

Decomposition of Homeomorphism on Topological Spaces

Ahmet Z. Ozelik, Serkan Narli

Abstract—In this study, two new classes of generalized homeomorphisms are introduced and shown that one of these classes has a group structure. Moreover, some properties of these two homeomorphisms are obtained.

Keywords—Generalized closed set, homeomorphism, gsg-homeomorphism, sgs-homeomorphism.

I. INTRODUCTION

LEVINE [9] has generalized the concept of closed sets to generalized closed sets. Bhattacharyya and Lahiri [2] have generalized the concept of closed sets to semi-generalized closed sets with the help of semi-open sets and obtained various topological properties. Arya and Nour [1] have defined generalized semi-open sets with the help of semi-openness and used them to obtain some characterizations of s -normal spaces. Devi, Balachandran and Maki [8] defined two classes of maps called semi-generalized homeomorphisms and generalized semi-homeomorphisms and also defined two classes of maps called sgc-homeomorphisms and gsc-homeomorphism. In this paper, we introduce two classes of maps called sgs-homeomorphisms and gsg-homeomorphisms and study their properties.

Throughout the present paper, (X, τ) and (Y, δ) denote topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of X . We denote the interior of A (respectively the closure of A) with respect to τ by $\text{Int}(A)$ (respectively $\text{Cl}(A)$)

II. PRELIMINARIES

Since we shall use the following definitions and some properties, we recall them in this section.

a. A subset B of a topological space (X, τ) is said to be semi-closed if there exists a closed set F such that $\text{Int}(F) \subset B \subset F$. A subset B of (X, τ) is called a semi-open set if its complement $X \setminus B$ is semi-closed in (X, τ) . Every closed (respectively open)

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set is semi-closed (respectively semi-open) [3,5].

b. A mapping $f : (X, \tau) \rightarrow (Y, \delta)$ is said to be semi-closed if the image $f(F)$ of each closed set F in (X, τ) is semi-closed in (Y, δ) . Every closed mapping is semi-closed [10].

c. Let (X, τ) be a topological space and A be a subset of X . Then, the semiinterior and semiclosure of A are defined by:

$$s\text{Int}(A) = \cup \{G_i : G_i \text{ is a semi-open in } X \text{ and } G_i \subset A\}$$

$$s\text{Cl}(A) = \cap \{K_i : K_i \text{ is a semi-closed in } X \text{ and } A \subset K_i\}$$

d. A subset B of a topological space (X, τ) is said to be semi-generalized closed (written in short as sg-closed) if $s\text{Cl}(B) \subset O$ whenever $B \subset O$ and O is semi-open [2]. The complement of a semi-generalized closed set is called a semi-generalized open. Every semi-closed set is sg-closed. The concepts of g-closed sets [7] and sg-closed sets are, in general, independent. The family of all sg-closed sets of any topological space (X, τ) is denoted by $\text{sgc}(X, \tau)$.

e. A subset B of a topological space (X, τ) is said to be generalized semi-open (written in short as gs-open) if $F \subset s\text{Int}(B)$ whenever $F \subset B$ and F is closed. B is generalized semi-closed (written in short as gs-closed) if and only if $X \setminus B$ is gs-open. Every closed set (semi-closed set, g-closed set and sg-closed set) is gs-closed. The family of all gs-closed sets of any topological space (X, τ) is denoted by $\text{gsc}(X, \tau)$ [1].

f. A map $f : (X, \tau) \rightarrow (Y, \delta)$ is called a semi-generalized continuous map (written in short as sg-continuous mapping) if $f^{-1}(V)$ is sg-closed in (X, τ) for every closed set V of (Y, δ) [5].

g. A map $f : (X, \tau) \rightarrow (Y, \delta)$ is called a generalized semi-continuous map (written in short as gs-continuous mapping) if $f^{-1}(V)$ is gs-closed in (X, τ) for every closed set V of (Y, δ) [8].

h. A map $f : (X, \tau) \rightarrow (Y, \delta)$ is called a semi-generalized closed map (respectively semi-generalized open map) if $f(V)$ is semi-generalized closed (respectively semi-generalized open) in (Y, δ) for every closed set (respectively open set) V of (X, τ) . Every semi-closed map is a semi-generalized closed map. A semi-generalized closed map (respectively semi-generalized open map) is written shortly as sg-closed map

(respectively sg open map) [7].

k. A map $f : (X, \tau) \rightarrow (Y, \delta)$ is called a generalized semi-closed map (respectively generalized semi-open map) if for each closed set (respectively open set) V of (X, τ) , $f(V)$ is gs-closed (respectively gs-open) in (Y, δ) . Every semi-closed map, every sg-closed map is a generalized semi-closed map. A generalized semi-closed map (respectively generalized semi-open map) is written shortly as gs-closed map (respectively gs open map) [7].

l. A map $f : (X, \tau) \rightarrow (Y, \delta)$ is said to be a semi-homeomorphism(B) (simply s.h. (B)) if f is continuous, f is semi-open (i.e. $f(U)$ is semi-open for every open set U of (X, τ)) and f is bijective [4].

m. A map $f : (X, \tau) \rightarrow (Y, \delta)$ is said to be a semi-homeomorphism (C.H) (simply s.h.(C.H)) if f is irresolute (i.e. $f^{-1}(V)$ is semi-open for every semi-open set V of (Y, δ)), f is pre-semi-open (i.e. $f(U)$ is semi-open for every semi-open set U of (X, τ)) and f is bijective [6].

n. A map $f : (X, \tau) \rightarrow (Y, \delta)$ is called a sg-irresolute map if $f^{-1}(V)$ is sg-closed in (X, τ) for every sg-closed set V of (Y, δ) [11].

o. A map $f : (X, \tau) \rightarrow (Y, \delta)$ is called a gs-irresolute map if $f^{-1}(V)$ is gs-closed in (X, τ) for every gs-closed set V of (Y, δ) [8].

p. A bijection $f : (X, \tau) \rightarrow (Y, \delta)$ is called a semi-generalized homeomorphism (abbreviated sg-homeomorphism) if f is both sg-continuous and sg-open [8].

r. A bijection $f : (X, \tau) \rightarrow (Y, \delta)$ is said to be a sgc-homeomorphism if f is sg-irresolute and its inverse f^{-1} is also sg-irresolute [8].

s. A bijection $f : (X, \tau) \rightarrow (Y, \delta)$ is called a generalized semi-homeomorphism (abbreviated gs-homeomorphism) if f is both gs-continuous and gs-open [8].

t. A bijection $f : (X, \tau) \rightarrow (Y, \delta)$ is said to be a gsc-homeomorphism if f is gs-irresolute and its inverse f^{-1} is also gs-irresolute [8].

u. A space (X, τ) is called a $T_{1/2}$ space if every g-closed set is closed, that is if and only if every gs-closed set is semi-closed [7,9].

v. A space (X, τ) is called a T_b space if every gs-closed set is closed [7].

III. GSG-HOMEOMORPHISM

In this section, the relations between semi-

homeomorphisms (B) and gsc-homeomorphisms are investigated and the diagram of implications is given. Also the gsg-homeomorphism is defined and some of its properties are obtained.

Remark 3.1. The following two examples show that the concepts of semi-homeomorphism (B) and gsc-homeomorphisms are independent of each other.

Example 3.2.

Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a, b\}, \{b, c\}, \{b, X\}, \delta = \{\emptyset, \{b\}, X\}$.

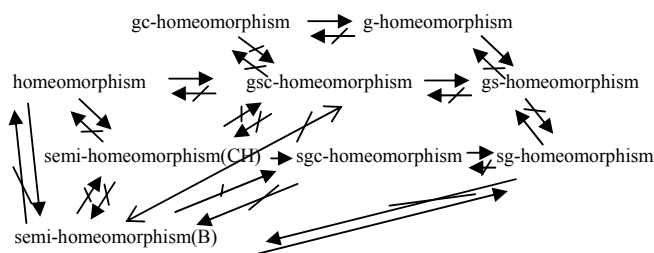
The identity map $I_x : (X, \tau) \rightarrow (X, \delta)$ is not gsc-homeomorphism. However I_x is a s.h. (B).

Example 3.3.

Let $X = \{a, b, c\}$, the topology τ on X be discrete and the topology δ on X be indiscrete.

The identity map $I_x : (X, \tau) \rightarrow (X, \delta)$ is not sh(B). However I_x is a gsc-homeomorphism.

Proposition 3.4. From remark 3.1 and remark 4.21 of R.Devi, K. Balachandran and H.Maki [8], we have the following diagram of implications.



Definition 3.5. A map $f : (X, \tau) \rightarrow (Y, \delta)$ is called a gsg-irresolute map if the set $f^{-1}(A)$ is sg-closed in (X, τ) for every gs-closed set A of (Y, δ) .

Definition 3.6. A bijection $f : (X, \tau) \rightarrow (Y, \delta)$ is called a gsg-homeomorphism if the function f and the inverse function f^{-1} are both gsg-irresolute maps. If there exists a gsg-homeomorphism from X to Y , then the spaces (X, τ) and (Y, δ) are said to be gsg-homeomorphic. The family of all gsg-homeomorphism of any topological space (X, τ) is denoted by $gsgh(X, \tau)$.

Remark 3.7. The following two examples show that the concepts of homeomorphism and gsg-homeomorphism are independent of each other.

Example 3.8.

Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{b\}, X\}$.

The identity map $I_x : (X, \tau) \rightarrow (X, \tau)$ is a homeomorphism but is not a gsg-homeomorphism on X .

Example 3.9.

Let X be any set which contains at least two elements; τ and δ be discrete and indiscrete topologies on X , respectively. The identity map $I_X: (X, \tau) \rightarrow (X, \delta)$ is a gsg-homeomorphism but is not a homeomorphism.

Remark 3.10. Every gsg-homeomorphism implies both a gsc-homeomorphism and a sg-homeomorphism.

However the converse is not true as shown by the following example.

Example 3.11.

Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{b\}, X\}$. Then

$sgc(X, \tau) = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$ and

$gsc(X, \tau) = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, X\}$.

The identity map $I_X: (X, \tau) \rightarrow (X, \tau)$ is both gsc-homeomorphism and sgc-homeomorphism. Since the set $\{b, c\}$ is gs-closed but the set $I_X^{-1}(\{b, c\}) = \{b, c\}$ is not sg-closed, then the identity map I_X is not a gsg-homeomorphism on X .

Proposition 3.12. Every gsg-homeomorphism implies both a gs-homeomorphism and a sg-homeomorphism. However its converse is not true.

Definition 3.13. Let (X, τ) and (Y, δ) be any topological spaces. If the following properties are satisfied

a) $sgc(X, \tau) = gsc(X, \tau)$ and $sgc(Y, \delta) = gsc(Y, \delta)$

b) there exists a bijective map

$\phi: gsc(X, \tau) \rightarrow gsc(Y, \delta)$ such that

$\forall A \in gsc(X, \tau) \quad \#(\phi(A)) = \#(A)$ ($\#(A)$ is cardinality of A).

then the spaces (X, τ) and (Y, δ) are called S-related

Theorem 3.14. The space (X, τ) and (Y, δ) are gsg-homeomorphic if and only if these spaces are S-related.

Proof. It follows from definition of gsg-homeomorphism and definitions 2.3, 2.4

Theorem 3.15.

a) Every gsc(sgc)-homeomorphism from $T_{1/2}$ space onto itself is a gsg-homeomorphism.

b) Every gs(sg)-homeomorphism from T_b space onto itself is a gsg-homeomorphism.

Proof. Since for any $T_{1/2}$ space (X, τ) the family of sg-closed sets is equal to the family of gs-closed sets, any gsc(sgc)-homeomorphism from X to X is a gsg-homeomorphism.

In any T_b space (X, τ) every gs-closed subset is a closed subset so (b) is obvious.

Result 3.16. Let (X, τ) and (Y, δ) be any topological spaces. If there exists any gsg-homeomorphism from X to Y , then every gsc(sgc)-homeomorphism from X to Y is a gsc(sgc)-

homeomorphism.

Proof. It is obtained by theorem 3.14

Theorem 3.17. For a topological space (X, τ) the following implications hold:

a) $gsgh(X, \tau) \subset gsch(X, \tau) \subset gsh(X, \tau)$ and

$gsgh(X, \tau) \subset sgch(X, \tau) \subset sgh(X, \tau)$

b) If $gsgh(X, \tau)$ is nonempty then $gsgh(X, \tau)$ is a group and

$sgch(X, \tau) = gsch(X, \tau) = gsgh(X, \tau)$

Proof. It follows from R. Devi, H. Maki [4], remark 3.10 and result 3.16.

Theorem 3.18. If $f: (X, \tau) \rightarrow (Y, \delta)$ is a gsg-homeomorphism, then it induces an isomorphism from the group $gsgh(X, \tau)$ onto $gsgh(Y, \delta)$.

Proof. The homomorphism $f_*: gsgh(X, \tau) \rightarrow gsgh(Y, \delta)$ is induced from f by $f_*(h) = f \circ h \circ f^{-1}$ for every $h \in gsgh(X, \tau)$. Then it easily follows that f_* is an isomorphism

IV. SGS-HOMEOMORPHISM

Definition 4.1. A map $f: (X, \tau) \rightarrow (Y, \delta)$ is called a sgs-irresolute map if the set $f^{-1}(A)$ is gs-closed in (X, τ) for every sg-closed set A of (Y, δ) .

Definition 4.2. A bijection $f: (X, \tau) \rightarrow (Y, \delta)$ is called a sgs-homeomorphism if the function f and its inverse function f^{-1} are both sgs-irresolute maps. If there exists a sgs-homeomorphism from X to Y , then the space (X, τ) and (Y, δ) are said to be sgs-homeomorphic spaces.

Remark 4.3. Every sgc-homeomorphism and gsc-homeomorphism implies a sgs-homeomorphism.

Example 4.4.

Let $X = Y = \{a, b, c\}$ and

$\tau = \{\{a\}, \{b\}, \{a, b\}, \{b, c\}, X, \emptyset\}$, $\delta = \{\emptyset, \{b\}, \{a, b\}, Y\}$.

Since $sgc(X, \tau) = gsc(X, \tau) = \mathcal{P}(X) \setminus \{\{b\}, \{a, b\}\}$ ($\mathcal{P}(X)$ is power set of X) and

$sgc(Y, \delta) = \{\{c\}, \{a\}, \{a, c\}, \emptyset, X\}$, $gsc(Y, \delta) = \mathcal{P}(Y) \setminus \{\{b\}, \{a, b\}\}$,

then the identity map $I_X: (X, \tau) \rightarrow (Y, \delta)$ is a sgs-homeomorphism but is not a gsc-homeomorphism.

Example 4.5.

Let $X = Y = \{a, b, c\}$ and

$\tau = \{\emptyset, \{a\}, X\}$, $\delta = \{\emptyset, \{b\}, \{a, b\}, Y\}$. Since

$sgc(X, \tau) = \{\{b\}, \{c\}, \{b, c\}, X, \emptyset\}$, $gsc(X, \tau) = \{\{b\}, \{c\},$

$\{b, c\}, \{a, c\}, \{a, b\}, X, \emptyset\}$ and

$sgc(Y, \delta) = \{\{c\}, \{a\}, \{a, c\}, Y, \emptyset\}$, $gsc(Y, \delta) = \{\{a\}, \{c\},$

$\{a, c\}, \{b, c\}, Y, \emptyset\}$ then the mapping

$f: (X, \tau) \rightarrow (Y, \delta)$, defined by $f(a) = b$, $f(b) = a$, $f(c) = c$

is a sgs-homeomorphism but is not a gsc-homeomorphism.

Result 4.6. Every homeomorphism is a sgs-homeomorphism but the converse is not true.

Remark 4.7. Every sgs-homeomorphism is a gs-homeomorphism and the converse is not true as seen from the following example:

Example 4.8.

Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a, b\}, X\}$, $\delta = \{\emptyset, \{b\}, \{a, b\}, Y\}$ since $sgc(X, \tau) = gsc(X, \tau) = \{\{c\}, \{a, c\}, \{b, c\}, X, \emptyset\}$, $sgc(Y, \delta) = \{\{c\}, \{a\}, \{a, c\}, Y, \emptyset\}$ and $gsc(Y, \delta) = \{\{a\}, \{c\}, \{b, c\}, \{a, c\}, Y, \emptyset\}$

Then, the identity mapping $I: (X, \tau) \rightarrow (Y, \delta)$ is a gs-homeomorphism but it is not sgs-homeomorphism.

Example 4.9.

The map $I: (X, \tau) \rightarrow (Y, \delta)$ is given by Example 4.8 is a sg-homeomorphism but is not a sgs-homeomorphism.

Result 4.10.

- a) From the example 4.9 we can see that any sg-homeomorphism is not a sgs-homeomorphism.
- b) Every gsg-homeomorphism is a sgs-homeomorphism and the converse is not true as seen from the following example.

Example 4.12.

Let $X = Y = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$, $\delta = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, Y\}$. Then the mapping $f: (X, \tau) \rightarrow (Y, \delta)$ defined by $f(a) = b$, $f(b) = a$ and $f(c) = c$ is a sgs-homeomorphism. However f is not a gsg-homeomorphism.

Theorem 4.13.

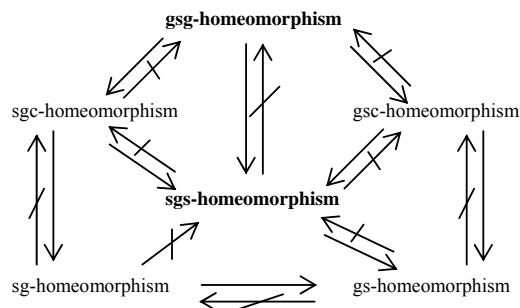
- a) Every sgs-homeomorphism from a $T_{1/2}$ space onto itself is a gsg-homeomorphism. This implies that sgs-homeomorphism is both a sgc-homeomorphism and gsc-homeomorphism.
- b) Every sgs-homeomorphism from a T_b space onto itself is a homeomorphism. This implies that sgs-homeomorphism is a gs-homeomorphism, a sg-homeomorphism, a sgc-homeomorphism, a gsc-homeomorphism and a gsg-homeomorphism.
- c) Every sgs-homeomorphism from a $T_{1/2}$ space onto itself is a sh (CH).

Proof.

- a) In a $T_{1/2}$ space, every gs-closed set is a semi-closed set.
- b) In a T_b space, every gs-closed set is a closed set.
- c) Follows from the definition of $T_{1/2}$ space.

V. CONCLUSION

In this paper, we introduce two classes of maps called sgs-homeomorphisms and gsg-homeomorphisms and study their properties. From all of the above statements, we have the following diagram:



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