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Delay-independent Stabilization of Linear Systems with Multiple Time-delays

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Abstract—The multidelays linear control systems described by difference differential equations are often studied in modern control theory. In this paper, the delay-independent stabilization algebraic criteria and the theorem of delay-independent stabilization for linear systems with multiple time-delays are established by using the *Lyapunov* functional and the *Riccati* algebra matrix equation in the matrix theory. An illustrative example and the simulation result, show that the approach to linear systems with multiple time-delays is effective.

Keywords—Linear system, Delay-independent stabilization, *Lyapunov* functional, *Riccati* algebra matrix equation.

I. Introduction

THE CE-2 spacecraft is the second of a series of Chinese missions to the moon. The tasks performed by the guidance, navigation and control (GNC) system are very complex due to the requirements for real time control, high reliability, and high accuracy, This is the time-delays of the GNC system.

Furthermore, the multidelays linear control systems described by difference differential equations are often researched in modern control theory (see, for example, [1], [2]). If this sore of control system is unstable, then its essential to introduce appropriate state feedback control law to make that the controlled closed-loop linear systems with multiple time-delays is asymptotically stable (see [3]), and its asymptotically stabilization does not rely on the free choosing of time-delays constants, which is the delay-independent stabilization of linear systems with multiple time-delays.

These time-delays problems had been discussed by Amemiya [1], Akazawa [2], Datk [4], Kamen [5] and Lewis & Anderson [6], etc. And an inherent limitation for the stabilization problem of state feedback stabilization for linear discrete systems with transmission delays is obtained (see [7]). The exact and approximate spectrum assignment properties associated with realizable output-feedback pole-placement-type controllers for single-input single-output linear time-invariant time-delay systems with commensurate point delays is investigated [8]. Further, Sen [8] dealted with the synthesis problem of pole-placement-based controllers for systems with point delays. A delay-dependent solution has been derived using a special *Lyapunov-Krasovskii* functional, the result is based on a sufficient condition and it thus entails an over design (see [9]). A delay-dependent stabilization criterion is devised by taking the relationship between the terms in the *Leibniz-Newton* formula into account based on the *Lyapunov* (see [10]). A characterization of delay-independent stability, stability of rays in the delay-interference phenomenon was made (see [11]).

But the most of the accomplished achievement is related to the freely choosing of time-delays constants and of which the problems discussed is relatively simple and the methods investigated is relatively complex, so it is far away from the development of modern spaceflight control theory.

Based on the above work, the main purpose of this paper is to provide a new design technique of state feedback control, and to show that the designed state feedback control ensures the dynamic responds of closed-loop linear systems with multiple time-delays to be asymptotically stable, which is extended to a general and large of linear systems. Then, in order to obtain the desirable properties, we combine the *Lyapunov* functional and the *Riccati* algebra matrix equation in the matrix theory. Next, the delay-independent stabilization algebraic criteria and the theorem of delay-independent stabilization for linear systems with multiple time-delays are established in this paper. At last, an illustrative example and the simulation result show that the approach to this sore of control system is effective and convenient and some open questions are given.

II. Problem Formulation

The modern trend in engineering systems is toward greater complexity, due mainly to the requirements of complex tasks and good accuracy. Complex systems may have multiple inputs and multiple outputs and may be time varying.

In the conventional approach to the design of a multiple inputs and multiple outputs control system, we design a controller (compensator) such that the dominant closed-loop poles have a design damping ratio ξ and an undamped natural frequency ω_n. In this approach, the order of the system may be raised by 1 or 2 unless pole-zero cancellation takes place. Note that in this approach we assume the effects on the responses of non-dominant closed-loop poles to be negligible.

Consider the following control system

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du, \end{cases} \tag{1}$$

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where $x \in R^n$ is the system state variables group, $u \in R^p$ is the system input variables group, $y \in R^q$ is the system output variables group, A is called the state matrix $(n \times n)$, B is called the input matrix $(n \times p)$, C is called the output matrix $(q \times n)$, D is called the direct transmission matrix $(q \times p)$.

A block diagram representation of (1) is show in Fig. 1. We shall choose the state feedback control law as follows

$$u(t) = Kx(t). \tag{2}$$

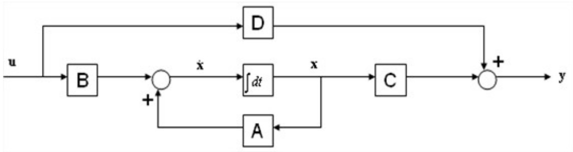


Fig. 1. Block diagram of the linear, continuous-time control system.

This means that the control signal $u(t)$ is determined by an instantaneous state. Such a scheme is called state feedback. The $(n \times n)$ matrix K is called the state feedback gain matrix. We assume that all state variables are available for feedback. In the sequel, we assume that $u(t)$ is unconstrained. A block diagram for this system is shown in Fig. 2.

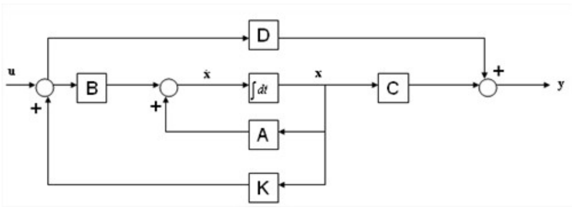


Fig. 2. Closed-loop control system with $u(t) = Kx(t)$.

This closed-loop system has no input. Its objective is to maintain the zero output. Because of the disturbances that may be present, the output will deviate from zero. The nonzero output will be returned to the zero reference input because of the state feedback scheme of system. Such a system where the reference input is always zero is called a regulator system (that is, the system is always a nonzero constant, the system is also called a regulator system.).

Substituting the state feedback control law (2) into the system (1), we have

$$\dot{x} = (A + BK)x,$$

which implies

$$x = e^{(A+BK)}x(0),$$

where $x(0)$ is the initial state caused by external disturbances. The stability and transient-response characteristics are determined by the eigenvalues of matrix $A + BK$. If matrix K is chosen properly, the matrix $A + BK$ can be made an asymptotically stable matrix, and for all $x(0) \neq 0$, it is possible to make $x(t)$ approach 0 as t approaches infinity. The eigenvalues of matrix of $A+BK$ are called the regulator poles. If these regulator poles are placed in the left-half s plane (see [3]), then $x(t)$ approaches 0 as t approaches infinity. The problem of placing the regulator poles (closed-loop poles) at the desired location is called a pole-placement problem.

In what follows, we shall prove that arbitrary pole placement for linear systems with multiple time-delays is possible and establish that delay-independent stabilization of linear systems with multiple time-delays if and only if the system is completely state controllable.

The following a general form of the dynamic equation of linear control system with multiple time-delays was introduced and considered in [1], [2] and [4]-[6]:

$$\begin{cases} \dot{x}(t) = Ax(t) + \sum_{i=1}^{m} x(t - \tau_i) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \tag{3}$$

where $x(t) \in R^n$ is the system state variables group, $u(t) \in R^p$ is the system input variables group, $y(t) \in R^q$ is the system output variables group, $A, A_i (i = 1, 2, \cdots, m)$ are respectively real matrixes of $(n \times n)$ dimensional, B is respectively real matrix of $(n \times p)$ dimensional, m is a any real and positive number and $\tau_i (i = 1, 2, \cdots, m)$ is a arbitrary nonnegative real number.

Suppose that the open-loop system (3) is unstable when $u(t) \equiv 0$, the task of this paper is to find the appropriate choice for the state feedback control law (2) such that the ordinary solution of the closed-loop system

$$\begin{cases} \dot{x}(t) = Ax(t) + \sum_{i=1}^{m} x(t - \tau_i) + BKx(t), \\ y(t) = Cx(t) + DKx(t). \end{cases} \tag{4}$$

is asymptotic stable and the stabilization is irrelevant with the selection of nonnegative time-delay constants $\tau_i (i = 1, 2, \cdots, m)$.

III. Preliminaries

Definition 1: For (1), if there is a matrix K which makes all characteristic values of $A + BK$ have negative real part, then (A, B) is able to be stable.

Definition 2: ([3]) For (1), (C, A) is observable if $rank[C^T, A^TC^T, \cdots, (A^T)^{n-1}C^T] = n$.

Definition 3: ([12]) For any time $t \geq 0$, let $x : K \to R^n$ be a continuous mapping. If $x_t(s) = x(t + s), s \in [-\tau, 0]$, then $x_t(s) \in C^0(J, R^n)$, that is, $x_t(s)$ is continuous on the interval (J, R^n). Denote the norm of $x_t(s)$ by $\|x_t(s)\| = sup_{t \in J}|x(t+s)|$ for all $J = [-\tau, 0]$ and $K = [-\tau, +\infty)$.

Lemma: ([13]) For (1), if the following conditions are satisfied:

(i) $C^TC = Q$ is positive definite symmetric matrix,

(ii) (A, B) is able to be stable,

(iii) (C, A) is observable,

then there is a unique solution matrix P, which is positive definite symmetry matrix for the *Riccati* algebra matrix equation

$$A^TP + PA - PBB^TP + Q = 0. \tag{5}$$

Definition 4: The system (3) is called delay-independent stable if and only if its zero solution is asymptotically stable for all $\tau_i (i = 1, 2, \cdots, m)$.

Definition 5: If there is a state feedback control law (2) such that the controlled closed-loop system (4) respect to the system (3) is asymptotically stable, and the stabilization is irrelevant with the selection of nonnegative real time-delays constants $\tau_i(i = 1, 2, \cdots, m)$, we say that the system (4) is the delay-independent stable and the state feedback control law (2) is the delay-independent stabilization.

IV. Main Results

Theorem: For (3), assume that the following conditions are satisfied:

(i) (A, B) is able to be stable,

(ii) (C, A) is observable,

(iii) there are a output matrix C and positive definite symmetric matrix $P_i(i = 1, 2, \cdots, m)$ such that

(a) $C^TC = Q$ is positive definite symmetric matrix,

(b)

$$\begin{bmatrix} Q - \sum_{i=1}^{m} P_1 & -PA_1 & -PA_2 & \cdots & -PA_m \\ -A_1^TP & P_1 & 0 & \cdots & 0 \\ -A_2^TP & 0 & P_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -A_m^TP & 0 & 0 & \cdots & P_m \end{bmatrix} = E \quad (6)$$

where P is the solution matrix of the *Riccati* algebra matrix equation (5), and E is a positive definite matrix.

Then there must be $K = -\frac{1}{2}B^TP$ in (2) such that the controlled closed-loop linear system (4) respect to (3) is delay-independent stable. That is, the state feedback control law (2) determined by such that (3) delay-independent stabilization.

Proof: From (i), (ii) and (iii) and the *lemma*, it's easy to know that about subpositive symmetric matrix Q, there is a unique positive definite and symmetric solution P for the *Riccati* algebra matrix equation (5). According to this, we can construct a *Lyapunov* functional

$$V(x_t) = x(t)^TPx(t) + \sum_{i=1}^{m} \int_{t-\tau_i}^{t} x(\xi)^TP_ix(\xi)d\xi. \quad (7)$$

Then

$$\begin{aligned} &\lambda_{min}(P)|x(t)|^2 \\ &\leq V(x_t) \\ &\leq \lambda_{max}(P)|x(t)|^2 + \sum_{i=1}^{m} \lambda_{max}(P_i) \int_{t-\tau_i}^{t} |x(\xi)|^2 d\xi, \quad (8) \end{aligned}$$

where $\lambda_{min}(P)$ is the smallest characteristic of the matrix P, $\lambda_{max}(P)$ is the maximum characteristic of the matrix P.

Letting $K = -\frac{1}{2}B^TP$ in (4), it follows from (i), (ii) and (iii) that

$$\begin{aligned} &\dot{V}_{(4)}(x_t) \\ &= x(t)^T(A^TP + PA - PBB^TP + \sum_{i=1}^{m} P_i)x(t) \\ &\quad + \sum_{i=1}^{m} x(t)^T(A_i^TP + PA_i)x(t-\tau_i) \\ &\quad - \sum_{i=1}^{m} x(t-\tau_i)^TP_ix(t-\tau_i) \\ &= -\omega^TE\omega, \quad (9) \end{aligned}$$

where $\dot{V}_{(4)}(x_t)$ is all derivatives of $V(x_t)$ along the track of (4) when the parameter t change, E determined by condition (iii) is a positive definite symmetric matrix and

$$\omega = \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \\ x_1(t-\tau_1) & x_2(t-\tau_1) & \cdots & x_n(t-\tau_1) \\ \vdots & \vdots & \ddots & \vdots \\ x_1(t-\tau_m) & x_2(t-\tau_m) & \cdots & x_n(t-\tau_m) \end{bmatrix}.$$

It follows from the *Cayley-Hamilton* theorem in [14] that and $\lambda_{min}(E) > 0$ and

$$\begin{aligned} &\dot{V}_{(4)}(x_t) \\ &\leq -\lambda_{min}(E)(\mid x(t) \mid^2 + \sum_{i=1}^{m} x(t-\tau_i)^2) \\ &\leq -\lambda_{min}(E) \mid x(t) \mid^2 . \quad (10) \end{aligned}$$

From (8), (10) and the results in [15], now we know that the controlled closed-loop linear system (4) respect to the system (3) is asymptotically stableand the stabilization does not rely on the selection of nonnegative time-delay constant $\tau_i(i = 1, 2, \cdots, m)$.

Hence, the state feedback control law (2) determined by $K = -\frac{1}{2}B^TP$ such that (3) delay-independent stabilization. This completes the proof.

Remark: We only make equation transformation at (9) without the inequality of expanding and shrinking for the proof of the above theorem, we can see that the condition of the criterion in this paper is expanded compared to pre-existing criterions [1], [2], [4]-[6], and the stabilization problem of (3) is become more generalized.

V. Illustrate

In this section, in order to show that the approach of this paper to this sore of control system is effective and convenient, we give an illustrative example and the simulation result.

Example: Considering the following second order system

$$\left\{\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} 8.5 & 12 \\ 12 & 8.5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ &+ \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t-\tau_1) \\ x_2(t-\tau_1) \end{bmatrix} \\ &+ \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t-\tau_2) \\ x_2(t-\tau_2) \end{bmatrix} \\ &+ \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \\ y(t) &= \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ &+ \begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \end{aligned}\right. \tag{11}$$

Solution: Firstly,

$$A = \begin{bmatrix} 8.5 & 12 \\ 12 & 8.5 \end{bmatrix}, B = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}, C = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix},$$

so it is easy to know (11) is not stable when $u(t) \equiv 0$.

Secondly, we can know that (A, B) is able to stable and (C, D) is observable.

Then, $Q = C^T C = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$ is positive definite symmetric matrix and we have the *Riccati* algebra matrix equation as follows

$$\begin{bmatrix} 8.5 & 12 \\ 12 & 8.5 \end{bmatrix}^T P + P \begin{bmatrix} 8.5 & 12 \\ 12 & 8.5 \end{bmatrix}$$
$$-P \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}^T P + \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} = 0$$

Thus, by using the soft *MATLAB*, we obtain

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Without loss of generality, let

$$P_1 = P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

in the condition (c) of *Theorem*. Then from (6), we have

$$E = \begin{bmatrix} 6 & 0 & -1 & 0 & 1 & 0 \\ 0 & 6 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

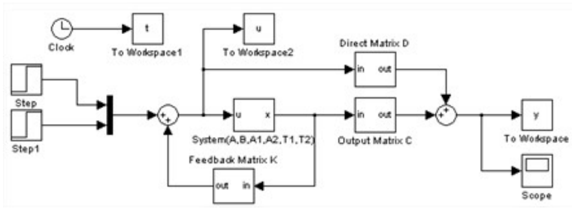
It's easy to know E is positive. Hence, if the state feedback control law

$$u(t) = -\frac{1}{2} B^T P x(t) = -\frac{1}{2} \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} x(t) \tag{12}$$

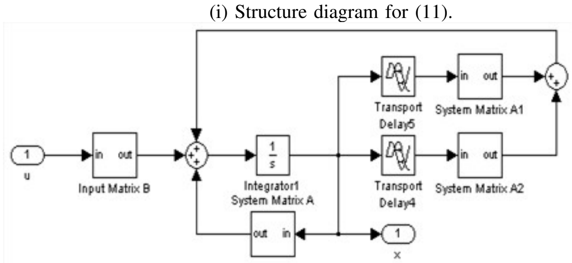
is designed, then (11) is delay-independent stable.

Simulation: Suppose that simulation time is 20 second and simulation step length is 0.01 second. We can intercalate that the first step signal action time is begining at 1 second and the second step signal action time is begining at 4 second. We can intercalate that the first time-delay is 1 second and the second time-delay is 2 second.

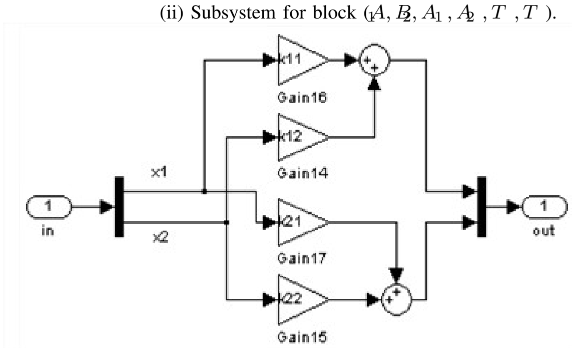
The simulation structure diagram of (11) is shown in Fig. 3.



(i) Structure diagram for (11).



(ii) Subsystem for block $(A, B, A_1, A_2, T_1, T_2)$.



(iii) Matrix structure.

Fig. 3. Simulation structure diagram of (11).

When $u(t) \equiv 0$, the dynamic response of (11) is shown in Fig. 4.

When $u(t) = -\frac{1}{2} \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} x(t) \triangleq u^*(t)$, the dynamic response of the system (11) is shown in Figure 5.

From the Fig. 4, we can know that the opened-loop linear system (3) is unstableand from the Fig. 5, we can know that the controlled closed-loop linear system (4) of (3) is asymptotically stable. That is, the approach presented in this paper to linear systems with multiple time-delays is effective.

VI. Open question

In this paper, there are still many problems remain to be dissolved.

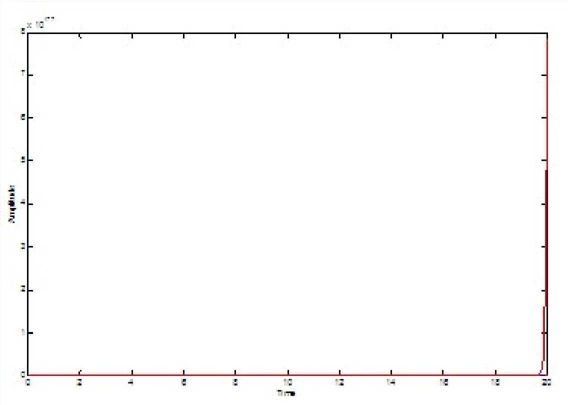


Fig. 4. Dynamic response of the system (11) at $u(t) \equiv 0$.

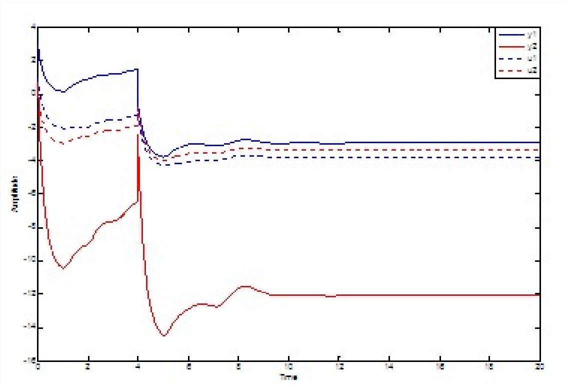


Fig. 5. Dynamic response of (11) at $u_*(t)$.

Question 1: For the matrix

$$E = \begin{bmatrix} Q - \sum_{i=1}^{m} P_1 & -PA_1 & -PA_2 & \cdots & -PA_m \\ -A_1^T P & P_1 & 0 & \cdots & 0 \\ -A_2^T P & 0 & P_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -A_m^T P & 0 & 0 & \cdots & P_m \end{bmatrix}$$

in which Q, P and $A_i (i = 1, 2, \cdots, m)$ are determined by character of (3). In order to as possible that E is a positive definite, we must choose proper $P_i(i = 1, 2, \cdots, m)$. If we can not find proper $P_i(i = 1, 2, \cdots, m)$, then we may be mistaken that (3) can't stabilization. So we must present the generally method of selection $P_i(i = 1, 2, \cdots, m)$ in the future.

Question 2: Combining the matrix

$$E = \begin{bmatrix} Q - \sum_{i=1}^{m} P_1 & -PA_1 & -PA_2 & \cdots & -PA_m \\ -A_1^T P & P_1 & 0 & \cdots & 0 \\ -A_2^T P & 0 & P_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -A_m^T P & 0 & 0 & \cdots & P_m \end{bmatrix}$$

with (3), we can know that E is a $(n^2 + n) \times (n^2 + n)$ matrix. The check computation of positive definite symmetric matrix is too heavy to practical engineering application at system order times rised. So we must present the generally succinctly method of check matrix E positive definite in the future.

Question 3: In condition (iii) of *Theorem*, $Q = C^T C$ is a positive definite symmetric matrix. Hence, the output matrix in this paper is still limited by some conditions. Therefore, it still needs to be further research.

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