International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:4, No:12, 2010

Effect of Implementation of Nonlinear Sequence Transformations on Power Series Expansion for a Class of Non-Linear Abel Equations

Javad Abdalkhani

Abstract—Convergence of power series solutions for a class of non-linear Abel type equations, including an equation that arises in nonlinear cooling of semi-infinite rods, is very slow inside their small radius of convergence. Beyond that the corresponding power series are wildly divergent. Implementation of nonlinear sequence transformation allow effortless evaluation of these power series on very large intervals..

Keywords—Nonlinear transformation, Abel Volterra Equations, Mathematica

I. Introduction

Nonlinear Volterra integral equations of the second kind with weakly singular kernels (or the Abel type equations) have the following general form:

$$x(t) = g(t) + \int_0^t \frac{K(t,s,x(s))}{(t-s)^\alpha} ds, 0 \le \alpha < 1, t \in [0, T_0], \tag{1}$$

T_0 is a fixed real number. For conditions on existence, uniqueness, and continuity of a solution for equation (1) see [19], [5], or [17]. Most often, it is not possible to find the analytical solution of equation (1), and one must find a numerical method to approximate the solution. However, finding an effective computational technique for (1) is challenging because (normally) the analytical solution of (1) is not differentiable at 0, see for example,[2], [5], [13], or [18]. There are essentially two different approaches to compensate for this lack of differentiability.

One approach tries to find an analytical method to downgrade the effect of the singularity and then find a numerical technique. For example, when $\alpha = 0.5$, Theorem 1.1 of [13] shows that when g and K satisfy certain differentiability criteria in their domains of definitions (see [5] page 29 for details), then

$$x(t) = u(t) + \sqrt{t}v(t), \tag{2}$$

where $u(t)$ and $v(t)$ are differentiable functions on the interval of integration. The authors then were able to find more accurate numerical results for (1) based on equation(2). Their analysis could not be extended to other cases where $\alpha \ne 0.5$. Theorem 2 page 89 of [18] shows that if g and K are analytic in their domains of definitions, then $x(t)$, can be written as

$$x(t) = X(t, t^{1-\alpha}), \tag{3}$$

Javad Abdalkhani is with the Department of Mathematics, Ohio State University,Lima, OH, 45804 USA e-mail: abdalkhani.1@osu.edu.

.

where $X(z_1, z_2)$ is real analytic at $(0,0)$. Based on equation(3) the author offers effective numerical techniques. In [2], for linear equations the singularity is extracted and as a result a numerical method that works well for approximation of a non-singular equation will produce accurate numerical approximation when applied to equation(1).

The other class of approach lessens the effect of singularity by choosing either non-polynomial instead of polynomial approximations, or graded meshes instead of the regular ones. In [7] approximation on graded grids and in ([8], [9]) the hybird collocation methods are used for linear equations. For nonlinear equations [6] uses graded grids, and [14] employs β-polynomial spline collocation. In [17] the product integration method is used to find accurate approximation for $x(t)$. For related works on the Fredholm integral equations see [4] pp 116-156 or [11] and references given there.

In this paper, for a particular class of Abel equations, we offer an alternative technique which completely avoids quadratures and solves this class of equations very accurately, in particular for ts close to 0. This subclass of equations are:

$$y(t) = 1 - \lambda \int_0^t \frac{y^n(s)}{(t-s)^\alpha} ds \;\; 0 \le \alpha < 1, \;\; \lambda > 0, \;\; n = 1, 2, 3, \ldots. \tag{4}$$

For a given set of α, λ, and n, the corresponding analytical solution $y(t)$ of equation (4) is unique, continuous, decreasing in t, $0 < y(t) \le 1$, and $y(t) \to 0$ as $t \to \infty$, see theorem 6.3 of [19]. In subsequent sections we first find the power series expansion of $y(t)$. For $n = 1$ this power series is always convergent. However, for $n \ge 2$ normally the power series solution obtained for equation (4) has a small radius of convergence. For example, if $\lambda = 1/\sqrt{\pi}, \alpha = 1/2$, and $n = 4$ (a case of practical importance), we will show that $r \approx 0.0258$. Moreover, the convergence of the power series for $t \in [0, r]$, is very slow. Outside the interval of convergence the corresponding power series are wildly divergent.

The main objective of this article is to show the remarkable effect of applying nonlinear sequence transformation to these power series. For instance, $y(t)$ in the above example can be evaluated (almost instantaneously) for each $t \in [0, 40]$, (a significant improvement when compared to the interval of (slow) convergence of only $[0, .0258]$). Contrary to quadrature methods; the approximation for each $y(a)$ is found directly and independent of evaluations of prior values of $y(t)$ for $t < a$.

II. Power expansion of $y(t)$

To obtain the (Neumann) series expansion for $x(t)$ one employs the Picard successive approximation using the following pair of recursive relations,

$$x_{n+1}(t) = g(t) + \int_0^t \frac{K(t,s,x_n(s))}{(t-s)^\alpha} ds, \quad \text{with} \quad x_0(t) = g(t). \tag{5}$$

For linear equations the Picard successive iterations is always convergent ($x_n(t) \to x(t)$ as $n \to \infty$, for $t \in [0, T_0]$), see theorem 10.15 page 152 of [16] also pp 92-95 of [22]; and [15] pp 34-35 for solved examples. The Picard method is simple to program, and is effective if the corresponding series converges after adding a sufficiently small number of terms.

For nonlinear equations $x_n(t) \to x(t)$ as $n \to \infty$, on a sufficiently small interval $[0, r]$, with $r \le T_0$. For more details see [19] , [12], or [17]. In addition, evaluation of successive steps of Picard's iterations in some cases is not practical (e.g. $n \ge 4$ in equation (4)). However, we are still able to obtain a power series expansion for equation (4). To do this we need the following two equations; the first equation is:

$$\int_0^t \frac{s^\gamma}{(t-s)^\alpha} ds = t^{\gamma+1-\alpha} B(\gamma+1, 1-\alpha), \tag{6}$$

where $\gamma > -1, \alpha < 1$, and B is the Beta function. Equation (6) can be established using the definition of beta function and the change of variable $s = tu$. The second equation we need is:

$$\begin{aligned}(x_1 + x_2 + \ldots + x_m)^n &= \sum \frac{n!}{n_1! n_2! \ldots n_m!} x_1^{n_1} x_2^{n_2} \ldots x_m^{n_m} \\ &= \sum_{|\delta|=n} \binom{n}{\delta} x^\delta,\end{aligned} \tag{7}$$

where $n_1 + n_2 + \ldots + n_m = n$, $\delta = (n_1, n_2, \ldots, n_m)$, and $x^\delta = x_1^{n_1} x_2^{n_2} \ldots x_m^{n_m}$. This is the "multinomial theorem" see for example [3] page 823.

For each given $n = n_0$, one can show by direct substitution, that the k^{th} iteration of the Picard method for equation (4) is given by $y_k(t) = \sum_{i=0}^{N_k} a_i t^{i(1-\alpha)}$, $N_k \ge k$, with $y_0(t) = 1$

Example 2.1: For $n = 2$ and $y_0(t) = 1$ we have $y_1(t) = 1 - \lambda \int_0^t \frac{1}{(t-s)^\alpha} ds = 1 - \frac{\lambda}{1-\alpha} t^{1-\alpha}$. Similarly, $y_2(t) = 1 - \lambda \int_0^t \frac{(1 - \lambda \frac{s^{1-\alpha}}{1-\alpha})^2}{(t-s)^\alpha} ds$, which implies $y_2(t) = 1 - \frac{\lambda}{1-\alpha} t^{1-\alpha}$ $+ 2\frac{\lambda^2 B(2-\alpha, 1-\alpha)}{1-\alpha} t^{2-2\alpha} - \frac{\lambda^3 B(3-\alpha, 1-\alpha)}{(1-\alpha)^2} t^{3-3\alpha}$, etc.

The convergence theory of Picard's iterations guarantees the existence of a positive r such that $y_k(t) \to y(t)$ uniformly as $k \to \infty$, for $t \in [0, r]$, see [19]. Therefore,

$$y(t) = \sum_{i=0}^{\infty} a_i t^{i(i-\alpha)}, \quad for \quad t \in [0, r]. \tag{8}$$

Remark 2.2: It has been shown in the literature that the power series solution of the (general) equation (1) for the most practical case; $\alpha = 1/2$, under proper conditions on K and g can be represented by $x(t) = \sum_{i=0}^{\infty} b_i t^{i/2}$; see[13].

Theorem 2.3: The coefficients a_ms, $m = 1, 2, \ldots$ in equation (8) are given by:

$$\begin{aligned}&a_m = -\lambda \sum \tfrac{n!}{n_0! n_1! \ldots n_{m-1}} a_0^{n_0} a_1^{n_1} \ldots a_{n_{m-1}}^{n_{m-1}} \\ &B((n_1 + 2n_2 + \ldots + (m-1)n_{m-1})(1-\alpha) + 1, 1-\alpha),\end{aligned} \tag{9}$$

together with the following pair of constrains:

$$\begin{cases} n_0 + n_1 + \ldots + n_{m-1} = n \\ n_1 + 2n_2 + 3n_3 \ldots + (m-1)n_{m-1} = m-1 \end{cases} \tag{10}$$

Proof: This proof is by induction. To find a_is the coefficients in equation (8) we substitute $y(t)$ by $\sum_{i=0}^{\infty} a_i t^{i(i-\alpha)}$ on both sides of (4) and obtain

$$\sum_{i=0}^{\infty} a_i t^{i(1-\alpha)} = 1 - \lambda \int_0^t \frac{(\sum_{i=0}^{\infty} a_i s^{i(1-\alpha)})^n}{(t-s)^\alpha} ds. \tag{11}$$

We will find $a_0, a_1, a_2 \ldots$, one-by-one, by matching the same powers of t from both sides of equation (11). We do this by ignoring all terms of higher order in t from both sides of equation (11). To find a_0 on left, we note that 1 is the only constant on the right and therefore,

$$a_0 = 1.$$

To find a_1, on the left of equation (11) we have $1 + a_1 t^{1-\alpha} +$ terms of higher in t. On the right we need to replace the infinite sum in the integrand by 1 to produce $t^{1-\alpha}$. Therefore, we have $1 + a_1 t^{1-\alpha} = 1 - \lambda \int_0^t 1/(t-s)^\alpha ds = 1 - \lambda(t^{(1-\alpha)}/(1-\alpha))$. Equating powers of $t^{1-\alpha}$ from both sides, we have:

$$a_1 = -\lambda/(1-\alpha).$$

To find a_2 the coefficient of $t^{2(1-\alpha)}$ we replace $y(s)$ by $a_0 + a_1 s^{1-\alpha}$ on the right hand side of equation (11) . We obtain $1 - \frac{\lambda}{1-\alpha} t^{1-\alpha} + a_2 t^{2(1-\alpha)} = 1 - \lambda \int_0^t \frac{(a_0 + a_1 s^{1-\alpha})^n}{(t-s)^\alpha} =$ $1 - \lambda \int_0^t \frac{\sum \frac{n!}{n_0! n_1!} a_0^{n_0} a_1^{n_1} s^{n_1(1-\alpha)}}{(t-s)^\alpha}$, with $n = n_1 + n_0$, recall equation (7). After integration (recall equation (6)) we have $1 - \frac{\lambda}{1-\alpha} t^{1-\alpha} + a_2 t^{2(1-\alpha)} = 1 -$ $\lambda \sum \frac{n!}{n_0! n_1!} a_0^{n_0} a_1^{n_1} t^{(n_1+1)(1-\alpha)} B(n_1(1-\alpha)+1, 1-\alpha)$. To find a_2 on the left we must have $n_1 = 1$ and $n_0 = n-1$ on the right. Equating equal powers of $t^{2(1-\alpha)}$ from both sides we obtain:

$$a_2 = -n\lambda a_0^{n-1} a_1 B((1-\alpha)+1, 1-\alpha) = n\lambda^2 B(2-\alpha, 1-\alpha)/(1-\alpha).$$

In general, suppose $a_1, a_2, \ldots, a_{m-1}$ are found; to find a_m we set

$$\begin{aligned}&a_0 + a_1 t^{1-\alpha} + \ldots + a_m t^{m(1-\alpha)} = 1 - \lambda \\ &\int_0^t \frac{\sum \frac{n!}{n_0! n_1! \ldots n_{m-1}} a_0^{n_0} a_1^{n_1} \ldots a_{n_{m-1}}^{n_{m-1}} s^{(n_1 + 2n_2 + \ldots + (m-1)n_{m-1})(1-\alpha)}}{(t-s)^\alpha} ds \\ &= 1 - \lambda \sum \tfrac{n!}{n_0! n_1! \ldots n_{m-1}} a_0^{n_0} a_1^{n_1} \ldots a_{n_{m-1}}^{n_{m-1}} \\ &t^{(1 + n_1 + 2n_2 + \ldots + (m-1)n_{m-1})(1-\alpha)} \\ &B((n_1 + 2n_2 + \ldots + (m-1)n_{m-1})(1-\alpha) + 1, 1-\alpha),\end{aligned} \tag{12}$$

with $n_0 + n_1 + \ldots + n_{m-1} = n$, and $n_1 + 2n_2 + 3n_3 \ldots + (m-1)n_{m-1} = m-1$. Equating the same power of $t^{m(1-\alpha)}$

from both sides, implies

$$a_m = -\lambda \sum \frac{n!}{n_0!n_1!\ldots n_{m-1}} a_0^{n_0} a_1^{n_1} \ldots a_{n_{m-1}}^{n_{m-1}} \\ B((n_1 + 2n_2 + \ldots + (m-1)n_{m-1})(1-\alpha) + 1, 1-\alpha), \tag{13}$$

■

One can find each a_i only in terms of λ, α and n given in equation (4). For example; we have;
$a_3 = -\lambda \left(n a_0^{n-1} a_2 + \dfrac{n(n-1)}{2!} a_0^{n-2} a_1^2 \right)$ $B(2(1-\alpha)+1, 1-\alpha)$. Substituting for a_0, a_1, and a_2 found in Theorem2.3 , we obtain a_3 as:

$$a_3 = \frac{n\lambda^3 B(3-2\alpha,1-\alpha)(1-n+2n(-1+\alpha)B(2-\alpha,1-\alpha))}{2(-1+\alpha)^2} \tag{14}$$

Other coefficients $a_4, a_5, \ldots$ can be found similarly.

Fortunately, it is possible to find a_ms (given by equation (9) under conditions (10)) easily for a given $n = N_0$ using the subsequent simple Mathematica program .

Mathematica Program:

$$a_0 = 1;\ a_1 = \frac{\lambda}{1-\alpha};$$
$$a_m = \Big[\text{Assuming}, \quad m > 1,$$
$$\lambda \cdot \text{Coefficient}\left[\left(\sum_{i=0}^{m-1} a_i \cdot t^{i(1-\alpha)}\right)^{N_0}, t^{(m-1)(1-\alpha)}\right].$$
$$\text{Beta}[(m-1)(1-\alpha)+1, 1-\alpha]\Big].$$

III. Linear Equations

For the linear case, $n = 1$ in equation (4), one can easily verify that a_is are given recursively by

$$\begin{cases} a_i = a_{i-1} B(i-(i-1)\alpha, 1-\alpha), \text{for} \quad i = 1, 2, 3, \ldots \\ \text{with} \quad a_0 = 1 \end{cases} \tag{15}$$

Therefore, the solution of

$$z(t) = 1 - \lambda \int_0^t \frac{z(s)}{(t-s)^\alpha} ds, \quad 0 \le \alpha < 1, \tag{16}$$

is given by:

$$z(t) = \sum_{i=0}^{\infty} (-\lambda)^i a_i t^{i-i\alpha}, \tag{17}$$

where a_is are given by (15). We note that $B(n-(n-1)\alpha, 1-\alpha) \to 0$ as $n \to \infty$ and therefore, the radius of convergence for this series is $\rho = \lim\limits_{n\to\infty} \dfrac{1}{\lambda B(n-(n-1)\alpha, 1-\alpha)} = \infty$. The solution of the general linear equation

$$u(t) = h(t) - \lambda \int_0^t \frac{u(s)}{(t-s)^\alpha} ds, \quad 0 \le \alpha < 1, \tag{18}$$

then is given by:

$$u(t) = h(0)z(t) + \int_0^t z(s)h'(t-s)ds, \tag{19}$$

see [12], page 39, Theorem 2.4.3.

Remark 3.1: For the particular case $y(t) = 1 - \int_0^t \dfrac{y(s)ds}{\sqrt{t-s}}$, the exact solution $y(t)$ can also be given as, $y(t) = e^{\pi t} ercf\sqrt{\pi t}$, see [19] page 23. This solution is consistent with the solution given by equation (17).

IV. Nonlinear Equations: An Application

To illustrate the difficulties with nonlinear equations we study

$$y(t) = 1 - \frac{1}{\sqrt{\pi}} \int_0^t \frac{y^4(s)}{\sqrt{t-s}} ds. \tag{20}$$

This equation arises in nonlinear cooling of a semi-infinite rod. O'Conner [21] studied the analytical, physical, and numerical aspects of this equation. To find the power series expansion for $y(t)$ rewrite $y(t)$ the exact solution of (20), for $t \in [0, r]$ for a sufficiently small $r > 0$, as

$$y(t) = \sum_{i=0}^{\infty} a_i t^{i/2}. \tag{21}$$

We then substitute (21) in both sides of (20) to obtain

$$\sum_{i=0}^{\infty} a_i t^{i/2} = 1 - \frac{1}{\sqrt{\pi}} \int_0^t (\sum_{i=0}^{\infty} a_i s^{i/2})^4 / \sqrt{t-s} \ \ ds. \tag{22}$$

We are able now to find a_i's successively by matching the same powers of t on both sides of (22). Equations (9) and (10) with $\alpha = \frac{1}{2}, \lambda = \frac{-1}{\sqrt{\pi}}$, and $n = 4$ will produce $a_0 = 1$, $a_1 = -\frac{2}{\sqrt{\pi}}$, $a_2 = 4, a_3 = -\frac{32}{3\sqrt{\pi}}(2 + \frac{3}{\pi}), a_4 = 4(17 + \dfrac{15}{\pi})$, $a_5 = -\dfrac{256}{15\sqrt{\pi}} \left(23 + \dfrac{59}{\pi} + \dfrac{49}{\pi^2} \right)$, $a_6 = \dfrac{2}{3}(2161 + \dfrac{3823}{\pi} + \dfrac{2288}{\pi^2})$, $a_7 = -\dfrac{256}{5\sqrt{\pi}} \left(\dfrac{3481}{21} + \dfrac{61819}{105\pi} + \dfrac{30104}{35\pi^2} + \dfrac{2432}{5\pi^3} \right)$, and $a_8 = \dfrac{803497}{24} + \dfrac{3557417}{40\pi} + \dfrac{499167}{5\pi^2} + \dfrac{224672}{5\pi^3} \ldots$ etc. Our Mathematica program provides the same values numerically; the first few a_i's are:
$1, -1.12838, 4, -17.7828, 87.0986, -450.1, 2406.48, -13172.3, 73352.5, 413881., 2.3596 * 10^6, -1.35657 * 10^7, 7.85335 * 10^7, -4.57291 * 10^8, 2.67596 * 10^9, -1.57261 * 10^{10}, 9.27631 * 10^{10}, -5.48969 * 10^{11}$. Therefore, the infinite sum in equation (21) is an alternating series with rapidly growing coefficients. The convergence of this series is extremely slow inside its radius of convergence. To find this radius; let the partial sums $s_n(t)$ be defined by

$$s_n(t) := \sum_{i=0}^{n} a_i t^{i/2}, \tag{23}$$

then we computationally find that $|a_{n+1}/a_n| \to 56/9$ as $n \to \infty$. That is, $|s_{n+1}(t) - s_n(t)| = |a_{n+1}| t^{(n+1)/2} < (56/9)^n t^{(n+1)/2}$. Therefore, $y(t)$ in equation (21) is convergent for $0 \le t \le (9/56)^2 \approx 0.025829$. However, $s_n(0.01)$ converges to 0.915147 for $n \ge 20$, and $s_n(0.02)$ converges to 0.890395 for $n \ge 75$, and more than 200 terms are necessary to approximate $s_n(.0258)$ with only 3-digits of accuracy. The remarkable effect of applying the nonlinear sequence transformations to $s_n(t)$ is discussed in the next section.

International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:4, No:12, 2010

Remark 4.1: The partial sum s_n given by (23)are the partial sums of the power series expansion of $y(t)$. These sums are different from $y_n(t)$, the n^{th} Picard iterations for $y(t)$, and are comparable to truncated Taylor's series expansion for a function. The first n-terms of s_n and y_n are equal, but y_n normally has (many) more terms,(see $y_2(t)$ in example (2.1)).

V. Nonlinear Transformations

Aitken's Δ^2-process is one of the oldest of non-linear sequence transformation, see [20] page 212. If $s_n(t), s_{n+1}(t), s_{n+2}(t)$ are three successive partial sums, then an improved estimate (see [23] theorem 5.10.4 page 313) is

$$g_{n,1}(t) = s_n(t) - \frac{(s_{n+1}(t) - s_n(t))^2}{s_{n+2}(t) - 2s_{n+1}(t) + s_n(t)}, \quad n = 1, 2, 3, \ldots \tag{24}$$

We can apply (24) to $g_{n,1}$ and find $g_{n,2}$ and so on. In general once $g_{n,1}(t)$ is given by (24) then

$$g_{n,i+1}(t) = g_{n,i}(t) - \frac{(g_{n+1,i}(t) - g_{n,i}(t))^2}{g_{n+2,i}(t) - 2g_{n+1,i}(t) + g_{n,i}(t)}, \quad n = 1, 2, \ldots, \quad i = 1, 2, \ldots \tag{25}$$

If $g_{n,i}(t)$ converges to its limit with order p, then $g_{n,i+1}(t)$ converges to the same limit with order $2p - 1$. If $p = 1$ then the rate of convergence for $g_{n,i+1}(t)$ is 2. For a proof see [23] page 315.

Other convergence properties of Aitken's Δ^2-process are discussed in the literature, see for example, [20],[23], [10], and [24]. We obtained our most accurate approximations for diagonal elements, $g_{k,k}(t)$, of Δ^2-process. In what follows (Levin transformation) s_n is a partial sum of the form

$$s_n = \sum_{k=0}^{n} a_k.$$

This is for simplicity in notation. For calculation of power series one need to make the necessary changes.

The Levin transformation is probably the best single sequence acceleration method currently known, see [20]. It is designed to be exact for model sequence of the form

$$s_n = s + \omega_n \sum_{j=0}^{\infty} c_j/(n+\beta)^j, \quad k, n \in \mathrm{N}_0, \tag{26}$$

see [24],page 39.

With $\beta + n \neq 0$, and $\beta > 0$. The remainder ω_n are essentially arbitrary functions of n, see equation (30) below. The general Levin transformation can be represented by:

$$\mathcal{L}_k^{(n)}(\beta, s_n, \omega_n) = \frac{\sum_{j=0}^{k} (-1)^k \,(kj)\, \frac{(\beta+n+j)^{k-1}}{(\beta+n+k)^{k-1}} \; \frac{s_{n+j}}{\omega_{n+j}}}{\sum_{j=0}^{k} (-1)^k \,(kj)\, \frac{(\beta+n+j)^{k-1}}{(\beta+n+k)^{k-1}} \; \frac{1}{\omega_{n+j}}}, \tag{27}$$

with $k, n \in \mathrm{N}_0$.

For derivation of equation (27) see [24] pp $39 - 40$, or consult the web notes for the Numerical Recipes Software 2007.

The numerator and denominator in (27) are not computed as written. The following recursive scheme allows convenient computation with different starting values:

$$D_{k+1}^n(\beta) = D_k^{n+1}(\beta) - \frac{(\beta+n)(\beta+n+k)^{k-1}}{(\beta+n+k+1)^k} D_k^n(\beta) \tag{28}$$

The starting values are

$$D_0^n = \begin{cases} s_n/\omega_n & \text{numerator} \\ 1/\omega_n & \text{denominator} \end{cases} \tag{29}$$

In the literature, there are normally 4 different choices for ω_n:

$$\omega_n = \begin{cases} (\beta+n)a_n, & u \text{ transformation} \\ a_n, & t \text{ transformation} \\ a_{n+1}, & \text{modified } t \text{ transformation} \\ \dfrac{a_n a_{n+1}}{a_n - a_{n+1}}, & v \text{ transformation,} \end{cases} \tag{30}$$

see [20] pp $214 - 215$.

In applying the Levin method we obtained our best results using the t-transform. That is, in equation (28) and (29) we substitute s_n by $s_n(t)$ given by equation (23) and the remainder term ω_n by $a_n t^{n/2}$.

TABLE I The following table compares the values obtained for $y(t)$ given by (20), approximated by the partial sums $s_k = \sum_{i=0}^{k} a_i t^{i/2}$, $g_{k,k}$ given by (25) and $\mathcal{L}_k^{(2)}(.5, s_2, \omega_2)$ given by (27)

t	s_k	$g_{k,k}$	$\mathcal{L}_k^{(2)}(.5, s_2, \omega_2)$
0.01	$0.915147, k = 20$	$0.915147, k = 3$	$0.915147, k = 3$
.1	Divergent	$0.816887, k = 5$	$0.816887, k = 5$
1	Divergent	$.686571, k = 6$	$.686571, k = 6$
10	Divergent	$.553207, k = 8$	$.553207, k = 8$
20	Divergent	$.515551, k = 9$	$.515551, k = 9$
40	Divergent	$.479554, k = 10$	$.479559, k = 10$

VI. Conclusion

To approximate $y(t)$ the solution of $y(t) = 1 - \lambda \int_0^t \frac{y^n(s)}{(t-s)^\alpha} ds$; the simple Mathematica program provided in section II can be used to evaluate the coefficients a_is in the sum $s_n = \sum_{i=0}^{n} a_i t^{i/2}$. Then either the Aitken or the Levin nonlinear transformation can be implemented to evaluate s_n. As discussed in introduction $y'(0)$ is undefined and normally quadrature methods do not provide very accurate approximations for $y(t)$ when t is close to 0. The method introduced in this paper provides exceptionally accurate answers for those ts. TABLE I shows that our method provides accurate approximation also for larger values of t.

REFERENCES

[1] J. Abdalkhani, *On Exact and Approximate Solutions of Integral Equations of the Second Kind Using Maple*, Proceedings Maple Conference (2005), 191-197.
[2] J. Abdalkhani, *A Modified Approach to the Numerical Solution of Linear Weakly Singular Volterra Integral Equations of the Second Kind*, Journal of Integral Equations and Applications, Vol. 5 (1993), no.2 149-166.
[3] M. Abramowitz, I,A. Stegun (editors) *Handbook of Mathematical Functions*. Dover, Canada, ninth printing 1964.
[4] K.E. Atkinson *The Numerical Solution of Integral Equations of the Second Kind*. Cambridge Monographs on Applied and Computational Mathematics. 1997
[5] H. Brunner and P.van der Houwen, *The Numerical Solution of Volterra Equations*. North-Holland, Amsterdam, 1986.
[6] H. Brunner, A. Pedas and G. Vainikko *The Piecewise Polynomial Collocation Method for Nonlinear Weakly Singular Volterra Equations*, Mathematics of Computations, Vol.5 (1999), no.227 1079-1095.
[7] H. Brunner, A. Pedas, G. Vainikko *Piecewise Polynomial Collocation Methods for Linear Volterra Integro-Differential Equations with Weakly Singular Kernels*, SIAM J. Numer. Anal., Vol.39 (2001), no.3 957-982.
[8] Y. Cao, G. Cerezo, T.Herdman, J.Turi *Singularity Expansion for A Class of Neutral Equations*, Journal of Integral Equations and Applications.Vo.19, Number 1, Spring 2007. 13-32
[9] Y. Cao, T. Herdman and Y. Xu *A Hybird Collocation Method for Volterra Integral Equations with Weakly Singular Kernels* . SIAM J. Numer. Anal., Vol.41 (2003), no.1 364-381.
[10] G. Dahlquist, Ä.Björck, Translated by N. Anderson *Numerical Methods*. Prentice-Hall, 1974.
[11] M. Golberg *Discrete Ploynomial-Based Galerkin Methods for Fredholm Integral Equation*, Journal of Integral Equations and Applications.Vo.6, Number 2, Spring 1994. 197-211
[12] W. Hackbusch, *Integral Equations Theory and Numerical treatment*. Birkhäuser, 1995.
[13] F. R. de Hoog and R. Weiss *Higher Order Methods for a Class of Volterra Integral Equations with Weakly Singular Kernels*, SIAM J.Numer.Anal.11(1974),1166-1180.
[14] Q. Hu *Stieltjes Derivatives and β-Polynomial Spline Collocation for Volterra Integro-Differential Equations with Singularities*, SIAM J.Numer.Anal.Vol.33, No.1 (1996),208-220.
[15] R. P. Kanwal, *Linear Integral Equations Theory & Technique, Second Edition*, Birkhäuser 1996.
[16] R. Kress, *Linear Integral Equations,* , Springer-Verlag,Berlin 1989.
[17] P. Linz, *Analytical and Numerical Methods for Volterra Equations*. SIAM Studies in Applied Mathematics, SIAM, 1985.
[18] C. Lubich, *Runge-Kutta Theory for Volterra and Abel Integral equations of the Second Kind*, Math.Comp., Vol.41 (1983), 87-103.
[19] R.K. Miller. , *Nonlinear Volterra Integral Equations*. W.A. Benjamin,Inc, 1971.
[20] W.H. Press, S.A. Teukolsky, W.T. Vetterling and B.P. Flannery, *Numerical Recipes. The Art of Scientific Computing*. Third Edition,Cambridge 2007.
[21] J. O'Conner, *The nonlinear cooling of semi-infinite solid-Pade approximation methods*, CTAC, Vol.83 (1984), 881–892
[22] A. Pipkin, *A Course on Integral Equations*. Springer-Verlag, 1991.
[23] J. Stoer and R. Bulirsch *Introduction to Numerical Analysis*. Second Edition Springer-Verlag, 1993.
[24] E.J. Weniger *Nonlinear Sequence Transformation for the Acceleration of Convergence and the Summation of Divergent Series* . Computer Physics Report Vol. 10,pp. 189- 371, 1989.

JavadAbdalkhani Received his Ph.D from Dalhousie University, and taught at Acadia University, Carleton University, and University of Western Ontario before joining the Ohio State University-Lima as a numerical analyst. He won the the Ohio State University Distinguished Alumni Teaching Award in 2001.