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Derivation of Darcy's Law using Homogenization Method

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Abstract—Darcy's Law is a well-known constitutive equation describing the flow of a fluid through a porous medium. The equation shows a relationship between the superficial or Darcy velocity and the pressure gradient which was first experimentally observed by Henry Darcy in 1855-1856. In this study, we apply homogenization method to Stokes equation in order to derive Darcy's Law. The process of deriving the equation is complicated, especially in multidimensional domain. Thus, for the sake of simplicity, we use the indicial notation as well as the homogenization. This combination provides a smooth, simple and easy technique to derive Darcy's Law.

Keywords—Darcy's Law, Homogenization method, Indicial notation.

I. Introduction

DARCY'S Law is one of the most important equations used to model porous media [3], [4], [6], [7]. Employing the equation without knowing the derivation may cause some problems to researchers. This disadvantage may hide them from having an appropriate equation for their research problems. Understanding the foundation of the equation may guide authors an appropriate way to adjust the equation and can lead them to have a useful model. In this article, we demonstrate the derivation of Darcy's Law using homogenization method and also employing the knowledge of indicial notation, see Section II, which is sometimes referred to as Einstein notation. In Section III, we introduce homogenization method, which is an upscaling technique from microscale to macroscale equations, and then apply this method to Stokes equation to derive Darcy's Law. We draw conclusions in Section IV.

II. Indicial Notation

Indicial notation is a convenient notation for manipulating vectors, tensors (a generalization of vectors to higher order), and equations involving these quantities. This notation provides a method of writing vector equations using indices and is valid only for an orthonormal coordinate system. For cylindrical or spherical coordinates we need a more general form of the indicial notation. For more details, see e.g. [1]. In this section, we begin with indicial notation rules and give some examples so that ones can clearly understand and extend this knowledge to more complicated tensors and equations.

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Indicial Notation Rule Number 1: Unless otherwise noted, all indices take on the values 1, 2, and 3.

Example 1: A vector $\mathbf{v}$ in three-dimensional Cartesian coordinates has components (v_1, v_2, v_3). In indicial notation, the components of the vector are denoted as $v_i, i =$ 1, 2, 3 unless otherwise noted.

Example 2: In indicial notation a *comma* denotes derivative, e.g.

$$u_{,i} = (u_{,1}, u_{,2}, u_{,3}) = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}\right) \quad (1)$$

Indicial Notation Rule Number 2: Repeated indices implies summation.

Example 3: The dot product of two vectors, $\mathbf{u}$ and $\mathbf{v}$, in direct notation is given by $\mathbf{u} \cdot \mathbf{v}$ while in indicial notation the dot product is:

$$u_i v_i = \sum_{i=1}^{3} u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3. \quad (2)$$

Example 4: The divergence of the vector field $\mathbf{u}$ is

$$\nabla \cdot \mathbf{u} = u_{i,i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}. \quad (3)$$

Indicial Notation Rule Number 3: In each term of an equation, the free indices must match.

Example 5: $u_i + v_i = w_i$ is correct, but $u_j + v_i = w_i$ is not. In another word, $u_j + v_i$ is not defined.

Example 6: Let $\underline{\underline{A}}$ be a second-order tensor or $\underline{\underline{A}}$ can be expressed as a matrix. In direct tensor notation, a linear system of equations is usually written as $\mathbf{v} = \underline{\underline{\mathbf{A}}} \cdot \mathbf{u}$, which in indicial notation is

$$v_i = A_{ij} u_j. \quad (4)$$

Example 7: Let $\underline{\underline{\underline{B}}}$ be a third-order tensor. The tensor $\underline{\underline{\underline{B}}} = B_{ijk}$ can send vectors into second-order tensors: $\underline{\underline{\underline{B}}} \cdot \mathbf{u} = \underline{\underline{A}}$ or $B_{ijk} u_k = A_{ij}$.

Note that, a dot product indicates one index is repeated. For Examples 6 and 7, the free indices are i and ij, respectively.

Indicial Notation Rule Number 4: Indices repeated more than two times in a single term are not allowed.

Example 8: $B_{ijk} A_{kl} u_k$ is not allowed (the subscript k is repeated three times).

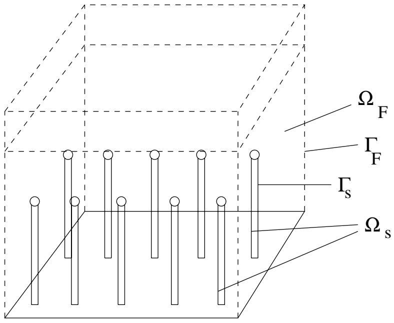

Fig. 1. Geometry: an cell of cylinders when the angle between cylinders and horizontal plane is 90 degrees.

III. Darcy's Law

Darcy's Law is a constitutive equation that describes the slow flow of a fluid through porous medium. In this section, we apply homogenization method to Stokes equation as well as the indicial notation to derive Darcy's Law. We summarize the homogenization method as following. For more details, see e.g. [2], [5], [8], [9].

Let Ω be the periodic cell domain which consists of a fluid phase Ω_F, a solid phase Ω_S and a piecewise continuous liquid-solid interface $\Gamma = \Gamma_S \cup \Gamma_F$ where Γ_S and Γ_F are the boundaries of the solid and liquid phases respectively (see Figure 1). We assume slow fluid flow, a fixed solid and a viscous incompressible fluid, so that the Stokes equations are applicable:

$$\nabla \cdot \mathbf{v} = 0 \qquad \text{in } \Omega_F \tag{5}$$

$$-\nabla p + \mu \triangle \mathbf{v} + \mathbf{f} = \mathbf{0} \qquad \text{in } \Omega_F \tag{6}$$

$$\mathbf{v} = \mathbf{0} \qquad \text{on } \Gamma_S \tag{7}$$

where we also assume a no-slip boundary condition on Γ_S; μ is the dynamic viscosity; $\mathbf{v}$ is the velocity; p is the pressure and $\mathbf{f}$ is a source term. Next, we assume the diameter of the cylinders, $a = 2r$, see Figure 1, is small compared to a macroscopic scale length L which can be the length of a lengthy pipe; i.e. if $\epsilon = a/L$, then $\epsilon \ll 1$. Let $\mathbf{x}$ be a macroscopic variable and define the microscopic variable or the stretched coordinate $\mathbf{y} = \mathbf{x}/\epsilon$. The dynamic viscosity coefficient μ is assumed fixed and independent of ϵ. Figure 2 shows a reference cell Y which is periodic and the period is ϵY. Figure 3 demonstrates the upscaling procedure where $\epsilon > 0$ is a spatial scale parameter. When ϵ tends to zero, we have the macroscopic scale.

Typically, with homogenization method, we begin and consider an asymptotic expansion of $\mathbf{v}$ and p, in the form,

$$\mathbf{v} = \epsilon^{\alpha}(\mathbf{v}^o(\mathbf{x},\mathbf{y}) + \epsilon \mathbf{v}^1(\mathbf{x},\mathbf{y}) + ...) \quad \text{and} \tag{8}$$

$$p = \epsilon^{\beta}(p^o(\mathbf{x},\mathbf{y}) + \epsilon p^1(\mathbf{x},\mathbf{y}) + ...) \tag{9}$$

where $\mathbf{v}^i$ and p^i are Ω-periodic in $\mathbf{y}$, and α and β are nonzero parameters that yield a physically meaningful solution. The choice of α and β yields different macroscopic models, and in this case we choose α and β to yield a nonzero macroscopic first-order pressure and a second-order velocity which can lead to obtaining Darcy's Law.

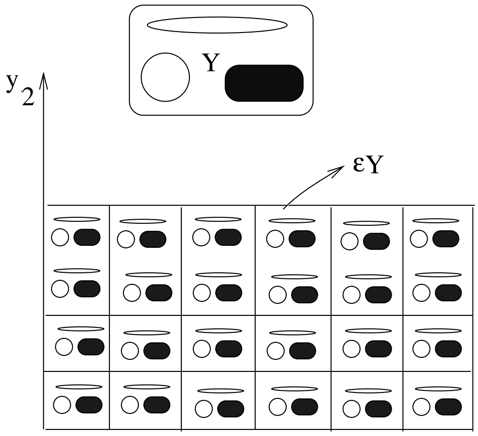

Fig. 2. A reference heterogeneities Y are periodic of period ϵY and their size is order of ϵ.

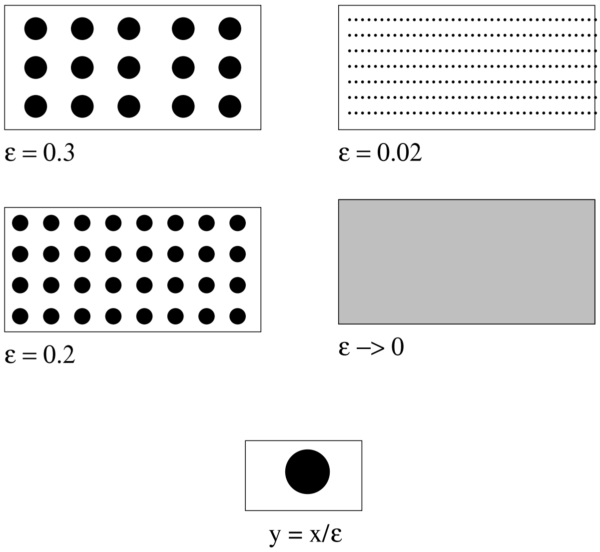

Fig. 3. Upscaling procedure where $\epsilon > 0$ is a spatial parameter.

It is known to be a reasonable equation for modeling slow flow through a porous medium. Recall that, for a three-dimensional domain,

$$\mathbf{x} = (x_1, x_2, x_3) \quad \text{and}$$
$$\mathbf{y} = (y_1, y_2, y_3) = \frac{1}{\epsilon}(x_1, x_2, x_3), \tag{10}$$

and then we apply the chain rule to (10), the first and second derivatives with respect to x_j are

$$\frac{d}{dx_j} = \frac{\partial}{\partial x_j} + \frac{1}{\epsilon}\frac{\partial}{\partial y_j}, \quad j = 1,2,3. \tag{11}$$

Recall that the notation in (11) is satisfied the indicial notation rule number 3, which is the matching of the free index j. For the second derivative, we have

$$\begin{aligned}\frac{d^2}{dx_j^2} &= \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_j}+\frac{1}{\epsilon}\frac{\partial}{\partial y_j}\right)+\frac{1}{\epsilon}\frac{\partial}{\partial y_j}\left(\frac{\partial}{\partial x_j}+\frac{1}{\epsilon}\frac{\partial}{\partial y_j}\right)\\ &= \frac{\partial^2}{\partial x_j^2}+\frac{1}{\epsilon}\frac{\partial^2}{\partial x_j\partial y_j}+\frac{1}{\epsilon}\left(\frac{\partial^2}{\partial y_j\partial x_j}+\frac{1}{\epsilon}\frac{\partial^2}{\partial y_j^2}\right)\\ &= \frac{1}{\epsilon^2}\frac{\partial^2}{\partial y_j^2}+\frac{1}{\epsilon}\left(\frac{\partial^2}{\partial x_j\partial y_j}+\frac{\partial^2}{\partial y_j\partial x_j}\right)+\frac{\partial^2}{\partial x_j^2}. \quad (12)\end{aligned}$$

Therefore, the vector Laplacian of the velocity field is

$$\begin{aligned}\Delta\mathbf{v} = \mathbf{v}_{,jj} &= (\frac{d^2v_1}{dx_j^2},\frac{d^2v_2}{dx_j^2},\frac{d^2v_3}{dx_j^2})\\ &= (\frac{1}{\epsilon^2}\frac{\partial^2 v_1}{\partial y_j^2}+\frac{1}{\epsilon}(\frac{\partial^2 v_1}{\partial x_j\partial y_j}+\frac{\partial^2 v_1}{\partial y_j\partial x_j})+\frac{\partial^2 v_1}{\partial x_j^2},\\ &\frac{1}{\epsilon^2}\frac{\partial^2 v_2}{\partial y_j^2}+\frac{1}{\epsilon}(\frac{\partial^2 v_2}{\partial x_j\partial y_j}+\frac{\partial^2 v_2}{\partial y_j\partial x_j})+\frac{\partial^2 v_2}{\partial x_j^2},\\ &\frac{1}{\epsilon^2}\frac{\partial^2 v_3}{\partial y_j^2}+\frac{1}{\epsilon}(\frac{\partial^2 v_3}{\partial x_j\partial y_j}+\frac{\partial^2 v_3}{\partial y_j\partial x_j})+\frac{\partial^2 v_3}{\partial x_j^2}), \quad (13)\\ &= \frac{1}{\epsilon^2}(\frac{\partial^2 v_1}{\partial y_j^2},\frac{\partial^2 v_2}{\partial y_j^2},\frac{\partial^2 v_3}{\partial y_j^2})\\ &+\frac{1}{\epsilon}(\frac{\partial^2 v_1}{\partial x_j\partial y_j}+\frac{\partial^2 v_1}{\partial y_j\partial x_j},\frac{\partial^2 v_2}{\partial x_j\partial y_j}+\frac{\partial^2 v_2}{\partial y_j\partial x_j},\\ &\frac{\partial^2 v_3}{\partial x_j\partial y_j}+\frac{\partial^2 v_3}{\partial y_j\partial x_j})+(\frac{\partial^2 v_1}{\partial x_j^2},\frac{\partial^2 v_2}{\partial x_j^2},\frac{\partial^2 v_3}{\partial x_j^2})\\ &= \frac{1}{\epsilon^2}\Delta_y\mathbf{v}+\frac{1}{\epsilon}\Delta_{xy}\mathbf{v}+\Delta_x\mathbf{v}\end{aligned}$$

where

$$\Delta_y\mathbf{v}=\left(\frac{\partial^2 v_1}{\partial y_j^2},\frac{\partial^2 v_2}{\partial y_j^2},\frac{\partial^2 v_3}{\partial y_j^2}\right);\Delta_x\mathbf{v}=\left(\frac{\partial^2 v_1}{\partial x_j^2},\frac{\partial^2 v_2}{\partial x_j^2},\frac{\partial^2 v_2}{\partial x_j^2}\right)$$

and

$$\begin{aligned}\Delta_{xy}\mathbf{v} =&(\frac{\partial^2 v_1}{\partial x_j\partial y_j}+\frac{\partial^2 v_1}{\partial y_j\partial x_j},\frac{\partial^2 v_2}{\partial x_j\partial y_j}+\frac{\partial^2 v_2}{\partial y_j\partial x_j},\\ &\frac{\partial^2 v_3}{\partial x_j\partial y_j}+\frac{\partial^2 v_3}{\partial y_j\partial x_j})\end{aligned}$$

where the repeated index j within each single term in (12) and (13) indicates summation such as

$$\frac{d^2v_1}{dx_j^2}=v_{1,jj}=\sum_{j=1}^{3}\frac{d^2v_1}{dx_j^2}.$$

Substituting (11) into (5) yields

$$0=\nabla\cdot\mathbf{v}=\frac{dv_j}{dx_j}=\frac{\partial v_j}{\partial x_j}+\frac{1}{\epsilon}\frac{\partial v_j}{\partial y_j}=\nabla_x\cdot\mathbf{v}+\frac{1}{\epsilon}\nabla_y\cdot\mathbf{v}, \quad (14)$$

where

$$\nabla_x\cdot\mathbf{v}=\frac{\partial v_j}{\partial x_j} \quad\text{and}\quad \nabla_y\cdot\mathbf{v}=\frac{\partial v_j}{\partial y_j}. \quad (15)$$

Similarly, substituting (11) and (12) into the Stokes equation (6) yields

$$-\nabla_x p-\frac{1}{\epsilon}\nabla_y p+\mu\left(\frac{1}{\epsilon^2}\Delta_y\mathbf{v}+\frac{1}{\epsilon}\Delta_{xy}\mathbf{v}+\Delta_x\mathbf{v}\right)+\mathbf{f}=\mathbf{0}. \quad (16)$$

Next, substituting (8) and (9) into (14) and (16), we have

$$\begin{aligned}&\nabla_x\cdot(\epsilon^\alpha\mathbf{v}^o(\mathbf{x},\mathbf{y})+\epsilon^{\alpha+1}\mathbf{v}^1(\mathbf{x},\mathbf{y})+...)\\ &+\frac{1}{\epsilon}\nabla_y\cdot(\epsilon^\alpha\mathbf{v}^o(\mathbf{x},\mathbf{y})+\epsilon^{\alpha+1}\mathbf{v}^1(\mathbf{x},\mathbf{y})+...)=0 \quad (17)\end{aligned}$$

and

$$\begin{aligned}&-\nabla_x(\epsilon^\beta p^o(\mathbf{x},\mathbf{y})+\epsilon^{\beta+1}p^1(\mathbf{x},\mathbf{y})+...)\\ &-\frac{1}{\epsilon}\nabla_y(\epsilon^\beta p^o(\mathbf{x},\mathbf{y})+\epsilon^{\beta+1}p^1(\mathbf{x},\mathbf{y})+...)\\ &+\mu\frac{1}{\epsilon^2}\Delta_y\epsilon^\alpha(\mathbf{v}^o(\mathbf{x},\mathbf{y})+\epsilon\mathbf{v}^1(\mathbf{x},\mathbf{y})+...)\\ &+\frac{\mu}{\epsilon}\Delta_{xy}\epsilon^\alpha(\mathbf{v}^o(\mathbf{x},\mathbf{y})+\epsilon\mathbf{v}^1(\mathbf{x},\mathbf{y})+...)\\ &+\mu\Delta_x\epsilon^\alpha(\mathbf{v}^o(\mathbf{x},\mathbf{y})+\epsilon\mathbf{v}^1(\mathbf{x},\mathbf{y})+...)+\mathbf{f}=\mathbf{0}. \quad (18)\end{aligned}$$

If we let both α and β be zeros, equations (17) and (18) become

$$\begin{aligned}&\nabla_x\cdot(\epsilon^o\mathbf{v}^o(\mathbf{x},\mathbf{y})+\epsilon^1\mathbf{v}^1(\mathbf{x},\mathbf{y})+...)\\ &+\frac{1}{\epsilon}\nabla_y\cdot(\epsilon^o\mathbf{v}^o(\mathbf{x},\mathbf{y})+\epsilon^1\mathbf{v}^1(\mathbf{x},\mathbf{y})+...)=0 \quad (19)\end{aligned}$$

and

$$\begin{aligned}&-\nabla_x(\epsilon^o p^o(\mathbf{x},\mathbf{y})+\epsilon^1 p^1(\mathbf{x},\mathbf{y})+...)\\ &-\frac{1}{\epsilon}\nabla_y(\epsilon^o p^o(\mathbf{x},\mathbf{y})+\epsilon^1 p^1(\mathbf{x},\mathbf{y})+...)\\ &+\mu\frac{1}{\epsilon^2}\Delta_y\epsilon^o(\mathbf{v}^o(\mathbf{x},\mathbf{y})+\epsilon\mathbf{v}^1(\mathbf{x},\mathbf{y})+...)\\ &+\frac{\mu}{\epsilon}\Delta_{xy}\epsilon^o(\mathbf{v}^o(\mathbf{x},\mathbf{y})+\epsilon\mathbf{v}^1(\mathbf{x},\mathbf{y})+...)\\ &+\mu\Delta_x\epsilon^o(\mathbf{v}^o(\mathbf{x},\mathbf{y})+\epsilon\mathbf{v}^1(\mathbf{x},\mathbf{y})+...)+\mathbf{f}=\mathbf{0}. \quad (20)\end{aligned}$$

Collecting the same orders of ϵ ($O(\epsilon^{-1})$ from (19), and $O(\epsilon^{-2})$ and $O(\epsilon^{-1})$ from (20)), we have the differential equations

$$\nabla_y\cdot\mathbf{v}^o(\mathbf{x},\mathbf{y})=0 \quad (21)$$

$$\mu\Delta_y\mathbf{v}^o(\mathbf{x},\mathbf{y})=0 \quad (22)$$

$$-\nabla_y p^o(\mathbf{x},\mathbf{y})+\mu\Delta_y\mathbf{v}^1(\mathbf{x},\mathbf{y})+\mu\Delta_{xy}\mathbf{v}^o(\mathbf{x},\mathbf{y})=\mathbf{0}. \quad (23)$$

Note the system of equations (21)-(23) does not provide a physically meaningful solution and we also lose the source term $\mathbf{f}$ in the momentum equation (23). Then, for the homogenization method, it is important that one needs to choose the right orders of epsilon, α and β, to have a physically meaningful solution. This is called state-of-the-art homogenization method.

After some trials and errors we found that in order to find a non-zero and physically meaningful solution, we let $\alpha=2$ and $\beta=0$ so that

$$\mathbf{v} = \epsilon^2(\mathbf{v}^o(\mathbf{x},\mathbf{y})+\epsilon\mathbf{v}^1(\mathbf{x},\mathbf{y})+...) \quad (24)$$

$$p = \epsilon^o(p^o(\mathbf{x},\mathbf{y})+\epsilon p^1(\mathbf{x},\mathbf{y})+...) \quad (25)$$

where, again, the function v^i and p^i are Ω-periodic in the microscale $\mathbf{y}$. Substituting (24) and (25) into (14) and (16), we have

$$\begin{aligned}&\nabla_x\cdot(\epsilon^2\mathbf{v}^o(\mathbf{x},\mathbf{y})+\epsilon^3\mathbf{v}^1(\mathbf{x},\mathbf{y})+...)\\ &+\frac{1}{\epsilon}\nabla_y\cdot(\epsilon^2\mathbf{v}^o(\mathbf{x},\mathbf{y})+\epsilon^3\mathbf{v}^1(\mathbf{x},\mathbf{y})+...)=0 \quad (26)\end{aligned}$$

and

$$\begin{aligned}
&-\nabla_x(p^o(\mathbf{x},\mathbf{y})+\epsilon p^1(\mathbf{x},\mathbf{y})+...)\\
&-\frac{1}{\epsilon}\nabla_y(p^o(\mathbf{x},\mathbf{y})+\epsilon p^1(\mathbf{x},\mathbf{y})+...)\\
&+\mu\frac{1}{\epsilon^2}\Delta_y\epsilon^2(\mathbf{v}^o(\mathbf{x},\mathbf{y})+\epsilon\mathbf{v}^1(\mathbf{x},\mathbf{y})+...)\\
&+\frac{\mu}{\epsilon}\Delta_{xy}\epsilon^2(\mathbf{v}^o(\mathbf{x},\mathbf{y})+\epsilon\mathbf{v}^1(\mathbf{x},\mathbf{y})+...)\\
&+\mu\Delta_x\epsilon^2(\mathbf{v}^o(\mathbf{x},\mathbf{y})+\epsilon\mathbf{v}^1(\mathbf{x},\mathbf{y})+...)+\mathbf{f}=\mathbf{0}.
\end{aligned}\tag{27}$$

Collecting the same orders of ϵ ($O(\epsilon)$ from (26), and $O(\epsilon^{-1})$ and $O(\epsilon^o)$ from (27)), we have the differential equations

$$\nabla_y\cdot\mathbf{v}^o(\mathbf{x},\mathbf{y})=0 \tag{28}$$
$$\nabla_y p^o(\mathbf{x},\mathbf{y})=0 \tag{29}$$
$$-\nabla_x p^o(\mathbf{x},\mathbf{y})-\nabla_y p^1(\mathbf{x},\mathbf{y})+\mu\Delta_y\mathbf{v}^o(\mathbf{x},\mathbf{y})+\mathbf{f}=\mathbf{0}. \tag{30}$$

Note that, from equation (29),

$$\left(\frac{\partial p^o}{\partial y_1},\frac{\partial p^o}{\partial y_2},\frac{\partial p^o}{\partial y_3}\right)=(0,0,0).$$

Then p^o depends only on x, i.e.

$$p^o=p^o(x). \tag{31}$$

For the no-slip boundary condition, we have $\mathbf{v}=\mathbf{0}$, i.e.,

$$\epsilon^2(\mathbf{v}^o(\mathbf{x},\mathbf{y})+\epsilon\mathbf{v}^1(\mathbf{x},\mathbf{y})+...)=0.$$

Hence,

$$\mathbf{v}^o=0 \qquad \text{on } \Gamma_S. \tag{32}$$

Since the domain is assumed to be periodic, we introduce a Hilbert space of Ω-periodic functions:

$$\begin{aligned}H(\Omega)=\{\omega=(\omega_1,\omega_2,\omega_3)\in(H^1(\Omega_F))^3:\\ \omega \text{ is } \Omega-\text{periodic}, \omega=0 \text{ on } \Gamma_S,\\ \text{and } \nabla_y\cdot\omega=0\}\end{aligned}\tag{33}$$

with scalar product:

$$(\mathbf{w},\omega)_{H(\Omega)}=\int_{\Omega_F}\frac{\partial w_j}{\partial y_k}\frac{\partial\omega_j}{\partial y_k}dy \tag{34}$$

where again the repeat indices j and k indicate summation and $H^1(\Omega_F)$ is the Sobolev spaces with the norm:

$$\|\omega\|_{H^1(\Omega_F)}=\left(\sum_{|\alpha\|\leqslant 1}\int_{\Omega_F}|D^\alpha\omega|^2\right)^{1/2}, \tag{35}$$

where

$$D^\alpha\omega=\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x_2}\right)^{\alpha_2}\left(\frac{\partial}{\partial x_3}\right)^{\alpha_3}\omega. \tag{36}$$

Note that the order of this derivative is given by $|\alpha|=\alpha_1+\alpha_2+\alpha_3$. Using the scalar product (34), we now define the norm of the space $H(\Omega)$ as follows

$$\|\omega\|_{H(\Omega)}=\sqrt{(\omega,\omega)}, \tag{37}$$

which will be used below. Equation (30) can be rewritten in the indicial notation as

$$-\frac{\partial p^1}{\partial y_i}+\mu\frac{\partial^2 v_i^o}{\partial y_j^2}-\frac{\partial p^o}{\partial x_i}+f_i=0. \tag{38}$$

To obtain a weak form of the equation, we multiply (38) by a test function $\omega_i\in H(\Omega)$ and then integrate the equation, we have

$$\begin{aligned}&-\int_{\Omega_F}\omega_i\frac{\partial p^1}{\partial y_i}dy+\mu\int_{\Omega_F}\omega_i\frac{\partial^2 v_i^o}{\partial y_j^2}dy-\int_{\Omega_F}\omega_i\frac{\partial p^o}{\partial x_i}dy\\ &+\int_{\Omega_F}f_i\omega_i dy=0\end{aligned}\tag{39}$$

where $\mathbf{f}=(f_1,f_2,f_3)$ and recall that the repeat indices i and j indicate the summation. Integrating by parts the first two terms, using the fact that $\omega_i=0$ on Γ_S; ω is Ω-periodic, and p^o and f_i are functions of x only, we have

$$\int_{\Omega_F}p^1\frac{\partial\omega_i}{\partial y_i}dy-\mu\int_{\Omega_F}\frac{\partial v_i^o}{\partial y_j}\frac{\partial\omega_i}{\partial y_j}dy-\left(\frac{\partial p^o}{\partial x_i}-f_i\right)\int_{\Omega_F}\omega_i dy=0$$

Employing the divergence-free property of the test function, $\frac{\partial\omega_i}{\partial y_i}=\nabla_y\cdot\omega=0$, the first integration is zero, and we have the simplified expression

$$\mu\int_{\Omega_F}\frac{\partial v_i^o}{\partial y_j}\frac{\partial\omega_i}{\partial y_j}dy=\left(f_i-\frac{\partial p^o}{\partial x_i}\right)\int_{\Omega_F}\omega_i dy.$$

Using the definition (34) of the inner product yields

$$\mu(\mathbf{v^o},\omega)_H=(\mathbf{f}-\nabla_x p^o)\cdot\int_{\Omega_F}\omega dy \quad \forall\omega\in H(\Omega). \tag{40}$$

Consequently, the problem given by equations (28), (30), and (32) is equivalent to the variational problem: Find $\mathbf{v}^o\in H(\Omega)$ satisfying equation (40).

To show that there exists a unique solution $\mathbf{v}^o\in H(\Omega)$ of (40) using Lax-Milgram Theorem, we define $a(\mathbf{w},\omega)=\mu(\mathbf{w},\omega)_{H(\Omega)}$ which is bilinear. Note that

$$a(\omega,\omega)=\mu(\omega,\omega)_{H(\Omega)}=\mu\|\omega\|^2_{H(\Omega)} \quad \forall\omega\in H(\Omega) \tag{41}$$

is also coercive and

$$\begin{aligned}|a(\mathbf{w},\omega)|=|\mu||(\mathbf{w},\omega)|_{H(\Omega)}\leqslant\mu\|\mathbf{w}\|_{H(\Omega)}\|\omega\|_{H(\Omega)}\\ \forall\mathbf{w},\omega\in H(\Omega)\end{aligned}\tag{42}$$

can be shown to be continuous by applying the Cauchy-Schwarz inequality to the last inequality. Define the linear functional

$$F(\omega)=(\mathbf{f}-\nabla_x p^o)\cdot\int_{\Omega_F}\omega dy \tag{43}$$

and note that

$$|F(\omega)|=|(\mathbf{f}-\nabla_x p^o)|\left|\int_{\Omega_F}\omega dy\right|\leqslant(\|\mathbf{f}\|+\|\nabla_x p^o\|)\,\|\omega\|_{H^1(\omega)}, \tag{44}$$

so it is continuous. By applying the Lax-Milgram Theorem, we know there exists a unique $\mathbf{v}^o$ satisfying (40).

The solution $\mathbf{v}^o$ of (40) can be used to derive Darcy's Law. Along the way to formulate Darcy's Law we obtain a system of equations that can be employed to calculate the permeability. Applying the linearity property, we can write the solution $\mathbf{v}^o$ in the form

$$\mathbf{v}^o = \frac{1}{\mu}\left(f_i - \frac{\partial p^o}{\partial x_i}\right)\mathbf{u}^i \tag{45}$$

where $\mathbf{u}^i$ is the only solution of the problem: Find $\mathbf{u}^i \in H(\Omega)$ such that

$$(\mathbf{u}^i, \omega)_{H(\Omega)} = \int_{\Omega_F} \omega_i dy \tag{46}$$

for all $\omega \in H(\Omega)$. Thus, $\mathbf{u}^i$ is the weak solution of the following strong formulation

$$\nabla_y \cdot \mathbf{u}^i = 0 \qquad \text{in } \Omega_F \tag{47}$$

$$-\nabla_y q^i + \triangle_y \mathbf{u}^i + \mathbf{e}^i = 0 \qquad \text{in } \Omega_F \tag{48}$$

$$\mathbf{u}^i = 0 \qquad \text{on } \Gamma, \tag{49}$$

where ∇_y and $\triangle_y$ represent the Gradient and Laplacian operators with respect to the microscopic scale y; $\mathbf{u}^i$ and q^i are Ω–periodic, and $\mathbf{e}^i$ is the unit vector in the direction of the y_i axis, $i = 1, 2, 3$. The solution $\mathbf{u}^i$ of the system of equations (47)-(49) can be used to compute the permeability. Note that $\mathbf{v}^o$ and $\mathbf{u}^i$ are defined on Ω_F. To obtain the Darcy's velocity, It is natural to extend them to Ω with zero values on Ω_S. Define

$$\widetilde{\mathbf{v}^o} = \frac{1}{|\Omega|}\int_{\Omega_F} \mathbf{v}^o dy, \quad \text{and} \quad \widetilde{\mathbf{u}^i} = \frac{1}{|\Omega|}\int_{\Omega_F} \mathbf{u}^i dy. \tag{50}$$

Integrating (45) and then dividing by the volume of the domain Ω, we have

$$\widetilde{\mathbf{v}^o} = \frac{1}{\mu}\left(f_i - \frac{\partial p^o}{\partial x_i}\right)\widetilde{\mathbf{u}^i} \tag{51}$$

which in the indicial notation, is

$$\widetilde{v^o_j} = \frac{1}{\mu}\left(f_i - \frac{\partial p^o}{\partial x_i}\right)\widetilde{u^i_j}. \tag{52}$$

Equation (52) is Darcy's Law and $\widetilde{u^i_j}$ is the permeability tenser which depends on the geometry of the periodic domain Ω. In general, we write equation (52) as

$$\mathbf{q} = -\frac{\mathbf{k}}{\mu}(\nabla p - \mathbf{f}) \tag{53}$$

where

$$k_{ij} = \widetilde{\mathbf{u}^i} = \frac{1}{|\Omega|}\int_{\Omega_F} u^i_j dy \tag{54}$$

is the permeability. Equation (53) is called **Darcy's Law**.

IV. Conclusion

We provide basic knowledge of indicial notation that can help one to write systems of complex vector fields in convenient and simple scalar forms. Indicial notation rules are introduced and examples are employed to explain how to use the notation. We derive Darcy's Law by beginning with the microscale Stokes equation and assume that the fluid is incompressible. We then apply the homogenization method that is an upscaling technique to the Stokes equation. In the mean time, the indicial notation is employed so that we can have simple expression while deriving Darcy's Law. Before having the Darcy's Law, we obtain a system of equations (47) - (49) which can be used to calculate the permeability and then the constitutive equation, Darcy's Law, is obtained by using the linearity property.

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