H_{3} \leqslant\|u\| \leqslant H_{4}$. This completes the sublinear part of the theorem. Therefore, the problem (1)-(2) has at least one positive solution.

## IV. Some examples

In this section, in order to illustrate our result, we consider some examples.
Example 1 Consider the BVP

$$
\begin{align*}
& \Delta^{2} u(t-1)+t^{2} u^{k}=0, \quad t \in N_{1,4},  \tag{27}\\
& u(0)=\frac{1}{4} \Delta u(0), \quad u(5)=\frac{2}{3} \sum_{s=1}^{2} u(s) . \tag{28}
\end{align*}
$$

Set $\alpha=\frac{2}{3}, \beta=\frac{1}{4}, \eta=2, T=4, a(t)=t^{2}, f(u)=u^{k}$. We can show that

$$
2(T+1)-\alpha \eta(\eta+1)-2 \beta(\alpha \eta-1)=\frac{40}{6}>0 .
$$

Case I : $k \in(1, \infty)$. In this case, $f_{0}=0, f_{\infty}=\infty$ and $(i)$ of theorem 2 holds. Then BVP (27)-(28) has at least one positive solution.
Case II : $k \in(0,1)$. In this case, $f_{0}=\infty, f_{\infty}=0$ and (ii) of theorem 2 holds. Then BVP (27)-(28) has at least one positive solution.

## Example 2 Consider the BVP

$$
\begin{align*}
\Delta^{2} u(t-1)+e^{t} t^{e}\left(\frac{\pi \sin u+2 \cos u}{u^{2}}\right) & =0, \\
u(0)=\frac{2}{5} \Delta u(0), \quad u(5) & =\frac{1}{3} \sum_{s=1}^{3} u(s), \tag{29}
\end{align*}
$$

Set $\alpha=\frac{1}{3}, \beta=\frac{2}{5}, \eta=3, T=4, a(t)=e^{t} t^{e}, \quad f(u)=$ $\frac{\pi \sin u+2 \cos u}{u^{2}}$.
We can show that

$$
\Lambda=2(T+1)-\alpha \eta(\eta+1)-2 \beta(\alpha \eta-1)=6>0
$$

Through a simple calculation we can get $f_{0}=\infty, f_{\infty}=0$. Thus, by (ii) of theorem 2, we can get BVP (29)-(30) has at least one positive solution.

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