Effects of Electric Potential on Thermo-Mechanical Behavior of Functionally Graded Piezoelectric Hollow Cylinder under Non-Axisymmetric Loads

Amir Atrian, Javad Jafari Fesharaki, Gh. Hossein Majzoobi, Mahsa Sheidaee

***Abstract*—**The analytical solution of functionally graded piezoelectric hollow cylinder which is under radial electric potential and non-axisymmetric thermo-mechanical loads, are presented in this paper. Using complex Fourier series and estimation of power law for variations of material characterizations through the thickness, the electro thermo mechanical behavior of the FGPM cylinder is obtained. The stress and displacement distributions and the effect of electric potential field on the cylinder behavior are also presented and some applicable results are offered at the end of the paper.

***Keywords*—**Analytical, FGM, Fourier series, Non-axisymmetric, Piezoelectric, Thermo-elasticity

I. INTRODUCTION

FUNCTIONALLY graded piezoelectric materials (FGPM) are a type of piezoelectric materials that combined of two materials which continuous transition between one material to another is in a specific gradient. The primary goal in designing such materials is to take advantage of the desirable properties. Piezoelectric materials due to their direct and inverse effects have affluent applications as sensors and actuators in many industries like aircraft and aerospace [1]. Continuous variations of material parameters are usually assumed to be in power law or exponential functions in radial direction [2]. There are many researches in symmetric conditions of FG problems but not very in asymmetric loadings. In most of them displacement and stress field in the FGM cases under thermal, mechanical, electrical or magnetic loads are obtained using different analytical approaches. Loghman *et al.* studied magneto thermo-elastic creep behavior of FG cylinders. They used time dependant creep stress redistributions analysis for a thick-walled FGM cylinder placed in uniform magnetic and temperature fields and subjected to an internal pressure [3]. By using the infinitesimal theory, Electro magneto-elastic behaviors of FG piezoelectric solid cylinder and sphere were studied by Dai *et al.* [4]. Thermo-elastic solution of functionally graded cylindrical shell bonded to thin

A. Atrian is with Department of Mechanical Engineering, Najafabad Branch, Islamic Azad University, Isfahan, Iran (phone: +98-913-210-8799; e-mail: atrian@pmc.iaun.ac.ir).

J. Jafari Fesharaki is with Department of Mechanical Engineering, Najafabad Branch, Islamic Azad University, Isfahan, Iran (e-mail: jafari@pmc.iaun.ac.ir).

Gh. Majzoobi is with Bu-Ali Sina University, Hamadan, Iran (e-mail: gh_majzoobi@yahoo.co.uk).

M. Sheidaee is with Najafabad Branch, Islamic Azad University, Isfahan, Iran (e-mail: mhs.sheidaee@gmail.com).

piezoelectric layers is another research work done by Alibeigloo. He studied infinitesimal axisymmetric deformations of the cylinder using Navier type solution and state space method to solve the equations [1].

Puoltangari *et al.* investigated analytically functionally graded hollow spheres under non-axisymmetric thermo-mechanical loads. They used the Legendre polynomials and the system of Euler differential equations to solve the Navier equations [5]. Jabbari and his coworkers presented the general solution using the separation of variables and complex Fourier series for functionally graded hollow cylinders under non-axisymmetric steady-state loads [6]. Jafari *et al.* also used the direct method to evaluate the effect of Lorentz force on non-axisymmetric thermo-mechanical behavior of functionally graded hollow cylinder [7].

In this research work, two dimensional (r, θ) distributions of displacement and stress in a thick walled FGPM cylinder subjected to non-axisymmetric loads are evaluated analytically. Also, Effects of electric potential field on thermo-mechanical behavior of such a cylinder is fully investigated and some applicable results are presented at the end of paper.

II. ANALYSIS

A. Derivations

Consider a thick walled functionally graded piezoelectric cylinder with "a" and "b" as inner and outer radius, respectively. All Material properties except Poisson's ratio (ν) which is assumed to be constant are approximated to obey the same power law through the thickness $(X = X_0 r^\beta)$. These parameters are the Young's modulus (E) and the coefficients of thermal expansion (α), thermal conduction (K), pyroelectric (p), dielectric (η), and piezoelectric (e).

To reach governing equations (Navier and Maxwell equations), It's needed at first to write equations of equilibrium and the charge equation of electrostatic (Maxwell equation). In the absence of body forces, these equations in the cylindrical coordinates are [8]:

$$\begin{cases} \dfrac{\partial \sigma_r}{\partial r} + \dfrac{1}{r}\dfrac{\partial \sigma_{r\theta}}{\partial \theta} + \dfrac{\sigma_r - \sigma_\theta}{r} = 0 \\[2mm] \dfrac{\partial \sigma_{r\theta}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \sigma_\theta}{\partial \theta} + \dfrac{2\sigma_{r\theta}}{r} = 0 \\[2mm] \dfrac{\partial D_r}{\partial r} + \dfrac{D_r}{r} + \dfrac{1}{r}\dfrac{\partial D_\theta}{\partial \theta} = 0 \end{cases} \qquad (1)$$

Strain-displacement and constitutive equations are also expressed as [6]

$$\begin{cases} \varepsilon_{rr} = \dfrac{\partial U}{\partial r} \\[2mm] \varepsilon_{\theta\theta} = \dfrac{U}{r} + \dfrac{1}{r}\dfrac{\partial V}{\partial \theta} \\[2mm] \varepsilon_{r\theta} = \dfrac{1}{2}\left(\dfrac{1}{r}\dfrac{\partial U}{\partial \theta} + \dfrac{\partial V}{\partial r} - \dfrac{V}{r}\right) \end{cases} \quad (2)$$

$$\begin{cases} \sigma_{rr} = (\lambda + 2\mu)\varepsilon_{rr} + \lambda\varepsilon_{\theta\theta} - (3\lambda + 2\mu)\alpha T(r,\theta) + e_{rr}\dfrac{\partial \varphi}{\partial r} \\[2mm] \sigma_{\theta\theta} = (\lambda + 2\mu)\varepsilon_{\theta\theta} + \lambda\varepsilon_{rr} - (3\lambda + 2\mu)\alpha T(r,\theta) + e_{r\theta}\dfrac{\partial \varphi}{\partial r} \\[2mm] \sigma_{r\theta} = 2\mu\varepsilon_{r\theta} \end{cases} \quad (3)$$

Where U and V are the displacement components in radial and circumferential directions. φ is the electric potential and T is thermal distribution in the cylinder. λ and μ are Lame' coefficients and electrical displacements D_r and D_θ are also represented as [9]

$$\begin{cases} D_r = e_{rr}\varepsilon_{rr} + e_{r\theta}\varepsilon_{\theta\theta} - \eta\dfrac{\partial \varphi}{\partial r} + p_{rr}T(r,\theta) \\[2mm] D_\theta = 2e_{\theta\theta}\varepsilon_{r\theta} + p_{\theta\theta}T(r,\theta) \end{cases} \quad (4)$$

Substitution (2)-(4) into (1) gives the final equations in terms of displacements and electrical potential components. All of constants in these three partial differential equations are described in the Appendix.

$$\begin{aligned} &\frac{\partial^2 U}{\partial r^2} + m_1 \frac{1}{r}\frac{\partial U}{\partial r} + m_2 \frac{1}{r^2}U + m_3 \frac{1}{r^2}\frac{\partial V}{\partial \theta} + m_4 \frac{1}{r}\frac{\partial^2 V}{\partial r \partial \theta} \\ &+ m_5 \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + m_6 \frac{1}{r}\frac{\partial \varphi}{\partial r} + m_7 \frac{\partial^2 \varphi}{\partial r^2} = m_8 r^{\beta-1}T + m_9 r^{\beta}\frac{\partial T}{\partial r} \end{aligned} \quad (5\text{-}1)$$

$$\begin{aligned} &\frac{\partial^2 V}{\partial r^2} + m_{10}\frac{1}{r}\frac{\partial V}{\partial r} + m_{11}\frac{1}{r^2}V + m_{12}\frac{1}{r^2}\frac{\partial U}{\partial \theta} + m_{13}\frac{1}{r}\frac{\partial^2 U}{\partial r \partial \theta} \\ &+ m_{14}\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + m_{15}\frac{1}{r}\frac{\partial^2 \varphi}{\partial r \partial \theta} = m_{16}r^{\beta-1}\frac{\partial T}{\partial \theta} \end{aligned} \quad (5\text{-}2)$$

$$\begin{aligned} &\frac{\partial^2 \varphi}{\partial r^2} + m_{17}\frac{1}{r}\frac{\partial \varphi}{\partial r} + m_{18}\frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + m_{19}\frac{\partial^2 U}{\partial r^2} + m_{20}\frac{1}{r}\frac{\partial U}{\partial r} \\ &+ m_{21}\frac{1}{r^2}U + m_{22}\frac{1}{r}\frac{\partial^2 V}{\partial r \partial \theta} + m_{23}\frac{1}{r^2}\frac{\partial V}{\partial \theta} = m_{24}r^{\beta-1}T \\ &+ m_{25}r^{\beta}\frac{\partial T}{\partial r} + m_{26}r^{\beta-1}\frac{\partial T}{\partial \theta} \end{aligned} \quad (5\text{-}3)$$

To solve these equations, U, V and φ are presumed to be in complex Fourier series form as below [6].

$$U(r,\theta) = \sum_{n=-\infty}^{\infty} U_n(r)e^{in\theta}$$

$$V(r,\theta) = \sum_{n=-\infty}^{\infty} V_n(r)e^{in\theta} \quad (6)$$

$$\varphi(r,\theta) = \sum_{n=0}^{0} \varphi_n(r)e^{in\theta} = \varphi_0(r)$$

Where $U_n(r)$ and $V_n(r)$ are the coefficients of the Fourier series. Electrical field, Due to symmetry, must not be dependent to circumferential coordinate. Therefore the third equation of (6) is valid only for n=0 and the third equation of (5) will be excluded from the system of equations (5) for n≠0.

The steady state heat conduction problem is already solved by Jabbari et al. [6] and is in the form of complex Fourier series too.

$$\begin{cases} T(r,\theta) = \sum_{n=-\infty}^{\infty} T_n(r)e^{in\theta} \\[2mm] T_n(r) = D_{n1}r^{\gamma_{n1}} + D_{n2}r^{\gamma_{n2}} \\[2mm] \gamma_{n1,2} = \dfrac{-\beta}{2} \pm \sqrt{\dfrac{\beta^2}{4} + n^2} \end{cases} \quad (7)$$

Replacement of (6) and (7) into Navier Equations (5) yield the system of equations which must be solved in homogenous and particular solution considering the boundary conditions. It is obvious that to get the complete answers, the solution for n=0 must be also done.

B. Boundary Conditions

To have non-axisymmetric loadings, the hollow cylinder is supposed to be under harmonic internal pressure and temperature as $\sigma_{rr}(a,\theta) = 40\cos^2\theta$ and $T(a,\theta) = 40\cos^2\theta$ respectively. There is no shear force in inner surface and the outer surface of cylinder is traction free and is constrained to have no radial and circumferential displacements. The electric potential in both inner and outer surfaces are also zero and the cylinder in its outside surface undergoes constant zero temperature.

III. Results and Discussion

Fig.1 and 2 represent the radial and circumferential displacement in the thick walled cylinder. Both of these quantities have zero value at outer surface of cylinder due of imposed boundary conditions. Whereas a fluctuation behavior is clear in each parameter at inside surface. As depicted in Fig.3 unlike radial displacement, the circumferential distribution has less variation and is almost constant.
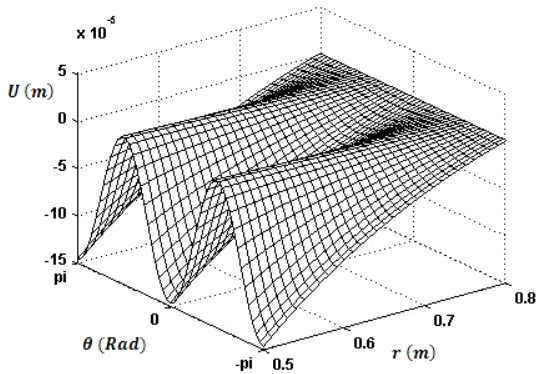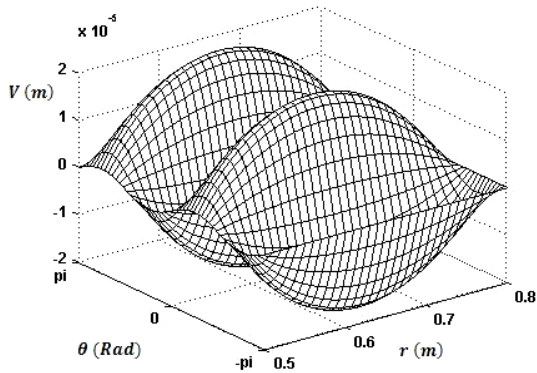
Fig. 1 Radial displacement
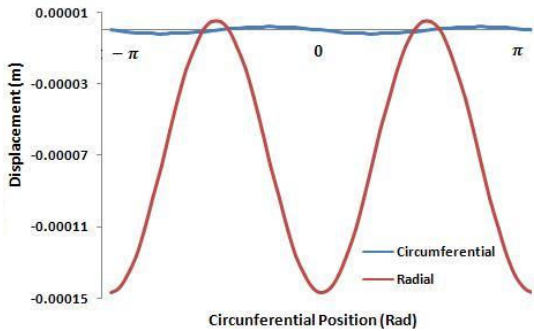

Fig. 2 Circumferential displacement

Fig. 3 Circumferential and radial displacement
at inner surface

As can be seen in Fig.4, the radial stress has a harmonic behavior which is in correspondence with the mechanical boundary conditions and is nearly uniform in each angular position (θ) along the radial direction. However, the circumferential distribution shows different behavior and its variations along the radial direction can not be considered uniform (Fig.5). It must be remembered that unlike the radial stress, there are not any imposed conditions for circumferential

stress in both sides. Shear stress nearly has a behavior like circumferential distribution of stress, but as showed by Fig.6 its maximum value is about 17.5 MPa in outside surface compared with 15.5 MPa of circumferential one.
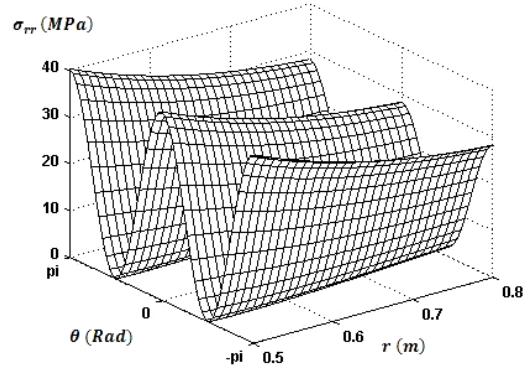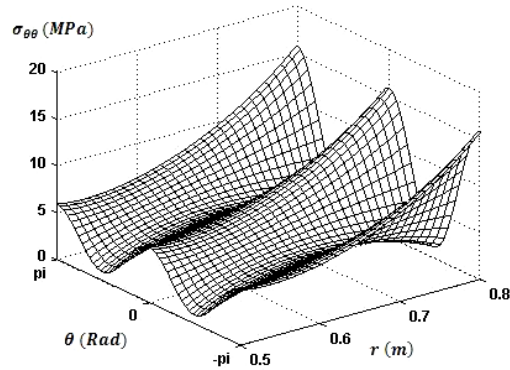

Fig. 4 Radial stress distribution

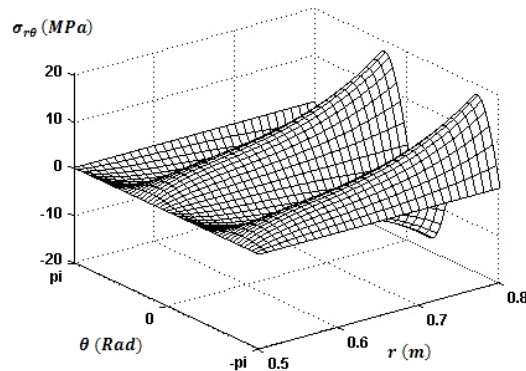Fig. 5 Circumferential stress distribution

Fig. 6 Shear stress distribution

Electric potential field causes a part of radial and circumferential displacement due of piezoelectricity nature of the material. Fig. 7 represents this part of total displacement in radial direction.
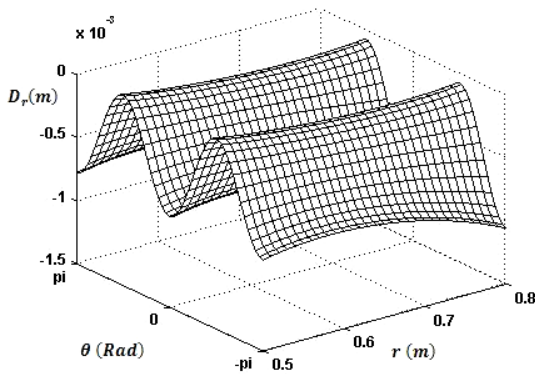
Fig. 7 Radial electric displacement

IV. CONCLUSION

An analytical solution using complex Fourier series is used to obtain electro thermo mechanical behavior of a functionally graded piezoelectric hollow cylinder which is under non-axisymmetric loads. Boundary conditions are assumed to be in harmonic form and the material parameters are considered to vary through the thickness based on the power law functions. Piezoelectricity nature of the material when is excited by electric potential field, increases the displacement components of cylinder. Therefore, Knowing the displacement and stress distributions in such parts, make one capable to apply these materials better in different industries as sensors and actuators. At the end, it should be noted that this solution method does not have any mathematical limitations to exert different boundary conditions.

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APPENDIX

$$m_1 = \beta + 1, \, m_2 = \frac{\beta v}{1-v} - 1, \, m_3 = \frac{2v(\beta+1)-1}{2(1-v)} - 1, \, m_4 = \frac{1-2v}{2(1-2v)(1-v)}$$

$$m_5 = \frac{1-2v}{2(1-v)}, \, m_6 = \frac{[e_{rr0}(\beta+1)-e_{r\theta 0}](1+v)(1-2v)}{(1-v)E_0}$$

$$m_7 = \frac{e_{rr0}(1+v)(1-2v)}{(1-v)E_0}, \, m_8 = \frac{2\beta\alpha_0(1+v)}{(1-v)}, \, m_9 = \frac{\alpha_0(1+v)}{(1-v)}, \, m_{10} = \beta + 1$$

$$m_{11} = -(\beta+1), \, m_{12} = \frac{2(1-v)}{(1-2v)} + (\beta+1), \, m_{13} = \frac{1}{(1-2v)}, \, m_{14} = \frac{2(1-v)}{(1-2v)}$$

$$m_{15} = \frac{2e_{r\theta 0}(1+v)}{E_0}, \, m_{16} = \frac{2\alpha_0(1+v)}{(1-2v)}, \, m_{17} = \beta + 1, \, m_{18} = \frac{-e_{\theta\theta 0}}{\eta_0}$$

$$m_{19} = \frac{-e_{r\theta 0}}{\eta_0}, \, m_{20} = \frac{-e_{rr0}(1+\beta)-e_{r\theta 0}}{\eta_0}, \, m_{21} = \frac{-e_{r\theta 0}\beta}{\eta_0}, \, m_{22} = \frac{-e_{rr0}-e_{\theta\theta 0}}{\eta_0}$$

$$m_{23} = \frac{-e_{r\theta 0}\beta + e_{\theta\theta 0}}{\eta_0}, \, m_{24} = \frac{p_{rr0}(1+2)\beta}{\eta_0}, \, m_{25} = \frac{p_{rr0}}{\eta_0}, \, m_{26} = \frac{p_{\theta\theta 0}}{\eta_0}$$