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# Dynamics of a Discrete Three Species Food Chain System

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*Abstract*—The main purpose of this paper is to investigate a discrete time three–species food chain system with ratio dependence. By employing coincidence degree theory and analysis techniques, sufficient conditions for existence of periodic solutions are established.

*Keywords*—Food chain; ratio–dependent; coincidence degree; periodic solutions

## I. INTRODUCTION

IN the past decade, the food chain systems in population dynamics have attracted new attraction because of their complex dynamical properties [1–7]. Many researchers focused on the the global stability, chaos, Hopf bifurcation, periodic solutions and permanence of those models governed by differential and difference equations.

Recently, Wang and Pang proposed the following three species food chain model with Holling II–type function response in [7]:

$$\begin{cases} \dot{x}(t) = r_1x(t) - a_1x^2(t) - a_2x(t)y(t), \\ \dot{y}(t) = r_2y(t) - \frac{d_1y^2(t)}{x(t)} - \frac{by(t)z(t)}{\delta+y(t)}, \\ \dot{z}(t) = \frac{kby(t)z(t)}{\delta+y(t)} - d_2z(t), \end{cases} \tag{1}$$

where all the coefficients are positive constants. The second species predate on the first species and the top species predate on the middle species. The detailed ecological meanings of this system can be found in [7].

Taking account of environmental periodic variation and time delay effect, the modification of (1) is the non–autonomous differential equations

$$\begin{cases} \dot{x}(t) = r_1(t)x(t) - a_1(t)x^2(t) - a_2(t)x(t)y(t), \\ \dot{y}(t) = r_2(t)y(t) - \frac{d_1(t)y(t)y(t-\tau)}{x(t-\tau)} - \frac{b(t)y(t)z(t)}{\delta(t)+y(t)}, \\ \dot{z}(t) = \frac{k(t)b(t)y(t-\tau)z(t)}{\delta(t)+y(t-\tau)} - d_2(t)z(t). \end{cases}$$

However, it is known that the discrete time model are more appropriate than the continuous ones when the populations have non–overlapping generations. Discrete time models can also provide efficient computational models of continuous for numerical simulations. Following the method in [8], we consider the following discrete analogue with the help of differential equations with piecewise constant arguments

$$\begin{cases} x(k+1) = x(k)\exp\{r_1(k) - a_1(k)x(k) - a_2(k)y(k)\}, \\ y(k+1) = y(k)\exp\{r_2(k) - \sum_{l=0}^{m} \frac{d_{1l}(k)y(k-l)}{x(k-l)} \\ \qquad - \frac{b(k)z(k)}{\delta(k)+y(k)}\}, \\ z(k+1) = z(k)\exp\{\sum_{l=0}^{m} \frac{h_l(k)y(k-l)}{\delta_l(k)+y(k-l)} - d_2(k)\}, \end{cases} \tag{2}$$

where all the coefficients are positive $\omega-$periodic sequences and $k$ is an integer. In the following, we shall explore the existence of periodic solutions for system (2).

## II. PRELIMINARIES

For simplicity, we use the following notations throughout this paper,

$$I_\omega = \{0, 1, 2, \dots, \omega-1\}, \quad \bar{f} = \frac{1}{\omega}\sum_{k=0}^{\omega-1} f(k).$$

According to the Theorem 2.1 in [9], we can easily obtain the following special case.

**Lemma 2.1** ([9]). Let $k_1, k_2 \in I_\omega$ and $k \in \mathbb{Z}$. If $g : \mathbb{Z} \to \mathbb{R}$ is $\omega-$periodic, then

$$g(k) \le g(k_1) + \frac{1}{2}\sum_{s=0}^{\omega-1} |g(s+1) - g(s)|$$

and

$$g(k) \ge g(k_2) - \frac{1}{2}\sum_{s=0}^{\omega-1} |g(s+1) - g(s)|,$$

the constant factor $\frac{1}{2}$ is the best possible.

Now, we introduce some concepts and a useful result from [10].

Let $X, Z$ be normed vector spaces, $L : \operatorname{Dom} L \subset X \to Z$ be a linear mapping, $N : X \to Z$ be a continuous mapping. The mapping $L$ will be called a Fredholm mapping of index zero if $\dim\ker L = \operatorname{codim}\operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in $Z$. If $L$ is a Fredholm mapping of index zero and there exist continuous projections $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \ker L$, $\operatorname{Im} L = \ker Q = \operatorname{Im}(I - Q)$, then it follows that $L|\operatorname{Dom} L \cap \ker P : (I - P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by $K_P$. If $\Omega$ is an open bounded subset of $X$, the mapping $N$ will be called $L$-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\ker L$, there exists an isomorphism $J : \operatorname{Im} Q \to \ker L$.

Next, we state the Mawhin's continuation theorem, which is a main tool in the proof of our theorem.

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**Lemma 2.2** (Continuation Theorem). Let $L$ be a Fredholm mapping of index zero and $N$ be $L$-compact on $\bar{\Omega}$. Suppose

(a) for each $\lambda \in (0,1)$, every solution $u$ of $Lu = \lambda Nu$ is such that $u \notin \partial\Omega$;

(b) $QNu \neq 0$ for each $u \in \partial\Omega \cap \ker L$ and the Brouwer degree $\deg\{JQN, \Omega \cap \ker L, 0\} \neq 0$.

Then the operator equation $Lu = Nu$ has at least one solution lying in $\mathrm{Dom}\, L \cap \bar{\Omega}$.

## III. Main results

We now prove our results on the existence of positive periodic solutions of system (2).

**Theorem 3.1.** Assume that

$$\bar{r}_2\omega e^{L_1} > e^{M_2}\sum_{k=0}^{\omega-1}\sum_{l=0}^{m} d_{1l}(k),$$

where $M_2 = \ln\frac{\bar{r}_1}{\bar{a}_2} + \bar{r}_2\omega$, $L_1 = \ln\frac{\sum_{k=0}^{\omega-1}\sum_{l=0}^{m} d_{1l}(k)e^{L_2}}{\bar{r}_2\omega} - \bar{r}_1\omega$ and $L_2 = \ln\frac{\bar{d}_2\omega}{\sum_{k=0}^{\omega-1}\sum_{l=0}^{m}\frac{h_l(k)}{\delta_l(k)}} - \bar{r}_2\omega$. Then system (2) has at least one positive $\omega-$periodic solution.

**Proof** Let $x(k) = e^{u_1(k)}$, $y(k) = e^{u_2(k)}$ and $z(k) = e^{u_3(k)}$, then system (2) is equivalent to the following form,

$$\begin{cases} u_1(k+1) - u_1(k) = r_1(k) - a_1(k)e^{u_1(k)} - a_2(k)e^{u_2(k)}, \\ u_2(k+1) - u_2(k) = r_2(k) - \sum_{l=0}^{m}\frac{d_{1l}(k)e^{u_2(k-l)}}{e^{u_1(k-l)}} \\ \qquad - \frac{b(k)e^{u_3(k)}}{\delta(k)+e^{u_2(k)}}, \\ u_3(k+1) - u_3(k) = \sum_{l=0}^{m}\frac{h_l(k)e^{u_2(k-l)}}{\delta_l(k)+e^{u_2(k-l)}} - d_2(k), \end{cases} \tag{3}$$

and we only need to establish the existence of $\omega-$periodic solutions for system (3).

To apply Lemma 2.2, we define

$$X = Z = \{(u_1(k), u_2(k), u_3(k))^T \in \mathbf{R}^3, u_i(k+\omega) = u_i(k)\},$$

$$\|(u_1,u_2,u_3)^T\| = \left(\sum_{i=1}^{3}\max_{k\in I_\omega}|u_i(k)|^2\right)^{\frac{1}{2}}.$$

Denote $u(k) = (u_1(k), u_2(k), u_3(k))^T$, then $X$ and $Z$ are both Banach spaces when they are endowed with the above norm $\|\cdot\|$.

Let

$$Nu = N\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} = \begin{bmatrix} r_1(k) - a_1(k)e^{u_1(k)} - a_2(k)e^{u_2(k)} \\ r_2(k) - \sum_{l=0}^{m}\frac{d_{1l}(k)e^{u_2(k-l)}}{e^{u_1(k-l)}} - \frac{b(k)e^{u_3(k)}}{\delta(k)+e^{u_2(k)}} \\ \sum_{l=0}^{m}\frac{h_l(k)e^{u_2(k-l)}}{\delta_l(k)+e^{u_2(k-l)}} - d_2(k)\end{bmatrix},$$

$$Lu = u(k+1) - u(k),$$

$$Pu = Qu = \frac{1}{\omega}\sum_{k=0}^{\omega-1}u(k).$$

Obviously, $\ker L = \mathbb{R}^3$, $\mathrm{Im}\, L = \left\{(u_1,u_2,u_3)^T \in Z : \bar{u}_1 = \bar{u}_2 = \bar{u}_3 = 0, t \in \mathbb{T}\right\}, \dim\ker L = 3 = \mathrm{codim}\,\mathrm{Im}\, L$.

Since $\mathrm{Im}\, L$ is closed in $Z$, then $L$ is a Fredholm mapping of index zero. It is easy to show that $P$ and $Q$ are continuous projections such that $\mathrm{Im}\, P = \ker L$ and $\mathrm{Im}\, L = \ker Q = \mathrm{Im}(I-Q)$. Furthermore, the generalized inverse (of $L$) $K_P : \mathrm{Im}\, L \to \ker P \cap \mathrm{Dom}\, L$ exists and is given by

$$Kp(u) = \sum_{s=1}^{k-1}u(s) - \frac{1}{\omega}\sum_{s=0}^{\omega-1}u(s)(\omega - s).$$

Clearly, $QN$ and $K_P(I-Q)N$ are continuous. According to the Arzela-Ascoli theorem, it is not difficulty to prove that $K_P(I-Q)N(\bar{\Omega})$ is compact for any open bounded set $\Omega \subset X$. In addition, $QN(\bar{\Omega})$ is bounded. Therefore, $N$ is $L$-compact on $\bar{\Omega}$ with any open bounded set $\Omega \subset X$.

Now, we shall search an appropriate open bounded subset $\Omega$ for the application of the continuation theorem, Lemma 2.2. For the operator equation $Lu = \lambda Nu$, where $\lambda \in (0,1)$, we have

$$\begin{cases} u_1(k+1) - u_1(k) = \lambda\left[r_1(k) - a_1(k)e^{u_1(k)} - a_2(k)e^{u_2(k)}\right], \\ u_2(k+1) - u_2(k) = \lambda\Big[r_2(k) - \sum_{l=0}^{m}\frac{d_{1l}(k)e^{u_2(k-l)}}{e^{u_1(k-l)}} \\ \qquad - \frac{b(k)e^{u_3(k)}}{\delta(k)+e^{u_2(k)}}\Big], \\ u_3(k+1) - u_3(k) = \lambda\left[\sum_{l=0}^{m}\frac{h_l(k)e^{u_2(k-l)}}{\delta_l(k)+e^{u_2(k-l)}} - d_2(k)\right]. \end{cases} \tag{4}$$

Assume that $(u_1(k), u_2(k), u_3(k))^T \in X$ is a solution of (4) for some $\lambda \in (0,1)$. Summing on both sides of system (5) over $I_\omega$ with respect to $k$, we can derive

$$\begin{cases} \bar{r}_1\omega = \sum_{k=0}^{\omega-1}a_1(k)e^{u_1(k)} + \sum_{k=0}^{\omega-1}a_2(k)e^{u_2(k)}, \\ \bar{r}_2\omega = \sum_{k=0}^{\omega-1}\sum_{l=0}^{m}\frac{d_{1l}(k)e^{u_2(k-l)}}{e^{u_1(k-l)}} + \sum_{k=0}^{\omega-1}\frac{b(k)e^{u_3(k)}}{\delta(k)+e^{u_2(k)}}, \\ \bar{d}_2\omega = \sum_{k=0}^{\omega-1}\sum_{l=0}^{m}\frac{h_l(k)e^{u_2(k-l)}}{\delta_l(k)+e^{u_2(k-l)}}. \end{cases} \tag{5}$$

Since $(u_1(k), u_2(k), u_3(k))^T \in X$, there exist $\xi_i, \eta_i \in I_\omega, i = 1,2,3$, such that

$$u_i(\xi_i) = \min_{k\in I_\omega}\{u_i(k)\}, \quad u_i(\eta_i) = \max_{k\in I_\omega}\{u_i(k)\}. \tag{6}$$

In view of (5) and (6), we have

$$\sum_{k=0}^{\omega-1}|u_1(k+1) - u(k)| < 2\bar{r}_1\omega,$$

$$\sum_{k=0}^{\omega-1}|u_2(k+1) - u(k)| < 2\bar{r}_2\omega,$$

$$\sum_{k=0}^{\omega-1}|u_3(k+1) - u(k)| < 2\bar{d}_2\omega.$$

From (6) and the first equation of (5), we have

$$u_1(\xi_1) < \ln\frac{\bar{r}_1}{\bar{a}_1}$$

and

$$u_2(\xi_2) < \ln\frac{\bar{r}_1}{\bar{a}_a},$$

then

$$\begin{aligned} u_1(k) &\leq u_1(\xi_1) + \frac{1}{2}\sum_{k=0}^{\omega-1}|u_1(k+1) - u_1(k)| \\ &< \ln\frac{\bar{r}_1}{\bar{a}_1} + \bar{r}_1\omega := M_1, \end{aligned}$$

$$\begin{aligned} u_2(k) &\le u_2(\xi_2) + \frac{1}{2}\sum_{k=0}^{\omega-1}|u_2(k+1)-u_2(k)| \\ &< \ln\frac{\bar{r}_1}{\bar{a}_2} + \bar{r}_2\omega := M_2. \end{aligned}$$

From (6) and the second equation of (5), we have

$$u_3(\xi_3) < \ln\frac{\bar{r}_2\omega}{\sum_{k=0}^{\omega-1}\frac{b(k)}{\delta(k)+e^{M_2}}},$$

and

$$\begin{aligned} u_3(k) &\le u_3(\xi_3) + \frac{1}{2}\sum_{k=0}^{\omega-1}|u_3(k+1)-u_3(k)| \\ &< \ln\frac{\bar{r}_2\omega}{\sum_{k=0}^{\omega-1}\frac{b(k)}{\delta(k)+e^{M_2}}} + \bar{d}_2\omega := M_3. \end{aligned}$$

From (6) and the third equation of (5), we have

$$u_2(\eta_2) > \ln\frac{\bar{d}_2\omega}{\sum_{k=0}^{\omega-1}\sum_{l=0}^{m}\frac{h_l(k)}{\delta_l(k)}}$$

and

$$\begin{aligned} u_2(k) &\ge u_2(\eta_2) - \frac{1}{2}\sum_{k=0}^{\omega-1}|u_2(k+1)-u_2(k)| \\ &> \ln\frac{\bar{d}_2\omega}{\sum_{k=0}^{\omega-1}\sum_{l=0}^{m}\frac{h_l(k)}{\delta_l(k)}} - \bar{r}_2\omega := L_2. \end{aligned}$$

Similarly, we can also get

$$\begin{aligned} u_1(k) &\ge u_1(\eta_1) - \frac{1}{2}\sum_{k=0}^{\omega-1}|u_1(k+1)-u_1(k)| \\ &> \ln\frac{\sum_{k=0}^{\omega-1}\sum_{l=0}^{m}d_{1l}(k)e^{L_2}}{\bar{r}_2\omega} - \bar{r}_1\omega := L_1. \end{aligned}$$

By the assumption of theorem, we can obtain

$$\begin{aligned} u_3(k) &\ge u_3(\eta_3) - \frac{1}{2}\sum_{k=0}^{\omega-1}|u_3(k+1)-u_3(k)| \\ &> \ln\frac{\bar{r}_2\omega e^{L_1} - e^{M_2}\sum_{k=0}^{\omega-1}\sum_{l=0}^{m}d_{1l}(k)}{\sum_{k=0}^{\omega-1}\frac{b(k)}{\delta(k)}} - \bar{d}_2\omega := L_3. \end{aligned}$$

From above, we have $\max_{k\in I_\omega}|u_i(k)| \le \max\{|M_i|,|L_i|\} := R_i, i = 1,2,3$ and $R_i$ is independent of $\lambda$. Let $R = R_1 + R_2 + R_3 + R_0$, where $R_0$ is taken sufficiently large such that every solution $\|(x^*,y^*,z^*)^T\|$ of the algebraic equations

$$\begin{cases} \bar{r}_1 - \bar{a}_1e^x - \bar{a}_2e^y = 0, \\ \bar{r}_2\omega - \omega\overline{\sum_{l=0}^{m}d_{1l}}e^{y-x} - \sum_{k=0}^{\omega-1}\frac{b(k)e^z}{\delta(k)+e^y} = 0, \\ \bar{d}_2\omega - \sum_{k=0}^{\omega-1}\frac{h_l(k)e^y}{\delta(k)+e^y} = 0 \end{cases}$$

satisfies $\|(x^*,y^*,z^*)^T\| < R$. Now, we define $\Omega = \{(u_1,u_2,u_3)^T \in X\}, \|(u_1,u_2,u_3)^T < R$. Then it is clear that $\Omega$ verifies the requirement (a) of Lemma 2.2. If $(u_1,u_2,u_3)^T \in \partial\Omega\cap\ker L = \partial\Omega\cap\mathbb{R}^3$, then $(u_1,u_2,u_3)^T$ is a constant vector in $\mathbb{R}^3$ with $\|(u_1,u_2,u_3)^T\| = |u_1|+|u_2|+|u_3| = R$, so we have $QNu \ne 0$.

By the invariance property of homotopy, direct calculation produces $\deg(JQN,\Omega\cap\ker L,0) = 1 \ne 0$. Now, we have proved that $\Omega$ satisfies all conditions of Lemma 2.2. Thus, system (2) has at least one positive $\omega-$periodic solution. This completes the proof.

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