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Evaluation of Linear and Geometrically Nonlinear Static and Dynamic Analysis of Thin Shells by Flat Shell Finite Elements

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***Abstract*—**The choice of finite element to use in order to predict nonlinear static or dynamic response of complex structures becomes an important factor. Then, the main goal of this research work is to focus a study on the effect of the in-plane rotational degrees of freedom in linear and geometrically non linear static and dynamic analysis of thin shell structures by flat shell finite elements. In this purpose: First, simple triangular and quadrilateral flat shell finite elements are implemented in an incremental formulation based on the updated lagrangian corotational description for geometrically nonlinear analysis. The triangular element is a combination of DKT and CST elements, while the quadrilateral is a combination of DKQ and the bilinear quadrilateral membrane element. In both elements, the sixth degree of freedom is handled via introducing fictitious stiffness. Secondly, in the same code, the sixth degrees of freedom in these elements is handled differently where the in-plane rotational *d.o.f* is considered as an effective *d.o.f* in the in-plane filed interpolation. Our goal is to compare resulting shell elements. Third, the analysis is enlarged to dynamic linear analysis by direct integration using Newmark's implicit method. Finally, the linear dynamic analysis is extended to geometrically nonlinear dynamic analysis where Newmark's method is used to integrate equations of motion and the Newton-Raphson method is employed for iterating within each time step increment until equilibrium is achieved. The obtained results demonstrate the effectiveness and robustness of the interpolation of the in-plane rotational *d.o.f.* and present deficiencies of using fictitious stiffness in dynamic linear and nonlinear analysis.

***Keywords*—**Flat shell, dynamic analysis, nonlinear, Newmark, drilling rotation.

I. Introduction

Numerical shell structures analysis became popular with the advancement has been made in finite element method and the advancements in digital computers since they allow possibility to solve large systems of equations quickly and efficiently. Then such a topic has been developed for several years and many research works have been made to attain effective and reliable shell finite elements for the numerical static linear, non linear and dynamic nonlinear analysis of shell structures.

The focus of this work is to develop and evaluate two kinds of flat shell elements for the numerical analysis of thin shell structures, which are triangular and quadrilateral flat shell elements with fictitious in-plane rotational stiffness and the same quadrilateral element with interpolated nodal in-plane rotational stiffness.

This paper is organized as follows: the flat shell finite elements are introduced in Section 2. Section 3 discuss the advantages of interpolate the rotational in-plane stiffness instead the use of fictitious stiffness for geometrically non linear analysis. Section 4 describes dynamic analysis for the proposed elements. Extension of linear dynamic to geometrically non linear dynamic analysis is given in Section 5. Section 6 represents numerical examples and discussions. Finally, section 7 concludes the paper.

II. Flat Shell Elements

Flat shell elements are developed by combining membrane elements with plate bending elements. Then the element stiffness matrix for a flat shell element is first assembled by superposing the membrane stiffness and bending stiffness at each node in the local coordinate system. Subsequently, it will be transformed from the local to the global coordinate system where the shell element has to deal with six *d.o.f* at each corner node.

An important aspect of this work is to implement four elements on a computer program:

1) The first one is a combination of the DKQ quadrilateral plate bending element of Batoz *et al* [1] which uses three out of plane *d.o.f* at corner node, and the four node isoparametric quadrilateral plane-stress element which uses two nodal translations *d.o.f* at corner node. The resulting element is a four node quadrilateral flat shell element with 5 *d.o.f* at each corner node, this element is noted "Quad". To avoid stiffness matrix singularity, the solution of fictitious stiffness is adopted

2) The second is a combination of the DKT triangular plate bending element of Batoz *et al* [2] which uses three out of plane *d.o.f* at corner node, and the three node isoparametric CST triangular plane-stress element which uses two nodal translations *d.o.f* at corner node. The resulting element is a triangular flat shell element with 5 *d.o.f* at each corner node, this element is noted "Trian". The solution of fictitious stiffness is adopted to avoid stiffness matrix singularity.

3) The third element noted "Qdrill" is a four node quadrilateral flat shell element with 6 *d.o.f* at corner node. It is a combination of the DKQ quadrilateral plate bending element Batoz *et al* [1] which uses three out of plane *d.o.f* at corner node, and Allman's quadrilateral plane-stress element of Ibrahimbegovic *et al* [3] which uses three nodal translations *d.o.f* at corner node. In this case the rotational degree of

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freedom θ_z is included in the plane-stress theory formulation as a parameter in the in-plane displacement field interpolation.

The elements "Train" and "Quad" use the classical approach that consist of associate a fictitious stiffness at the sixth *d.o.f* for each node. It is a relatively small constant inserted at appropriate places within the elementary stiffness matrix to avoid system singularity when all the elements meeting at one node are coplanar [4,5]. Zienkiewicz suggested the following overall form approximation of the moment-rotation relationship for triangular element [4]:

$$\begin{Bmatrix} M_{z1} \\ M_{z2} \\ M_{z3} \end{Bmatrix} = \frac{\alpha \cdot E \cdot h^n \cdot A}{36} \begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 1 & -0.5 \\ -0.5 & -0.5 & 1 \end{bmatrix} \begin{Bmatrix} \theta_{z1} \\ \theta_{z2} \\ \theta_{z3} \end{Bmatrix} \tag{1}$$

Even that solution can solve the problem of stiffness matrix singularity, but it doesn't represent real behavior because of the fact that a fictitious stiffness has been added. Furthermore, some research works on the effect of varying α constant determined that satisfactory results are obtained as α is smaller value [6,5,7]. Recently, much works was directed to include the in-plane rotation as an effective *d.o.f* [8,9,10,6,3]. The main goal behind including normal in-plane rotation *d.o.f* at corner nodes as a real *d.o.f* is to avoid the well-known sixth degree of freedom problem for shell elements (stiffness matrix singularity) and to improve the in-plane behviour of the plane-stress element because a higher order plane-stress element is formulated by this way.

III. Geometrically Nonlinear Analysis

The nonlinear analysis is carried out using the incremental method. It is based upon the progressive increase of the applied loads to obtain the non linear response in an incremental way satisfying the equilibrium equations in successive force increments. In purpose to avoid large rotations, the updated Lagrangian corotational formulation is an efficient approach to be chosen. In such approach the rigid body motion is eliminated with the movement of the system axis following the new configuration. Whereas, during each time step between the time t and $t+\Delta t$ which represent one force increment, the configuration $C_{t+\Delta t}$ to be calculated is obtained starting from the configuration C_t considered as known. Therefore the finite elements developed for linear analysis in small displacements, can be applied for large displacements and large rotations nonlinear analysis.

In this context, reference position of the triangular element which has 3 nodes is easily updated in the corotational system, that's not the case of quadrilateral elements which has 4 nodes that must be coplanar in the reference position that is not assured at the deformed configuration C_t which will be used as reference to obtain the configuration $C_{t+\Delta t}$. In purpose to overcome the problems related to the probably non planarity of the deformed quadrilateral, we proposed to use a local reference plane and technics as described by Boutagouga [11].

A. Tangent Stiffness Matrix

Since corotational coordinate system makes rotations and translations with the element, large translations and large rotations are absorbed by the motion of the corotational system axis. Therefore the deformation is always measured on the element local reference level. In incremental way, Green's strain tensor that representing shell element, is calculated in reference position that's always related to the corotational system axis, as:

$$\Delta e_{ij} = \frac{1}{2}\left(\frac{\partial \Delta u_j}{\partial x_j} + \frac{\partial \Delta u_j}{\partial x_i}\right) - z\frac{\partial^2 \Delta w}{\partial x_i \partial x_j} + \frac{1}{2}\left(\frac{\partial \Delta w}{\partial x_i} \cdot \frac{\partial \Delta w}{\partial x_j}\right) \tag{2}$$

The equilibrium equation is obtained by application of the virtual work principle in incremental form between the configurations C_t and $C_{t+\Delta t}$:

$$\int_{\overline{V}} (T_{ij}\delta(\Delta\varepsilon_{ij}^*) + D_{ijkl}\Delta\varepsilon_{ij}\delta(\Delta\varepsilon_{ij}))d\overline{V} = W_{ext} - \int_{\overline{V}} T_{ij}\delta(\Delta\varepsilon_{ij})d\overline{V} \tag{3}$$

$\Delta\varepsilon_{ij}$, $\Delta\varepsilon^*_{ij}$: are linear and nonlinear part of the incremental Green's strain tensor

T_{ij} : is the Cauchy stress , D_{ijkl} : Hooke Matrix

The tangent stiffness matrix $[K_T]$, with the same shape functions defined for linear analysis is written:

$$\int_{\overline{V}} (T_{ij}\delta(\Delta\varepsilon_{ij}^*) + D_{ijkl}\Delta\varepsilon_{ij}\delta(\Delta\varepsilon_{ij}))d\overline{V} = \{\delta(\Delta q)\}^T [K_T]\{\Delta q\} \tag{4}$$

Where:

$$[K_T] = [K_0] + [K_\sigma] \tag{5}$$

$[K_T]$: Tangent stiffness matrix, it must be formed at each iteration

$[K_0]$: small displacement matrix for linear analysis

$[K_\sigma]$: initial stress matrix

The internal forces $\{F_{int}\}$ are such as:

$$\int_{\overline{V}} T_{ij}\delta(\Delta\varepsilon_{ij})d\overline{V} = \{\delta(\Delta q)\}^T \{F_{\text{int}}\} \tag{6}$$

B. Newton-Raphson Algorithm

The nonlinear process is solved in an incremental way with correction of equilibrium by the standard iterative Newton-Raphson method associated to the arc-length control or load control techniques.

In discretized form, the equilibrium equation is written by considering an incremental load factor $\Delta\lambda$:

$$[K_T]\{\Delta q\} = \{R\} + \Delta\lambda\{P_{ext}\} \tag{7}$$

Where :$\{R\}$ represents unbalanced forces, and $\{P_{ext}\}$ the initial loads.

IV. Linear Dynamic Analysis

Physical equilibrium of structural dynamic behavior is approximated to the following set of second-order, linear, differential equations, written on matrix form as:

$$M\ddot{u} + C\dot{u} + Ku = P \tag{8}$$

Where u : Nodal displacement vector.
M : Structural mass matrix
C : Structural damping matrix
K : Structural stiffness matrix
P : Applied load vector

We solved this system of equations using direct numerical integration to obtain direct transient response since the solution and the excitation are time varying. Many algorithms of direct time integration numerical resolution have been developed and presented in literature where they differ by the manner used to express the relationship between displacement, velocity and acceleration [12, 13, 14]. However, all methods can fundamentally be classified as either explicit or implicit integration methods. Generally, implicit algorithms are most effective for structural dynamic problems, and large time step can be used since they are can be numerically stable algorithms, while explicit algorithms are very efficient for wave propagation problems and such high frequency modes, but they are numerically instable algorithms.

A. Newmark Algorithm

Shell structure's systems are generally resolved using Newmark's method as in this investigation. It is a single step method which assumes that the acceleration is not smooth function (because of some phenomenon in real structures like: buckling of elements, nonlinear hysteresis behaviour of material, and contacts between parts of structure). Subsequently, dynamic equations of motion are solved step by step by satisfying dynamic equilibrium at discrete points in time. Therefore, the eqn (8) is considered at time *(t+Δt)* as:

$$M\ddot{u}_{t+\Delta t} + C\dot{u}_{t+\Delta t} + Ku_{t+\Delta t} = F_{t+\Delta t} \tag{9}$$

Where: $u_{t+\Delta t}$, $\dot{u}_{t+\Delta t}$ and $\ddot{u}_{t+\Delta t}$ are respectively the $u(t+\Delta t),\ \dot{u}(t+\Delta t),\ and\ \ddot{u}(t+\Delta t)$ approximations at time *(t+Δt)*.

The Newmark process allows the dynamic equilibrium of the system at time *(t+Δt)* to be written in terms of the unknown nodal displacements $u_{t+\Delta t}$ as:

$$\begin{aligned}(b_1M + b_4C + K)u_{t+\Delta t} &= F_{t+\Delta t} + M(b_1u_t - b_2\dot{u}_t - b_3\ddot{u}_t) + \ldots \\ &\ldots + C(b_4u_t - b_5\dot{u}_t - b_6\ddot{u}_t\end{aligned} \tag{10}$$

Where the constants b_1 to b_6 are defined as:

$$b_1 = \frac{1}{\beta\Delta t^2},\quad b_2 = \frac{-1}{\beta\Delta t},\quad b_3 = (1 - \frac{1}{2\beta}),\quad b_4 = \gamma\cdot\Delta t\cdot b_1$$

$b_5 = 1 + \gamma\cdot\Delta t\cdot b_2$ and $b_6 = \Delta t(1 + \gamma b_3 - \gamma)$

β and γ are parameters which control the stability and accuracy of the algorithm.

B. Mass Matrix

The elementary consistent mass matrix is derived from the kinetic energy expression using the virtual work principle. It is given by:

$$[M] = \int_A [N]^T[P][N]\cdot dA \tag{11}$$

Where, [*P*] is the inertia matrix, and [*N*] is the shape function matrix that can be the same shape function adopted for the displacement interpolation, or it can be the linear geometric shape function as adopted in this work.

V. Newmark's Method for Nonlinear Dynamic Analysis

Once the finite element models of geometrically nonlinear analysis and linear dynamic analysis has been created, nonlinear dynamic analysis will be easily obtained using Newmark direct time integration algorithm. The dynamic equilibrium equations can easily be solved using iterations within each time step *(Δt)* to obtain the dynamic nonlinear response of shell structure, i.e., the equilibrium equation of motion can be solved by Newmark's step by step integration method, where the Newton-Raphson algorithm is employed for iterating within each time step increment until equilibrium is achieved.

Also, in order to extend the linear dynamic scheme to taking account for geometrically nonlinear behavior, the nodal internal (nonlinear) elastic forces must be taken as:

$$N(u) = \int_v [B]^T \{\sigma(\varepsilon)\}\cdot dv \tag{12}$$

Then, the equilibrium equation of motion is written as:

$$M\ddot{u}_{t+\Delta t} + C\dot{u}_{t+\Delta t} + N(u)_{t+\Delta t} = R_{t+\Delta t} \tag{13}$$

Where: $R_{t+\Delta t}$ is the nodal residual forces vector at time *(t+Δt)*

$N(u)_{t+\Delta t}$ is the equivalent internal forces vector at time (*t*+*Δt*), it is written as:

$$N(u)_{t+\Delta t} = \int_v [B_{t+\Delta t}]^T \{\sigma(\varepsilon)_{t+\Delta t}\}\cdot dv \tag{14}$$

Tangent stiffness matrix is used, and it must be formed and triangularized at each iteration within each time step.

VI. Results and Discussions

A finite element analysis program that calculates deflections was developed in order to verify the accuracy of the tree elements considered in this study. This program was written in FORTRAN77.

Two example problems were selected for the verification of displacements obtained by the deferent developed shell elements. Also the results were compared with those from literature. A description of each example problem and the discussion of the results are presented in the following section:

A. Cylindrical Shell

This example is widely used in literature to validate shell finite elements. It's a thin cylindrical shell subjected to a concentrated load applied in its centre. Geometry of the shell is defined by: radius of curvature *R*, longitudinal length *L*, thickness *h*, and opening angle θ as shown in "Fig. 1". Its curved edges are free while its straight edges are hinged. Because of the double symmetry of geometry, loading, and boundary conditions, only one quarter of the shell is modelized. The geometrical and mechanical characteristics are as follows:

R=2540 *mm*; *L*=2540 *mm*; θ= 0,1 *rad*; *h*=6.35 *mm*; ν= 0,30 and ρ= 3210.05 Kg/m^3.

1) Geometrically Non Linear Response

First, geometrically non linear analysis was carried out for E = 3.10275 kN/mm^2 and for two thicknesses:

- For *h*=12.7 *mm*: The results obtained by the "*Qdrill*" element, "Fig. 2" present a precise convergence towards the reference solution of Surana [15]. The total number of iterations which requires "*Qdrill*" to plot the curve presented at "Fig. 2" is 28 iterations, while the triangular element "*Trian*" requires 60 iterations and the quadrilateral "*Quad*" element requires more than 70 iterations.

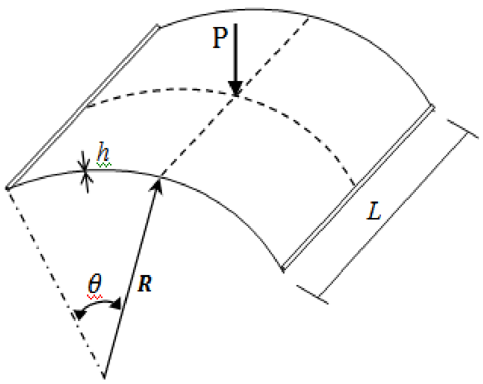

Fig. 1 Geometry of the cylindrical shell

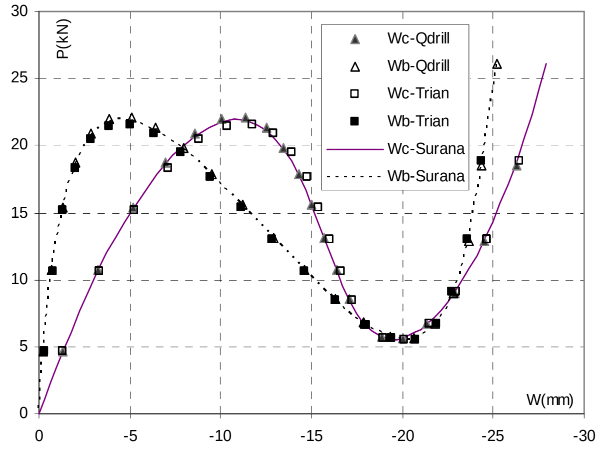

Fig. 2 Load-displacement curve (*h=12.7 mm*)

- For *h*=6.35 *mm*: In this case the shell presents a very sensible behaviour "Fig. 3", and a very marked snap-through is noted. For testing our results we refer to the load-displacement curve of Ramm [16]. The necessary total iteration number to plot the curve at "Fig. 3" is 46 iterations using "*Qdrill*" element, and 71 iterations for "*Trian*" element, and it's delicate to handle it with the "*Quad*" element.

The comparison of the results obtained by "*Qdrill*" element with those of the other elements shows that "*Qdrill*" element present some more flexibility, because in this case, the rotational *d.o.f* around local z-axis is natural, while that's not the case for the elements with fictitious rigidity like "*Quad*" and "*Trian*". But the most important remark is that "Qdrill" element required less number of iterations, compared with the very large number of iterations and numerical difficulties and some times divergence that the shell elements with fictitious rigidity suffer to undergo the unstable branch of the load-displacement curve, because rotational in-plane displacement results from correction of equilibrium by minimization of residual force can not be well controlled.

By consequence α value must be as large as possible to minimize non controlled displacement, that is in contradiction with static linear analysis where accurate results are obtained as α is smaller.

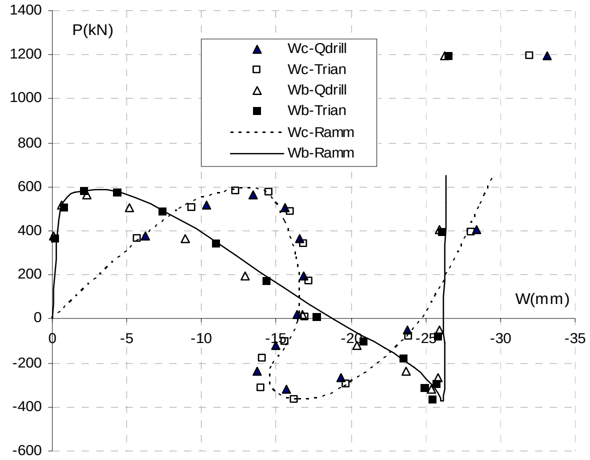

Fig. 3 Load-displacement curve (*h=6.35 mm*)

2) Dynamic Linear Response

Second: dynamic linear analysis was carried out for E=0.310275 kN/mm^2.

The time history displacement curves result using 8x8 *"Quad"* and *"Qdrill"* elements given in "Fig. 4", and using 8x8x(2) "Trian" elements given in "Fig. 5", show that a considerable attention must accorded to the value of α constant to be chosen when fictitious stiffness is adopted. One can deduce that there's a range between 10^{-2} and 10^{-5} where the solutions are very close, we conclude that α value must be chosen among values of this range. From the other side, nonlinear analyses obligate the use of the biggest value of the fictitious stiffness to unsure numerical stability and to perform analyses with the least numerical cost.

When "*Qdrill*" element always present a good convergence to the solution in [17], It can be seen from "Fig. 6" that a good convergence rate was obtained by the two other elements using $\alpha=10^{-3}$.

3) Dynamic Non Linear Response

Finally: geometrically non linear dynamic analysis was carried out for E = 0.310275 kN/mm^2. The deflection time history curves obtained using "*Qdrill*", "*Quad*" and "*Trian*" elements given in "Fig. 7", show that an excellent agreement with the solution in [17] is obtained using 8x8 quadrilateral meshing. The most important factor presented in this study is, the iterations number required by each one of the three shell

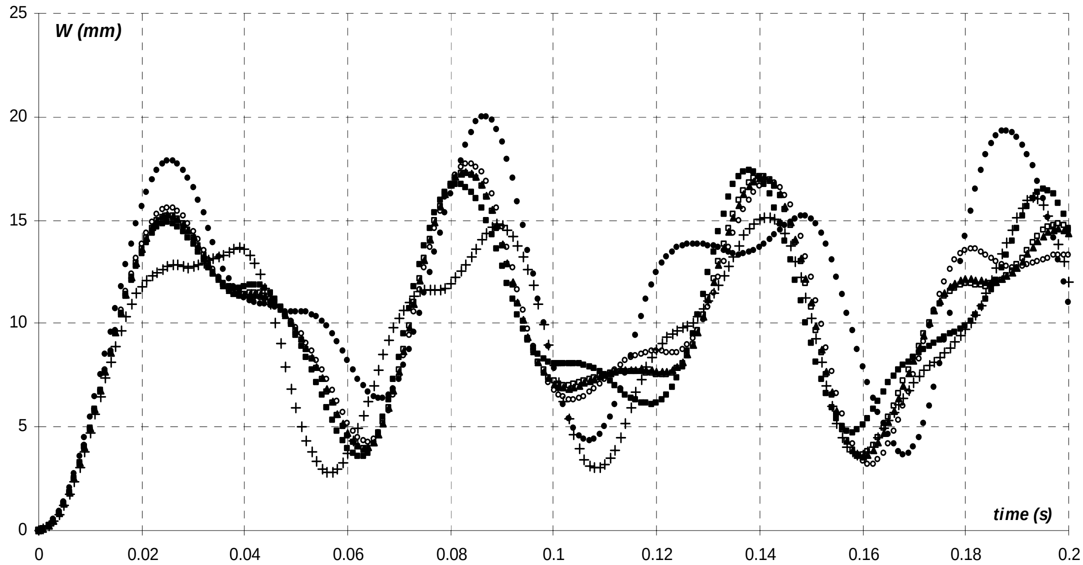

Fig. 4 Dynamic response of the cylindrical shell using "*Quad*" element

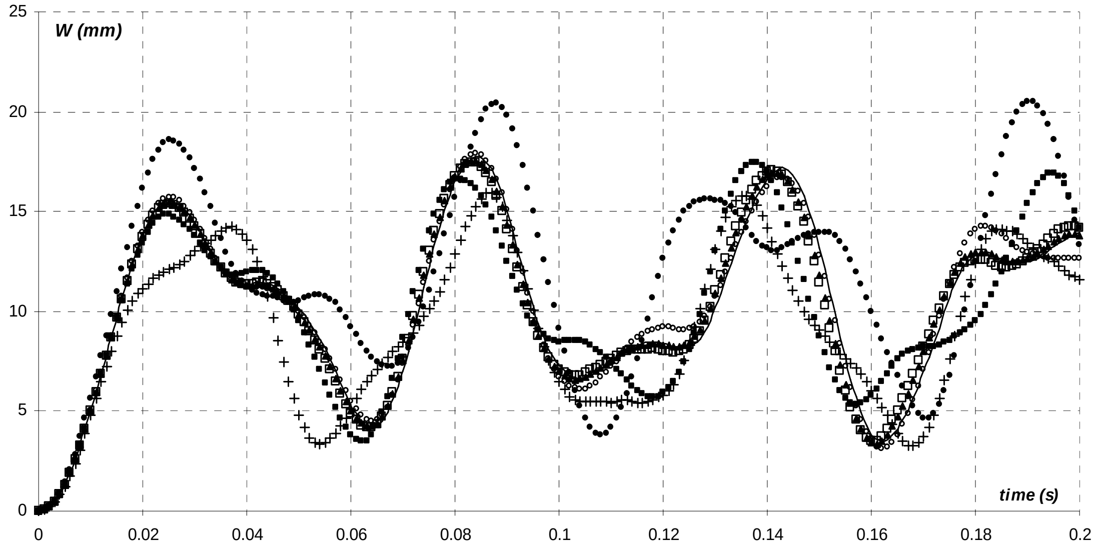

Fig. 5 Dynamic response of the cylindrical shell using "*Trian*" element

elements presented herein to perform the example. "*Qdrill*" element performs 522 iterations while "*Quad*" and "*Trian*" elements perform 625 and 626 iterations respectively to complete the whole curve. Then "*Qdrill*" element is 0.2 time less costly. So it's easy to demonstrate that the solution obtained using "*Qdrill*" element was very faster than those obtained by using "*Trian*" or "*Quad*" elements. Also the shell element "*Qdrill*" ensure more numerical stability especially when large nonlinearity is involved.

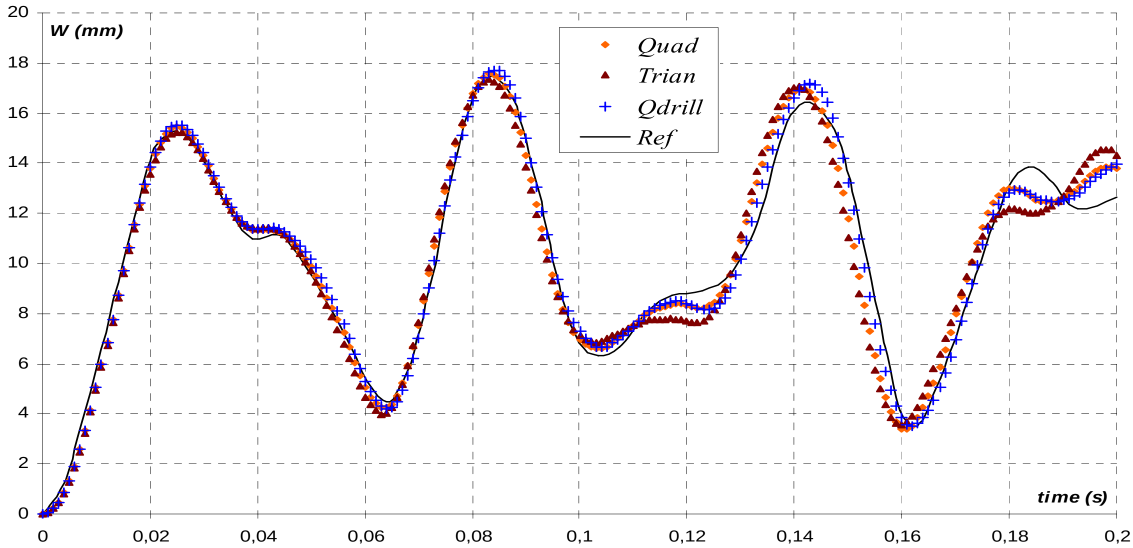

Fig. 6 Vertical displacement time history of the cylindrical shell $\alpha=10^{-3}$

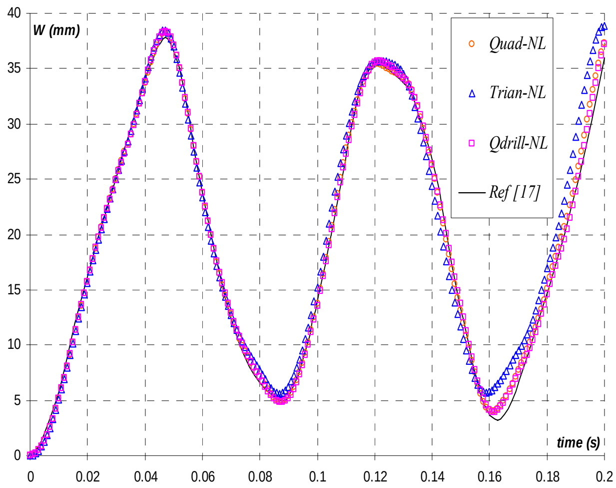

Fig. 7 Central point deflection time history of the cylindrical shell

VII. Conclusion

The Quadrilateral shell element with drilling rotation presented for nonlinear dynamic analysis seems to be a powerful element to use for geometrically nonlinear dynamic analysis by direct time integration.

As the results illustrated, on the one hand "*Qdrill*" shell element shows a grate stability and less numerical cost in critical situations when the structure response involves large nonlinear behaviour, on the other hand, "*Trian*" and "*Quad*" shell elements with fictitious stiffness present deficiency and grater numerical cost. In conclusion, as we can see, the interpolation of the in-plane rotational *d.o.f* has a major advantage in shell structures analysis by flat shell elements.

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