International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:13, No:5, 2019

Effect of Supplementary Premium on the Optimal Portfolio Policy in a Defined Contribution Pension Scheme with Refund of Premium Clauses

Edikan E. Akpanibah Obinichi C. Mandah Imoleayo S. Asiwaju

***Abstract*—**In this paper, we studied the effect of supplementary premium on the optimal portfolio policy in a defined contribution (DC) pension scheme with refund of premium clauses. This refund clause allows death members' next of kin to withdraw their relative's accumulated wealth during the accumulation period. The supplementary premium is to help sustain the scheme and is assumed to be stochastic. We considered cases when the remaining wealth is equally distributed and when it is not equally distributed among the remaining members. Next, we considered investments in cash and equity to help increase the remaining accumulated funds to meet up with the retirement needs of the remaining members and composed the problem as a continuous time mean-variance stochastic optimal control problem using the actuarial symbol and established an optimization problem from the extended Hamilton Jacobi Bellman equations. The optimal portfolio policy, the corresponding optimal fund size for the two assets and also the efficient frontier of the pension members for the two cases was obtained. Furthermore, the numerical simulations of the optimal portfolio policies with time were presented and the effect of the supplementary premium on the optimal portfolio policy was discussed and observed that the supplementary premium decreases the optimal portfolio policy of the risky asset (equity). Secondly we observed a disparity between the optimal policies for the two cases.

***Keywords*—**Defined contribution pension scheme, extended Hamilton Jacobi Bellman equations, optimal portfolio policies, refund of premium clauses, supplementary premium.

I. Introduction

THE importance of pension scheme in planning the old age income of retirees cannot be over emphasized. Already in existence is two types of pension schemes and these include the defined benefit (DB) pension scheme and the DC pension scheme. The DB pension scheme is a scheme where members' benefits are predetermined based on salary histories of the members, years in service and age. Although most members of this scheme are comfortable with the scheme since the burden of contribution is only on the employers, it has in recent years generated controversies and delay in payment after retirement. The later scheme known as the DC pension scheme depends mostly on members' contributions and involvement and requires members to contribute a specific percentage of their earnings into the retirement savings account (RSA). This scheme is much more lucrative and dependable than the older scheme since members are fully involved in the contribution and investment process and their benefits depend mostly on the returns of the investments during the accumulation period. These expected returns are influenced by some factors such as investment efficiency, inflation, mortality risk etc. Although this scheme looks attractive, it requires members to know how to invest in different assets available in the financial market. Hence, the study of optimal portfolio policy has become very useful especially to financial institutions.

There are several research works on the study of optimal portfolio policies, some of such include [1], which studied the optimal investment strategy to DC members with asset, salary and interest rate risk; they proposed a novel form of terminal utility function by incorporating habit formulation. Reference [2] proposed and investigated a model of optimal allocation for DC pension plan with a minimum guarantee in the continuous-time setting. In [3], asset allocation problem under a stochastic interest rate was studied, [4] investigated optimal investment strategy for a DC pension with stochastic interest rate. Reference [5] investigated a case where the interest rate was of Vasicek model; [6] studied the effect of extra contribution on the optimal investment strategies for DC pension with a stochastic salary under affine interest rate model which includes the Cox-Ingersoll-Ross (CIR) model and Vasicek model. Lately, the study of constant elasticity of Variance (CEV) model in DC pension fund investment strategies have taken center stage in modeling the stock price. Reference [7] investigated the impact of additional voluntary contribution on the optimal investment strategy; also the optimal investment strategies in DC pension scheme with multiple contributors studied using Legendre transformation method to obtain the explicit solution for CRRA and CARA see [12]. Reference [8] studied the CEV model and the Legendre transform-dual solution for annuity contracts. Reference [9] obtained explicit solutions of the optimal investment strategy for investor with CRRA and CARA utility function by extending the work of [8]. Reference [10] studied stochastic strategies of the optimal investment for DC pension fund with multiple contributors; here the authors assumed the rate of contributions to be stochastic.

Recently, a number of work have been done on the optimal investment strategy with refund of premium clause; some of which include [11], which investigated optimal investment

E.E. Akpanibah is with the Department of Mathematics and Statistics, Federal University Otuoke, Bayelsa State, Nigeria (e-mail: edikanakpanibah@gmail.com).
O. C. Mandah is with the Department of Mathematics, University of Ibadan, Oyo State, Nigeria (e-mail: m.obinichi@gmail.com).
I. S. Asiwaju is with the Department of Mathematics, Cypher Crescent Limited, Port Harcourt, Rivers State, Nigeria (e-mail: samuelasiwaju@gmail.com).

strategy for a DC pension scheme with return of premium clauses in a mean-variance utility function. Reference [14] investigated equilibrium investment strategy for DC pension plan with default risk and return of premium clauses under CEV model. Reference [16] investigated the optimal time-consistent investment strategy for a DC pension scheme with the return of premium clauses and annuity contracts. Reference [13] studied mean variance optimization problem with return of premium in a DC pension with multiple contributors. Reference [15] studied the optimal Portfolio Selection for a DC pension fund with return of premium clause with predetermined interest rate under mean-variance utility; in their work, they assumed that the return is with predetermined interest.

From the available literature and to the best of our knowledge, mandatory and supplementary contributions have not been merged together to study the effect of supplementary premium on the optimal portfolio policy with refund of premium clause. Hence, this forms the basis of this research where we study the effect of supplementary premium on the optimal portfolio policy with refund of premium clause. We assume that the supplementary premium is stochastic and the price process of the equity follows a geometric Brownian motion.

II. Financial Market and Investment Model

Let us consider a financial market which is complete, frictionless and continuously open over a fixed time interval $t \in [0,T]$. T is the time frame of the accumulation period. Let(Ω, F, P) be a complete probability space where Ω is a real space and P a probability measure, $\{B_0(t) : \mathrm{t} \geq 0\}$ is a standard Brownian motion. $F$is the filtration and represents the information generated by the Brownian motion $\{B_0(t)\}$.

Let $C_t(t)$ and $E_t(t)$ represent the price of the risk-free asset (cash) and the risky asset (equity) respectively, and their models are given as follows:

$$\frac{dC_t(t)}{C_t(t)} = rdt, \tag{1}$$

$$\frac{dE_t(t)}{E_t(t)} = \alpha dt + \beta dB_0. \tag{2}$$

where r is the risk-free interest rate, α is the expected instantaneous rate of return of equity and satisfies the general condition $\alpha > r$ and β is the instantaneous volatility of equity. Also, let b be the contribution paid to the member's pension account at a given time t, $\omega_0$the initial age of accumulation phase, $\vartheta_0 + T$ is the end age, $\frac{1}{i}K_{\vartheta_0+t}$ is the mortality rate from time $t$to $t+\frac{1}{i}$, tb is the premium accumulated at time t, $tb\frac{1}{i}K_{\vartheta_0+t}$ is the returned premium to the death member's family. Also, we assume that there is a supplementary premium φ introduced to amortize the pension fund which is assumed to be stochastic.

Let μ represent the proportion of the wealth to be invested in risky assets and $1-\mu$, the proportion to be invested in the risk-free asset.

Once return of premium occurs, the pension fund manager's interest will be to increase the fund size of the surviving members and simultaneously reduce the risk on the accumulated wealth. There is need for the pension fund manager to formulate an optimal investment problem under the mean-variance criterion as follows:

$$\sup_{\mu}\{E_{t,l}L^{\mu}(T) - Var_{t,l}L^{\mu}(T)\} \tag{3}$$

III. Main Result

A. Optimal Portfolio Policy and the Efficient Frontier When the Remaining Wealth Are Equally Distributed Among Members

Considering the time interval $[t, t+\frac{1}{i}]$, the differential form associated with the fund size, when the remaining wealth is shared evenly among the surviving members of the scheme, is given as:

$$L\left(t+\frac{1}{i}\right) = \left(L(t)\left(\mu\frac{E_{t+\frac{1}{i}}}{E_t} + (1-\mu)\frac{C_{t+\frac{1}{i}}}{C_t}\right) + b\left(\frac{1}{i}\right) + \varphi dB_0 - tb\frac{1}{i}K_{\vartheta_0+t}\right)\frac{1}{1-\frac{1}{i}K_{\vartheta_0+t}} \tag{4}$$

$$L\left(t+\frac{1}{i}\right) = \left(L(t)\left(\mu\left(\frac{E_{t+\frac{1}{i}}}{E_t} - \frac{E_t}{E_t} + \frac{E_t}{E_t}\right) + (1-\mu)\left(\frac{C_{t+\frac{1}{n}}}{C_t} - \frac{C_t}{C_t} + \frac{C_t}{C_t}\right)\right) + b\left(\frac{1}{i}\right) + \varphi dB_0 - tb\frac{1}{i}K_{\vartheta_0+t}\right)\left(1 + \frac{\frac{1}{i}K_{\vartheta_0+t}}{1-\frac{1}{i}K_{\vartheta_0+t}}\right) \tag{5}$$

$$L\left(t+\frac{1}{i}\right) = \left(L(t)\left(\mu + 1 - \mu + \mu\left(\frac{E_{t+\frac{1}{i}}-E_t}{E_t}\right) + (1-\mu)\left(\frac{C_{t+\frac{1}{i}}-C_t}{C_t}\right)\right) + b\left(\frac{1}{i}\right) + \varphi dB_0 - tb\frac{1}{i}K_{\vartheta_0+t}\right)\left(1 + \frac{\frac{1}{i}K_{\vartheta_0+t}}{1-\frac{1}{i}K_{\vartheta_0+t}}\right) \tag{6}$$

$$\frac{1}{i}K_{\vartheta_0+t} = 1 - \exp\{-\int_0^{\frac{1}{i}}\pi(\vartheta_0+t+s)ds\} = \pi(\vartheta_0+t)\frac{1}{i} + O(\frac{1}{i})$$
$$i \to \infty, \frac{1}{i}K_{\vartheta_0+t} = \pi(\vartheta_0+t)dt,\ b\left(\frac{1}{i}\right) \to bdt,\ \frac{E_{t+\frac{1}{i}}-E_t}{E_t} \to \frac{dE_t(t)}{E_t(t)},\ \frac{C_{t+\frac{1}{i}}-C_t}{C_t} \to \frac{dC_t(t)}{C_t(t)} \tag{7}$$

Substituting (7) into (6) we have

$$dL(t) = L(t)\left(\mu\left(\frac{dE_t(t)}{E_t(t)}\right) + (1-\mu)\left(\frac{dC_t(t)}{C_t(t)}\right) + \frac{1}{\vartheta-\vartheta_0-t}\right) + bdt + \varphi dB_0 - tb\pi(\vartheta_0+t)dt \tag{8}$$

$$dL(t) = L(t)\left(\mu(\alpha dt + \beta dB_0) + (1-\mu)(rdt) + \frac{1}{\vartheta-\vartheta_0-t}\right) + bdt + \varphi dB_0 - tb\pi(\vartheta_0+t)dt \tag{9}$$

$$dL(t) = \left\{L(t)\left(\mu[\alpha - r] + r + \frac{1}{\vartheta-\vartheta_0-t}\right) + b\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right)\right\}dt + (\mu L(t)\beta + \varphi)dB_0 L(0) = l_0 \tag{10}$$

The force function $\pi(t)$ is given as

$$\pi(t) = \frac{1}{\vartheta-t}\ 0 \leq t < \vartheta \tag{11}$$

where ϑ is the maximal age of the life table.

If we apply the variational inequality method cited in [20], the mean-variance control problem (3) is similar to the Markovian time inconsistent optimal control problem with value function $A(t,l)$ see [16]. Our interest here is to determine the optimal portfolio policy for the two assets using the mean-variance utility function.

$$\begin{cases} B(t,l,\mu) = E_{t,l}[L^{\mu}(T)] - \frac{\gamma}{2} Var_{t,l}[L^{\mu}(T)] \\ B(t,l,\mu) = E_{t,l}[L^{\mu}(T)] - \frac{\gamma}{2}\left(E_{t,l}[L^{\mu}(T)^2] - (E_{t,l}[L^{\mu}(T)])^2\right) \\ A(t,l) = \sup_{\mu} B(t,l,\mu) \end{cases} \quad (12)$$

Following [20] the optimal portfolio policyμ^* satisfies:

$$A(t,l) = \sup_{\mu} B(t,l,\mu^*) \quad (13)$$

γ is a constant representing risk aversion coefficient of the members. Let $u^{\mu}(t,l) = E_{t,l}[L^{\mu}(T)]$, $v^{\mu}(t,l) = E_{t,l}[L^{\mu}(T)^2]$ then $A(t,l) = \sup_{\mu} x\left(t,l,u^{\mu}(t,l),v^{\mu}(t,l)\right)$ where,

$$x(t,l,u,v) = u - \frac{\gamma}{2}(v-u^2) \quad (14)$$

Theorem 1 (verification theorem). If there exist three real functions X,Y,Z $[0,T]\times R \to R$ satisfying the following extended Hamilton Jacobi Bellman equation equations

$$\begin{cases} \sup_{\mu} \begin{cases} X_t - x_t \\ +(X_l - x_l)\begin{bmatrix} l\left(\mu[\alpha - r] + r + \frac{1}{\vartheta - \vartheta_0 - t}\right) \\ +b\left(\frac{\vartheta - \vartheta_0 - 2t}{\vartheta - \vartheta_0 - t}\right) \end{bmatrix} \\ +\frac{1}{2}(X_{ll} - U_{ll})(\mu l\beta + \varphi)^2 \end{cases} = 0 \\ X(T,l) = x(t,l,l,l^2) \end{cases} \quad (15)$$

where,

$$U_{ll} = x_{ll} + 2x_{lu}u_l + 2x_{lv}v_l + x_{uu}u_l^2 + 2x_{uv}u_l v_l + x_{vv}v_l^2 = \gamma u_l^2$$

$$\begin{cases} \begin{cases} Y_t \\ +Y_l\begin{bmatrix} l\left(\mu[\alpha - r] + r + \frac{1}{\vartheta - \vartheta_0 - t}\right) \\ +b\left(\frac{\vartheta - \vartheta_0 - 2t}{\vartheta - \vartheta_0 - t}\right) \end{bmatrix} \\ +\frac{1}{2}Y_{ll}(\mu l\beta + \varphi)^2 \end{cases} = 0 \\ Y(T,l) = l \end{cases} \quad (16)$$

$$\begin{cases} \begin{cases} Z_t \\ +Z_l\begin{bmatrix} l\left(\mu[\alpha - r] + r + \frac{1}{\vartheta - \vartheta_0 - t}\right) \\ +b\left(\frac{\vartheta - \vartheta_0 - 2t}{\vartheta - \vartheta_0 - t}\right) \end{bmatrix} \\ +\frac{1}{2}Z_{ll}(\mu l\beta + \varphi)^2 \end{cases} = 0 \\ Y(T,l) = l^2 \end{cases} \quad (17)$$

Then $A(t,l) = X(t,l), u^{\mu^*} = Y(t,l), v^{\mu^*} = Z(t,l)$for the optimal investment strategy μ^*

Proof. The details of the proof can be found in [17]-[19].

Next, we find the optimal investment policy for the both assets and also the efficient frontier by solving (15)-(17).

Recall that $x(t,l,u,v) = u - \frac{\gamma}{2}(v-u^2)$

$$x_t = x_l = x_{ll} = x_{lu} = x_{lv} = x_{uv} = x_{vv} = 0, x_u = 1 + \gamma u, \quad x_{uu} = \gamma, x_v = -\frac{\gamma}{2} \quad (18)$$

Substituting (18) into (15) and differentiating (15) with respect to μ and solving for μ we have:

$$\mu^* = -\left[\frac{(\alpha - r)X_l + \varphi\beta(X_{ll} - \gamma Y_l^2)}{(X_{ll} - \gamma Y_l^2)l\beta^2}\right] \quad (19)$$

Substituting (19) into (15) and (16) we have

$$X_t + X_l\left[\left(r + \frac{1}{\vartheta - \vartheta_0 - t}\right)l + b\left(\frac{\vartheta - \vartheta_0 - 2t}{\vartheta - \vartheta_0 - t}\right)\right] - X_l^2\frac{(\alpha - r)^2}{2(X_{ll} - \gamma Y_l^2)\beta^2} - \frac{\varphi}{\beta}X_l[\alpha - r] = 0 \quad (20)$$

$$Y_t + Y_l\left[\left(r + \frac{1}{\vartheta - \vartheta_0 - t}\right)l + b\left(\frac{\vartheta - \vartheta_0 - 2t}{\vartheta - \vartheta_0 - t}\right) + \frac{1}{\vartheta - \vartheta_0 - t}\right] - X_l Y_l\frac{(\alpha - r)^2}{2(X_{ll} - \gamma Y_l^2)\beta^2} - \frac{\varphi}{\beta}Y_l[\alpha - r] + \frac{Y_{ll}}{2}\left[-X_l\left[\frac{(\alpha - r)X_l + \varphi\beta(X_{ll} - \gamma Y_l^2)}{(X_{ll} - \gamma Y_l^2)\beta}\right] + \varphi\right]^2 = 0 \quad (21)$$

Next, we assume a solution for $X(t,l)$ and $Y(t,l)$ as:

$$\begin{cases} X(t,l) = F(t)l + G(t)F(T) = 1, G(T) = 0 \\ Y(t,l) = H(t)l + I(t)H(T) = 1, I(T) = 0 \\ X_t = lF(t) + G(t), X_l = F(t), X_{ll} = 0, \\ Y_t = lH(t) + I(t), Y_l = H(t), Y_{ll} = 0 \end{cases} \quad (22)$$

Substituting (22) into (20) and (21)

$$\begin{cases} F_t(t) + \left(r + \frac{1}{\vartheta - \vartheta_0 - t}\right)F(t) = 0 \\ G_t(t) + F(t)b\left(\frac{\vartheta - \vartheta_0 - 2t}{\vartheta - \vartheta_0 - t}\right) + F^2(t)\frac{[\alpha - r]^2}{2\gamma H^2(t)\beta^2} - \frac{\varphi}{\beta}F(t)[\alpha - r] = 0 \end{cases} \quad (23)$$

$$\begin{cases} H_t(t) + \left(r + \frac{1}{\vartheta - \vartheta_0 - t}\right)H(t) = 0 \\ I_t(t) + H(t)b\left(\frac{\vartheta - \vartheta_0 - 2t}{\vartheta - \vartheta_0 - t}\right) + F(t)\frac{[\alpha - r]^2}{\gamma H(t)\beta^2} - \frac{\varphi}{\beta}H(t)[\alpha - r] = 0 \end{cases} \quad (24)$$

Solving (23) and (24), we have

$$F(t) = \left(\frac{\vartheta - \vartheta_0 - t}{\vartheta - \vartheta_0 - T}\right)e^{r(T-t)} \quad (25)$$

$$H(t) = \left(\frac{\vartheta - \vartheta_0 - t}{\vartheta - \vartheta_0 - T}\right)e^{r(T-t)} \quad (26)$$

$$G(t) = \frac{1}{2\gamma\beta^2}(\alpha - r)^2(T - t) + \frac{\varphi[\alpha - r]}{\beta(\vartheta - \vartheta_0 - T)}\left[\frac{1}{r^2}e^{r(T-t)} - \frac{1}{r^2} + \frac{\vartheta - \vartheta_0 - T}{r} - \frac{\vartheta - \vartheta_0 - t}{r}e^{r(T-t)}\right] + \frac{b}{\vartheta - \vartheta_0 - T}\left[\frac{2}{r^2} - \frac{2}{r^2}e^{r(T-t)} + \frac{\vartheta - \vartheta_0 - 2t}{r}e^{r(T-t)} - \frac{\vartheta - \vartheta_0 - 2T}{r}\right] \quad (27)$$

$$I(t) = \frac{1}{\gamma\beta^2}(\alpha - r)^2(T - t) + \frac{\varphi[\alpha - r]}{\beta(\vartheta - \vartheta_0 - T)}\left[\frac{1}{r^2}e^{r(T-t)} - \frac{1}{r^2} + \frac{\vartheta - \vartheta_0 - T}{r} - \frac{\vartheta - \vartheta_0 - t}{r}e^{r(T-t)}\right] + \frac{b}{\vartheta - \vartheta_0 - T}\left[\frac{2}{r^2} - \frac{2}{r^2}e^{r(T-t)} + \frac{\vartheta - \vartheta_0 - 2t}{r}e^{r(T-t)} - \frac{\vartheta - \vartheta_0 - 2T}{r}\right] \quad (28)$$

$$X(t,l) = l\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)e^{r(T-t)} + \frac{1}{2\gamma\beta^2}(\alpha-r)^2(T-t) + \frac{\varphi[\alpha-r]}{\beta(\vartheta-\vartheta_0-T)}\left[\frac{1}{r^2}e^{r(T-t)} - \frac{1}{r^2} + \frac{\vartheta-\vartheta_0-T}{r} - \frac{\vartheta-\vartheta_0-t}{r}e^{r(T-t)}\right] + \frac{b}{\vartheta-\vartheta_0-T}\left[\frac{2}{r^2} - \frac{2}{r^2}e^{r(T-t)} + \frac{\vartheta-\vartheta_0-2t}{r}e^{r(T-t)} - \frac{\vartheta-\vartheta_0-2T}{r}\right] \quad (29)$$

$$Y(t,l) = l\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)e^{r(T-t)} + \frac{1}{\gamma\beta^2}(\alpha-r)^2(T-t) + \frac{\varphi[\alpha-r]}{\beta(\vartheta-\vartheta_0-T)}\left[\frac{1}{r^2}e^{r(T-t)} - \frac{1}{r^2} + \frac{\vartheta-\vartheta_0-T}{r} - \frac{\vartheta-\vartheta_0-t}{r}e^{r(T-t)}\right] + \frac{b}{\vartheta-\vartheta_0-T}\left[\frac{2}{r^2} - \frac{2}{r^2}e^{r(T-t)} + \frac{\vartheta-\vartheta_0-2t}{r}e^{r(T-t)} - \frac{\vartheta-\vartheta_0-2T}{r}\right] \quad (30)$$

Result 1. The optimal investment strategy for the risky asset is given as

$$\mu^* = \frac{\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)(\alpha-r)e^{-r(T-t)} - \gamma\beta\varphi}{\gamma l\beta^2} \quad (31)$$

Proof. From (19) and (22), we have

$$\mu^* = -\left[\frac{(\alpha-r)X_l + \varphi\beta\left(X_{ll} - \gamma Y_l^2\right)}{\left(X_{ll} - \gamma Y_l^2\right)l\beta^2}\right]$$

$$X_l = F(t) = \left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)e^{r(T-t)}, X_{ll} = 0,$$

$$Y_l = H(t) = \left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)e^{r(T-t)}$$

then

$$\mu^* = -\left[\frac{(\alpha-r)\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)e^{r(T-t)} - \left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)^2\varphi\beta\gamma e^{-2r(T-t)}}{\gamma e^{-2r(T-t)}l\beta^2\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)^2}\right]$$

$$\mu^* = \frac{\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)(\alpha-r)e^{-r(T-t)} - \gamma\beta\varphi}{\gamma l\beta^2}$$

Result 2. The optimal fund size is given as

$$L(t) = \frac{(\alpha-r)^2}{\gamma\beta^2}\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)te^{r(T-t)} - \frac{b}{r}\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right) + \frac{1}{r^2}\left(\frac{2b}{\vartheta-\vartheta_0-t}\right) - \frac{\varphi}{\beta}[\alpha-r]\left(\frac{1}{r^2(\vartheta-\vartheta_0-t)} - \frac{1}{r}\right) + \left(l_0(\vartheta-\vartheta_0) + \frac{b(\vartheta-\vartheta_0)}{r} - \frac{2b}{r^2} + \frac{\varphi}{\beta}[\alpha-r]\left(\frac{1}{r^2} - \frac{(\vartheta-\vartheta_0)}{r}\right)\right)\frac{e^{rt}}{\vartheta-\vartheta_0-t} \quad (32)$$

Proof. Recall that (9) and (31) is given respectively as

$$dL(t) = \left\{L(t)\left(\mu[\alpha-r] + r + \frac{1}{\vartheta-\vartheta_0-t}\right) + b\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right)\right\}dt + (\mu L(t)\beta + \varphi)dB_0 L(0) = l_0$$

$$\mu^* = \frac{(\alpha-r)e^{-r(T-t)} - \varphi\beta\gamma}{\gamma l\beta^2}$$

Substituting (31) into (9), we have

$$L_t(t) - (r + \frac{1}{\vartheta-\vartheta_0-t})L = \frac{(\alpha-r)^2}{\gamma\beta^2}e^{r(T-t)}\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right) - \frac{\varphi}{\beta}[\alpha-r] + b\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right) \qquad L(0) = l_0 \quad (33)$$

Solving (33) for $L(t)$ with initial condition we have

$$L(t) = \frac{(\alpha-r)^2}{\gamma\beta^2}\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)te^{r(T-t)} - \frac{b}{r}\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right) + \frac{1}{r^2}\left(\frac{2b}{\vartheta-\vartheta_0-t}\right) - \frac{\varphi}{\beta}[\alpha-r]\left(\frac{1}{r^2(\vartheta-\vartheta_0-t)} - \frac{1}{r}\right) + \left(l_0(\vartheta-\vartheta_0) + \frac{b(\vartheta-\vartheta_0)}{r} - \frac{2b}{r^2} + \frac{\varphi}{\beta}[\alpha-r]\left(\frac{1}{r^2} - \frac{(\vartheta-\vartheta_0)}{r}\right)\right)\frac{e^{rt}}{\vartheta-\vartheta_0-t}$$

Result 3. The efficient frontier of the pension fund is given as

$$E_{t,l}[L^{\mu^*}(T)] = l\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)e^{r(T-t)} + \frac{\varphi[\alpha-r]}{\beta(\vartheta-\vartheta_0-T)}\left[\frac{1}{r^2}e^{r(T-t)} - \frac{1}{r^2} + \frac{\vartheta-\vartheta_0-T}{r} - \frac{\vartheta-\vartheta_0-t}{r}e^{r(T-t)}\right] + \frac{b}{\vartheta-\vartheta_0-T}\left[\frac{2}{r^2} - \frac{2}{r^2}e^{r(T-t)} + \frac{\vartheta-\vartheta_0-2t}{r}e^{r(T-t)} - \frac{\vartheta-\vartheta_0-2T}{r}\right] + \frac{[\alpha-r]}{\beta}\sqrt{(T-t)\left(Var_{t,l}[L^{\mu^*}(T)]\right)} \quad (34)$$

Proof. Recall that

$$Var_{t,l}[L^{\mu^*}(T)] = E_{t,l}[L^{\mu}(T)^2] - (E_{t,l}[L^{\mu}(T)])^2$$

$$Var_{t,l}[L^{\mu^*}(T)] = \frac{2}{\gamma}(Y(t,l) - X(t,l)) \quad (35)$$

Substituting (29) and (30) into (35), we have

$$Var_{t,l}[L^{\mu^*}(T)] = \frac{1}{\gamma^2\beta^2}(\alpha-r)^2(T-t) \quad (36)$$

$$\frac{1}{\gamma} = \frac{\beta}{(\alpha-r)}\sqrt{\frac{Var_{t,l}[L^{\mu^*}(T)]]}{(T-t)}} \quad (37)$$

$$E_{t,l}[L^{\mu^*}(T)] = Y(t,l) \quad (38)$$

Substituting (30) into (38), we have

$$E_{t,l}[L^{\mu^*}(T)] == l\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)e^{r(T-t)} + \frac{1}{\gamma\beta^2}(\alpha-r)^2(T-t) + \frac{\varphi[\alpha-r]}{\beta(\vartheta-\vartheta_0-T)}\left[\frac{1}{r^2}e^{r(T-t)} - \frac{1}{r^2} + \frac{\vartheta-\vartheta_0-T}{r} - \frac{\vartheta-\vartheta_0-t}{r}e^{r(T-t)}\right] + \frac{b}{\vartheta-\vartheta_0-T}\left[\frac{2}{r^2} - \frac{2}{r^2}e^{r(T-t)} + \frac{\vartheta-\vartheta_0-2t}{r}e^{r(T-t)} - \frac{\vartheta-\vartheta_0-2T}{r}\right] \quad (39)$$

Substitute (37) in (39), we have:

$$E_{t,l}[L^{\mu^*}(T)] = l\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)e^{r(T-t)} + \frac{\varphi[\alpha-r]}{\beta(\vartheta-\vartheta_0-T)}\left[\frac{1}{r^2}e^{r(T-t)} - \frac{1}{r^2} + \frac{\vartheta-\vartheta_0-T}{r} - \frac{\vartheta-\vartheta_0-t}{r}e^{r(T-t)}\right] + \frac{b}{\vartheta-\vartheta_0-T}\left[\frac{2}{r^2} - \frac{2}{r^2}e^{r(T-t)} + \frac{\vartheta-\vartheta_0-2t}{r}e^{r(T-t)} - \frac{\vartheta-\vartheta_0-2T}{r}\right] + \frac{[\alpha-r]}{\beta}\sqrt{(T-t)\left(Var_{t,l}[L^{\mu^*}(T)]\right)}$$

Remark 1. If there is no supplementary premium, i.e. $\varphi = 0$, then the optimal portfolio policy, optimal fund size, and efficient frontier reduce to the following

$$\mu^* = \frac{\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)(\alpha-r)e^{-r(T-t)}}{\gamma l\beta^2}$$

$$L(t) = \frac{(\alpha-r)^2}{\gamma\beta^2}\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)te^{r(T-t)} - \frac{b}{r}\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right) +$$

$$\frac{1}{r^2}\left(\frac{2b}{\vartheta-\vartheta_0-t}\right)+\left(l_0(\vartheta-\vartheta_0)+\frac{b(\vartheta-\vartheta_0)}{r}-\frac{2b}{r^2}\right)\frac{e^{rt}}{\vartheta-\vartheta_0-t}$$

$$E_{t,l}[L^{\mu^*}(T)]=l\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)e^{r(T-t)}+\frac{b}{\vartheta-\vartheta_0-T}\left[\frac{2}{r^2}-\frac{2}{r^2}e^{r(T-t)}+\frac{\vartheta-\vartheta_0-2t}{r}e^{r(T-t)}-\frac{\vartheta-\vartheta_0-2T}{r}\right]+\frac{[\alpha-r]}{\beta}\sqrt{(T-t)\left(Var_{t,l}[L^{\mu^*}(T)]\right)}$$

B. Optimal Portfolio Policy and the Efficient Frontier When the Remaining Wealth Are not Equally Distributed among Members

Considering the time interval $[t,t+\frac{1}{i}]$, the differential form associated with the fund size when the remaining wealth is not equally distributed among the remaining members is given as:

$$L\left(t+\tfrac{1}{i}\right)=L(t)\left(\mu\frac{E_{t+\frac{1}{i}}}{E_t}+(1-\mu)\frac{C_{t+\frac{1}{i}}}{C_t}\right)+b\left(\tfrac{1}{i}\right)+\varphi dB_0-tb\tfrac{1}{i}K_{\vartheta_0+t} \quad (40)$$

$$L\left(t+\tfrac{1}{i}\right)-L(t)=L(t)\left(\mu(\frac{E_{t+\frac{1}{i}}-E_t}{E_t})+(1-\mu)(\frac{C_{t+\frac{1}{i}}-C_t}{C_t})\right)+b\left(\tfrac{1}{i}\right)+\varphi dB_0-tb\tfrac{1}{i}K_{\vartheta_0+t} \quad (41)$$

Substituting (6) into (41) we have

$$dL(t)=L(t)\left(\mu\left(\frac{dE_t(t)}{E_t(t)}\right)+(1-\mu)\left(\frac{dC_t(t)}{C_t(t)}\right)\right)+bdt+\varphi dB_0-tb\pi(\vartheta_0+t)dt \quad (42)$$

Substituting (1) and (2) into (42), we have

$$dL(t)=L(t)\big(\mu(\alpha dt+\beta dB_0)+(1-\mu)(rdt)\big)+bdt+\varphi dB_0-tb\pi(\vartheta_0+t)dt \quad (43)$$

$$dL(t)=\left\{L(t)(\mu[\alpha-r]+r)+b\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right)\right\}dt+(\mu L(t)\beta+\varphi)dB_0 L(0)=l_0 \quad (44)$$

According to [20], the mean-variance control problem (3) is similar to the Markovian time inconsistent stochastic optimal control problem with value function $A^*(t,l)$, see [16]. Our interest here is to determine the optimal portfolio policy for the two assets using the mean-variance utility function.

$$\begin{cases} B^*(t,l,\mu)=E_{t,l}[L^{\mu}(T)]-\frac{\gamma}{2}Var_{t,l}[L^{\mu}(T)] \\ B^*(t,l,\mu)=E_{t,l}[L^{\mu}(T)]-\frac{\gamma}{2}\left(E_{t,l}[L^{\mu}(T)^2]-(E_{t,l}[L^{\mu}(T)])^2\right) \\ A^*(t,l)=\sup_{\mu}B^*(t,l,\mu)\end{cases} \quad (45)$$

Following [1] the optimal portfolio policyμ^* satisfies:

$$A^*(t,l)=\sup_{\mu}(t,l,\mu^*) \quad (46)$$

$\gamma$is a constant representing risk aversion coefficient of the members. Let $u^{\mu}(t,l)=E_{t,l}[L^{\mu}(T)]$, $v^{\mu}(t,l)=E_{t,l}[L^{\mu}(T)^2]$ then $A^*(t,l)=\sup_{\mu}x^*\left(t,l,u^{\mu}(t,l),v^{\mu}(t,l)\right)$ where,

$$x^*(t,l,u,v)=u-\frac{\gamma}{2}(v-u^2) \quad (47)$$

Theorem 2 (verification theorem). If there exist three real functions X^*,Y^*,Z^* [0,T]$\times R\to R$ satisfying the following extended Hamilton Jacobi Bellman equation equations:

$$\begin{cases}\sup_{\mu}\begin{cases}{X^*}_t-{x^*}_t \\ +({X^*}_l-{x^*}_l)\begin{bmatrix}l(\mu[\alpha-r]+r)+\\ b\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right)\end{bmatrix} \\ +\frac{1}{2}({X^*}_{ll}-U_{ll})(\mu l\beta+\varphi)^2\end{cases}=0 \\ X^*(T,l)=x^*(t,l,l,l^2)\end{cases} \quad (48)$$

where,

$$U_{ll}={x^*}_{ll}+2{x^*}_{lu}u_l+2{x^*}_{lv}v_l+{x^*}_{uu}u_l^2+2{x^*}_{uv}u_lv_l+{x^*}_{vv}v_l^2=\gamma u_l^2$$

$$\left\{\begin{cases}{Y^*}_t \\ +{Y^*}_l\begin{bmatrix}l(\mu[\alpha-r]+r)\\ +b\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right)\end{bmatrix}+\frac{1}{2}{Y^*}_{ll}(\mu l\beta+\varphi)^2\end{cases}=0 \\ Y^*(T,l)=l\right\} \quad (49)$$

$$\left\{\begin{cases}{Z^*}_t \\ +{Z^*}_l\begin{bmatrix}l(\mu[\alpha-r]+r)\\ +b\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right)\end{bmatrix}+\frac{1}{2}{Z^*}_{ll}(\mu l\beta+\varphi)^2\end{cases}=0 \\ Z^*(T,l)=l^2\right\} \quad (50)$$

Then $A^*(t,l)=X^*(t,l),u^{\mu^*}=Y^*(t,l),v^{\mu^*}=Z^*(t,l)$ for the optimal portfolio policyμ^*

Proof. For the details of the proof, see [17]-[19].

Next, we determine the optimal investment policy for the two available assets and also the efficient frontier by solving (48)-(50).

Recall that $x(t,l,u,v)=u-\frac{\gamma}{2}(v-u^2)$

$${x^*}_t={x^*}_l={x^*}_{ll}={x^*}_{lu}={x^*}_{lv}={x^*}_{uv}={x^*}_{vv}=0,{x^*}_u=1+\gamma u,{x^*}_{uu}=\gamma,{x^*}_v=-\frac{\gamma}{2} \quad (51)$$

Substituting (51) into (48) and differentiating (48) with respect to μ and solving for μ we have:

$$\mu^*=-\left[\frac{(\alpha-r){X^*}_l+\varphi\beta\left({X^*}_{ll}-\gamma{Y^*}_l^2\right)}{({X^*}_{ll}-\gamma{Y^*}_l^2)l\beta^2}\right] \quad (52)$$

Substituting (52) into (48) and (49) we have

$${X^*}_t+{X^*}_l\left[rl+b\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right)\right]-{X^*}_l^2\frac{(\alpha-r)^2}{2({X^*}_{ll}-\gamma{Y^*}_l^2)\beta^2}-\frac{\varphi}{\beta}{X^*}_l[\alpha-r]=0 \quad (53)$$

$${Y^*}_t+{Y^*}_l\left[rl+b\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right)\right]-{X^*}_l{Y^*}_l\frac{(\alpha-r)^2}{2({X^*}_{ll}-\gamma{Y^*}_l^2)\beta^2}-\frac{\varphi}{\beta}Y_l[\alpha-r]+\frac{{Y^*}_{ll}}{2}\left[-{X^*}_l\left[\frac{(\alpha-r){X^*}_l+\varphi\beta({X^*}_{ll}-\gamma{Y^*}_l^2)}{({X^*}_{ll}-\gamma{Y^*}_l^2)\beta}\right]+\varphi\right]^2=0 \quad (54)$$

Next, we assume a solution for $X^*(t,l)$ and $Y^*(t,l)$ as:

$$\begin{cases} X^*(t,l) = F^*(t)l + G^*(t)F^*(T) = 1, G^*(T) = 0 \\ Y^*(t,l) = H^*(t)l + I^*(t)H^*(T) = 1, I^*(T) = 0 \\ X^*{}_t = lF^*(t) + G^*(t), X^*{}_l = F^*(t), \\ X^*{}_{ll} = 0, Y^*{}_t = lH^*(t) + I^*(t), Y^*{}_l = H^*(t), Y^*{}_{ll} = 0 \end{cases} \quad (55)$$

Substituting (55) into (53) and (54)

$$\begin{cases} F^*{}_t(t) + rF^*(t) = 0 \\ G^*{}_t(t) + F^*(t)b\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right) + F^{*2}(t)\frac{[\alpha-r]^2}{2\gamma H^{*2}(t)\beta^2} \\ -\frac{\varphi}{\beta}F^*(t)[\alpha-r] = 0 \end{cases} \quad (56)$$

$$\begin{cases} H^*{}_t(t) + rH^*(t) = 0 \\ I^*{}_t(t) + H^*(t)b\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right) + F^*(t)\frac{[\alpha-r]^2}{\gamma H^*(t)\beta^2} \\ -\frac{\varphi}{\beta}H^*(t)[\alpha-r] = 0 \end{cases} \quad (57)$$

Solving (56) and (57), we have

$$F^*(t) = e^{r(T-t)} \quad (58)$$

$$H^*(t) = e^{r(T-t)} \quad (59)$$

$$G^*(t) = \frac{1}{2\gamma\beta^2}(\alpha-r)^2(T-t) + (\frac{\varphi}{\beta}[\alpha-r] - \frac{b}{r})(1-e^{r(T-t)}) + b\int_t^T \frac{\tau}{\vartheta-\vartheta_0-\tau}e^{-r\tau}d\tau \quad (60)$$

$$I^*(t) = \frac{1}{\gamma\beta^2}(\alpha-r)^2(T-t) + (\frac{\varphi}{\beta}[\alpha-r] - \frac{b}{r})(1-e^{r(T-t)}) + b\int_t^T \frac{\tau}{\vartheta-\vartheta_0-\tau}e^{-r\tau}d\tau \quad (61)$$

$$X^*(t,l) = le^{-r(T-t)} + \frac{1}{2\gamma\beta^2}(\alpha-r)^2(T-t) + (\frac{\varphi}{\beta}[\alpha-r] - \frac{b}{r})(1-e^{r(T-t)}) + b\int_t^T \frac{\tau}{\vartheta-\vartheta_0-\tau}e^{-r\tau}d\tau \quad (62)$$

$$Y^*(t,l) = le^{-r(T-t)} + \frac{1}{\gamma\beta^2}(\alpha-r)^2(T-t) + (\frac{\varphi}{\beta}[\alpha-r] - \frac{b}{r})(1-e^{r(T-t)}) + b\int_t^T \frac{\tau}{\vartheta-\vartheta_0-\tau}e^{-r\tau}d\tau \quad (63)$$

Result 4. The optimal investment strategy for the risky asset is given as

$$\mu^* = \frac{(\alpha-r)e^{-r(T-t)}-\gamma\beta\varphi}{\gamma l\beta^2} \quad (64)$$

Proof. From (52) and (55), we have

$$\mu^* = -\left[\frac{(\alpha-r)X^*{}_l + \varphi\beta\left(X^*{}_{ll} - \gamma Y^{*2}_l\right)}{\left(X^*{}_{ll} - \gamma Y^{*2}_l\right)l\beta^2}\right]$$

$$X^*{}_l = F^*(t) = e^{-r(T-t)}, X^*{}_{ll} = 0, Y^*{}_l = H^*(t) = e^{-r(T-t)}$$

then

$$\mu^* = -\left[\frac{(\alpha-r)e^{-r(T-t)} - \varphi\beta\gamma e^{-2r(T-t)}}{\gamma e^{-2r(T-t)}l\beta^2}\right]$$

$$\mu^* = \frac{(\alpha-r)e^{-r(T-t)}-\varphi\beta\gamma}{\gamma l\beta^2}$$

Result 5. The optimal fund size is given as

$$L(t) = \frac{(\alpha-r)^2}{\gamma\beta^2}(t-T)e^{r(T-t)} + \frac{\frac{\varphi}{\beta}[\alpha-r]-b}{r}(1-e^{r(t-T)}) - b\int_t^T \frac{\tau e^{-r\tau}}{\vartheta-\vartheta_0-\tau}d\tau \quad (65)$$

Proof. Recall that (44) and (52) are given respectively as

$$dL(t) = \left\{L(t)(\mu[\alpha-r]+r) + b\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right)\right\}dt + (\mu L(t)\beta + \varphi)dB_0 L(0) = l_0$$

$$\mu^* = \frac{(\alpha-r)e^{-r(T-t)}-\varphi\beta\gamma}{\gamma l\beta^2}$$

Substituting (52) into (44), we have

$$L_t(t) - rL = \frac{(\alpha-r)^2}{\gamma\beta^2}e^{r(T-t)} - \frac{\varphi}{\beta}[\alpha-r] + b\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right)$$

$$L(0) = l_0 \quad (66)$$

Solving (66) for $L(t)$ with initial condition we have

$$L(t) = \frac{(\alpha-r)^2}{\gamma\beta^2}(t-T)e^{r(T-t)} + \frac{\frac{\varphi}{\beta}[\alpha-r]-b}{r}(1-e^{r(t-T)}) - b\int_t^T \frac{\tau e^{-r\tau}}{\vartheta-\vartheta_0-\tau}d\tau$$

Result 6. The efficient frontier of the pension fund is given as

$$E_{t,l}[L^{\mu^*}(T)] = le^{-r(T-t)} + (\frac{\varphi}{\beta}[\alpha-r] - \frac{b}{r})(1-e^{r(T-t)}) + b\int_t^T \frac{\tau}{\vartheta-\vartheta_0-\tau}e^{-r\tau}d\tau + \frac{(\alpha-r)}{\beta}\sqrt{\left(Var_{t,l}[L^{\mu^*}(T)]\right)(T-t)} \quad (67)$$

Recall that

$$Var_{t,l}[L^{\mu^*}(T)] = E_{t,l}[L^{\mu}(T)^2] - (E_{t,l}[L^{\mu}(T)])^2$$

$$Var_{t,l}[L^{\mu^*}(T)] = \frac{2}{\gamma}(Y^*(t,l) - X^*(t,l)) \quad (68)$$

Substituting (62) and (63) into (68), we have

$$Var_{t,l}[L^{\mu^*}(T)] = \frac{1}{\gamma^2\beta^2}(\alpha-r)^2(T-t) \quad (69)$$

$$\frac{1}{\gamma} = \frac{\beta}{(\alpha-r)}\sqrt{\frac{Var_{t,l}[L^{\mu^*}(T)]}{(T-t)}} \quad (70)$$

$$E_{t,l}[L^{\mu^*}(T)] = Y^*(t,l) \quad (71)$$

Substituting (63) into (71), we have

$$E_{t,l}[L^{\mu^*}(T)] == le^{-r(T-t)} + \frac{1}{\gamma\beta^2}(\alpha-r)^2(T-t) + (\frac{\varphi}{\beta}[\alpha-r] - \frac{b}{r})(1-e^{r(T-t)}) + b\int_t^T \frac{\tau}{\vartheta-\vartheta_0-\tau}e^{-r\tau}d\tau \quad (72)$$

Substitute (70) in (72), we have:

$$E_{t,l}[L^{\mu^*}(T)] = le^{-r(T-t)} + (\frac{\varphi}{\beta}[\alpha - r] - \frac{b}{r})(1 - e^{r(T-t)}) + b\int_t^T \frac{\tau}{\vartheta - \vartheta_0 - \tau} e^{-r\tau} d\tau + \frac{(\alpha - r)}{\beta}\sqrt{(Var_{t,l}[L^{\mu^*}(T)])(T-t)}$$

IV. Numerical Simulation

In this section we present numerical simulations of the optimal investment policy with respect to time using the following data: $\vartheta = 100$; $\vartheta_0 = 20$; $\gamma = 0.05$; $r = 0.02$; $\alpha = 0.05$; $\beta = 1$;$l = L(t)$; $l_0 = 1$; $T = 40$; $t = 0{:}5{:}20$; $\varphi = 0.05$.

V. Discussion

From Remark 1, we observed that when $\varphi = 0$, the optimal portfolio policy, optimal fund size, and efficient frontier reduced to the one obtained in [11]. Also, (31) and (64) show there are disparity between the optimal portfolio policies of equity for the two cases.

In Fig. 1, we observed that the optimal portfolio policy with supplementary premium is lower compared to the optimal portfolio policy without supplementary premium. This is because with the supplementary premium, the overall pension wealth is increased and the pension manager will prefer to invest more in a riskless asset rather than investing more in risky asset. Figs. 2 and 4 show that the optimal portfolio policy is inversely proportional to the supplementary premium i.e. as the supplementary premium increases, the optimal portfolio policy decreases which implies that with more funds less risk is taken and if there are lesser funds the fund manager increases investment in equity. In Figs. 1and 2, the optimal investment policy decreases with time; this is because at the early stage of investment, the optimal fund size which corresponds to optimal portfolio policy was used. We also observed that with time, the fund manager reduces the fraction of his wealth invested in equity to avoid his members of losing what he or she has accumulated over time and invest more in cash as retirement age approaches. Similarly, in Figs. 3 and 4, we observed that the optimal portfolio policy increases with time; this is because the fund manager started with the initial wealth and as retirement age draws near, the willingness to invest in equity increases in order to increase the expected returns of his members.

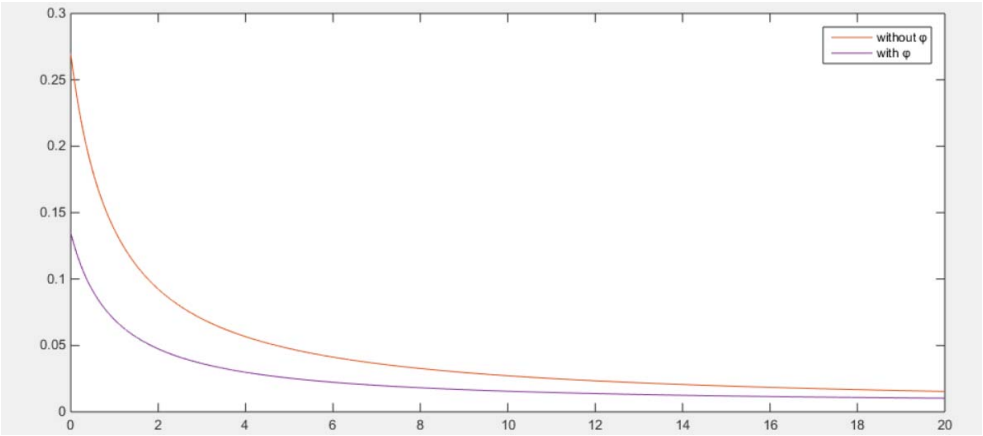

Fig. 1 Time evolution of optimal portfolio policy with φ and without $\varphi$when the remaining wealth is divided equally

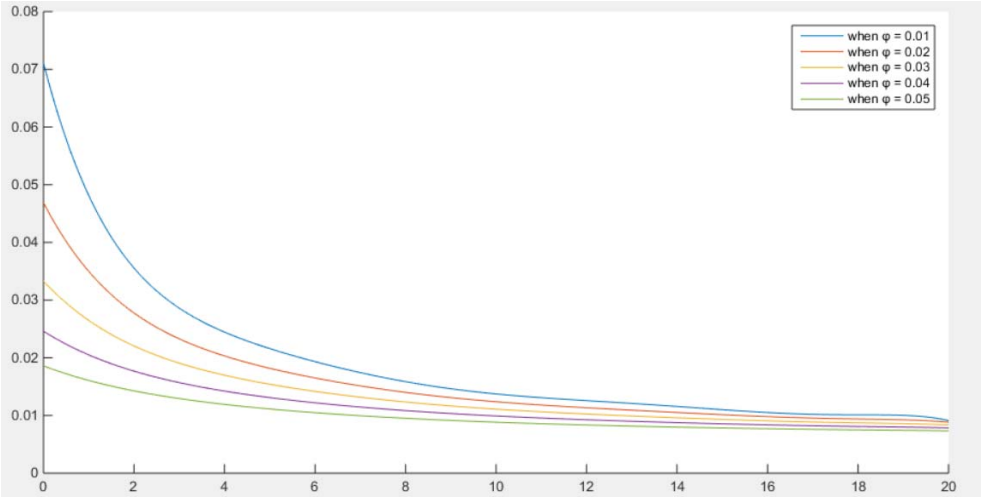

Fig. 2 Time evolution of optimal portfolio policy with different $\varphi$when the remaining wealth is divided equally

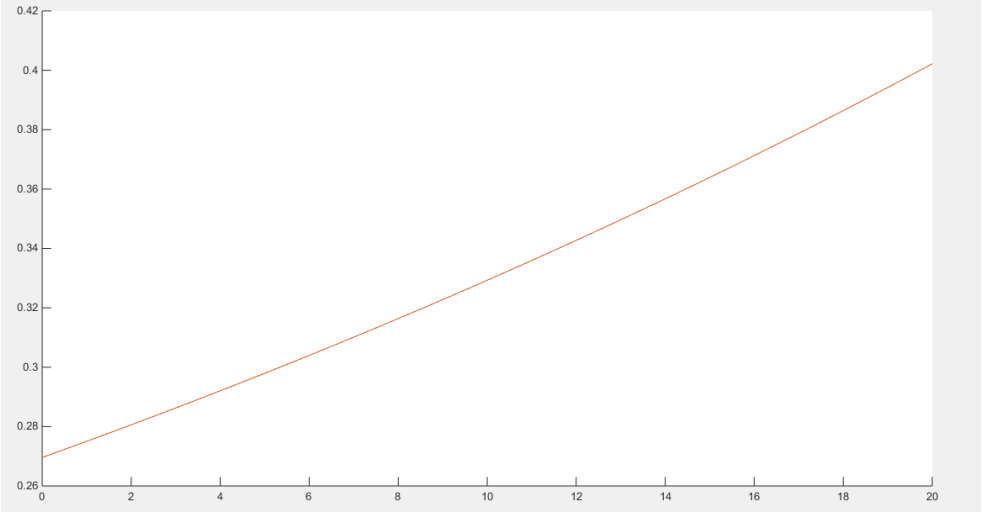

Fig. 3 Time evolution of optimal portfolio policy with φ when the remaining wealth is not equally divided

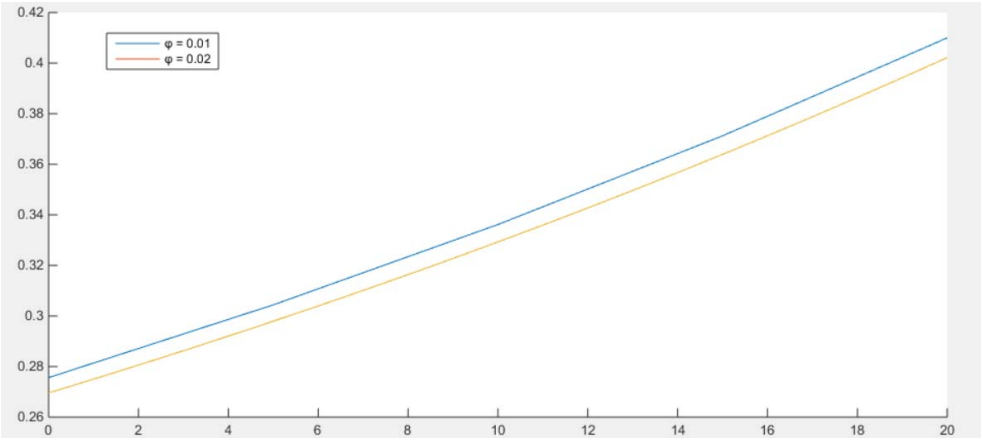

Fig. 4 Time evolution of optimal portfolio policy with different φ when the remaining wealth is not divided equally

VI. Conclusion

The effect of supplementary premium on the optimal portfolio policy in a DC pension scheme with refund of premium clauses was studied. The clause enables death members' next of kin to claim the accumulated wealth of the death members during the accumulation phase. We considered two cases: (1) when the remaining wealth is equally distributed among the remaining members of the pension scheme and (2) when it is not equally distributed among the remaining members. The supplementary premium which is to help sustain the scheme is assumed stochastic. We considered investments in cash and equity to help increase the accumulated funds of the remaining members to meet their retirement needs. Also, we composed the problem as a continuous time mean-variance stochastic optimal control problem using the actuarial symbol and an optimized problem is established from the extended Hamilton Jacobi Bellman equations. We obtained the optimal portfolio policy, the corresponding optimal fund size for the two assets and also the efficient frontier of the pension members for the two cases. Furthermore, the effect of the supplementary premium on the optimal portfolio policy with numerical simulations were discussed and observed that the supplementary premium decreases the optimal portfolio policy of the risky asset (equity).

References

[1] A. J. G. Cairns, D. Blake, K. Dowd. Stochastic life styling: optimal dynamic assetallocation for defined contribution pension plans, Journal of Economic Dynamics &Control 30(5) (2006) 843–877.
[2] M. Giacinto, F. Gozzi, S. Federico, 2011. Pension funds with a minimum guarantee: a stochastic control approach. Finance and Stochastic. 15, 297-342
[3] J. Gao. Stochastic optimal control of DC pension funds, Insurance, 42(3) (2008), 1159–1164.
[4] J. F. Boulier, S. Huang, G. Taillard G. Optimal management under stochastic interest rates: the case of a protected defined contribution pension fund, Insurance 28(2) (2001), 173–189.
[5] P. Battocchio, F. Menoncin. Optimal pension management in a stochastic framework, Insurance34(1) (2004) 79–95.
[6] Njoku, K.N. C. Osu, B. O, Akpanibah, E. E. and Ujumadu, R. N. (2017) 'Effect ofExtraContribution on Stochastic Optimal Investment Strategies for DC Pension with Stochastic Salary under the Affine Interest Rate Model' Journal of Mathematical Finance, 7, 821-833.

[7] E. E. Akpanibah, and O. Okwigbedi (2018) '' Optimal Portfolio Selection in a DC Pension withMultiple Contributors and the Impact of Stochastic Additional Voluntary Contribution on the Optimal Investment Strategy'' International journal of mathematical and computational sciences, 12(1) (2018), 14-19.
[8] J. Xiao, Z. Hong, C. Qin. The constant elasticity of variance (CEV) model and the Legendre transform-dual solution for annuity contracts, Insurance, 40(2) (2007), 302–310.
[9] J. Gao. Optimal portfolios for DC pension plans under a CEV model, Insurance: Mathematics and Economics 44 (2009), 479-490.
[10] E. E. Akpanibah, S. K. Samaila**.**Stochastic strategies for optimal investment in a defined contribution (DC) pension fund, International Journal of Applied Science and Mathematical Theory, 3(3) (2017), 48-55.
[11] L. He, Z. Liang. The optimal investment strategy for the DC plan with the return of premiums clauses in a mean-variance framework, Insurance, 53(2013), 643-649.
[12] B. O. Osu, E. E. Akpanibah, B I. Oruh. Optimal investment strategies for defined contribution (DC) pension fund with multiple contributors via Legendre transform and dual theory, International journal of pure and applied researches, 2(2) (2017), 97-105
[13] B. O. Osu, E. E. Akpanibah, O. Olunkwa. Mean-Variance Optimization of portfolios with return of premium clauses in a DC pension plan with multiple contributors under constant elasticity of variance model, International journal of mathematical and computational sciences pure and applied researches, 12(5) (2018), 85-90.
[14] D Li, X. Rong, H. Zhao, B. Yi. Equilibrium investment strategy for DC pension plan with default risk and return of premiums clauses under CEV model, Insurance 72(2017), 6-20
[15] E. E. Akpanibah, B. O. Osu. Optimal Portfolio Selection for a Defined Contribution Pension Fund with Return Clauses of Premium with Predetermined Interest Rate under Mean-variance Utility, Asian Journal of Mathematical Sciences 2(2).(2018) 19-29
[16] D. Sheng, X. Rong. Optimal time consistent investment strategy for a DC pension with the return of premiums clauses and annuity contracts, Hindawi Publishing Corporation vol (2014) http://dx.doi.org/10.1155/2014/862694. 1-13
[17] Y. Zeng, Z. Li. Optimal time consistent investment and reinsurance policies for mean-variance insurers. Insurance: Mathematics & Economics 49(2011), 145–154.
[18] L. He, Z. Liang. Optimal financing and dividend control of the insurance company with fixed and proportional transaction costs. Insurance: Mathematics & Economics 44(2009), 88–94.
[19] Z. Liang, J. Huang. Optimal dividend and investing control of an insurance company with higher solvency constraints. Insurance: Mathematics & Economics 49(2011), 501–511.
[20] T. Björk, A. Murgoci. A general theory of Markovian time inconsistent stochastic control problems. Working Paper. Stockholm School of Economics. http://ssrn.com/abstract=1694759. (2009).