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Decomposition of Graphs into Induced Paths and Cycles

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Abstract—A *decomposition* of a graph G is a collection ψ of subgraphs $H_1, H_2, \ldots, H_r$ of G such that every edge of G belongs to exactly one H_i. If each H_i is either an induced path or an induced cycle in G, then ψ is called an *induced path decomposition* of G. The minimum cardinality of an induced path decomposition of G is called the *induced path decomposition number* of G and is denoted by $\pi_i(G)$. In this paper we initiate a study of this parameter.

Keywords—Path decomposition, Induced path decomposition, Induced path decomposition number.

I. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [8]. All graphs in this paper are assumed to be connected and non-trivial. It is easy to see that

If $P = (v_0, v_1, \ldots, v_r)$ is a path in a graph G, then $v_1, v_2, \ldots, v_{r-1}$ are called *internal vertices* of P and v_0, v_r are called *external vertices* of P. If $P = (v_0, v_1, \ldots, v_r)$ and $Q = (v_r = w_0, w_1, \ldots, w_s)$ are two paths in G, then the walk obtained by concatenating P and Q at v_r is denoted by $P \circ Q$ and the path $(v_r, v_{r-1}, \ldots, v_0)$ is denoted by P^{-1}. For a unicyclic graph G with cycle C, if w is a vertex of degree > 2 on C, then the maximal subtree T of G such that $V(T) \cap V(C) = \{w\}$ is called the *subtree rooted* at w.

A *decomposition* of a graph G is a collection of subgraphs $H_1, H_2, \ldots, H_r$ of G such that every edge of G belongs to exactly one H_i. Various types of decompositions and corresponding parameters have been studied by several authors by imposing conditions on the members of the decomposition. Some such decomposition parameters are path decomposition number, acyclic path decomposition number and simple acyclic path decomposition number which are defined as follows.

Let $\psi = \{H_1, H_2, \ldots, H_r\}$ be a decomposition of a graph G. If each H_i is either a path or a cycle, then ψ is called a *path decomposition* of G. If each H_i is a path, then ψ is called an *acyclic path decomposition* of G. Further, an acyclic path decomposition in which any two paths have at most one vertex in common is called a *simple acyclic path decomposition* of G. The minimum cardinality of a path decomposition (acyclic path decomposition, simple acyclic path decomposition) of G is called the *path decomposition number (acyclic path decomposition number, simple acyclic path decomposition number)* of G and is denoted by $\pi(G)(\pi_a(G), \pi_{as}(G))$.

The parameter π_a was introduced by Harary [9] and further studied by Harary and Schwenk [10], Peroche [11], Stanton *et al.* [12] and Arumugam and Suresh Suseela [7] who used the notation π for the acyclic path decomposition number of G and called an acyclic path decomposition as a path cover. The parameter π_{as} was introduced by Arumugam and Sahul Hamid [5] who used π_s for simple acyclic path decomposition number and called a simple acyclic path decomposition as a simple path cover and the parameter π was introduced by Arumugam *et al.* [6].

Further, by imposing on each of the decomposition defined above the condition that every vertex of G is an internal vertex of at most one member of the decomposition, we get another set of path covering parameters namely graphoidal covering number $\eta(G)$, acyclic graphoidal covering number $\eta_a(G)$, simple graphoidal covering number $\eta_s(G)$ and simple acyclic graphoidal covering number $\eta_{as}(G)$ and all these parameters can be found respectively in [1], [7], [4] and [3].

Arumugam and Sahul Hamid [5] observed that every member of a simple acyclic path decomposition of a graph G is an induced path in G. However, a collection ψ of induced paths such that every edge of G is in exactly one path in ψ need not be a simple acyclic path decomposition of G. Motivated by this observation, Arumugam [2] introduced the concept of induced path decomposition and induced path decomposition number of a graph.

In this paper we initiate a study of this parameter and determine the value of the parameter for several families of graphs. Also, we obtain some bounds and characterize the graphs attaining the bounds.

II. Main Results

Definition 2.1. An *induced path decomposition* of a graph G is a path decomposition ψ of G such that every member of ψ is either an induced path or an induced cycle in G. The minimum cardinality of an induced path decomposition of G is called the *induced path decomposition number* of G and is denoted by $\pi_i(G)$. An induced path decomposition ψ of G with $|\psi| = \pi_i(G)$ is called a minimum induced path decomposition of G.

Obviously, in a tree every path decomposition is an induced path decomposition. The following theorem says, in fact, that trees are the only graphs in which every path decomposition is induced.

Theorem 2.2. Every path decomposition of G is an induced path decomposition of G if and only if G is a tree.

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Proof. Since every path in a tree is an induced path, every path decomposition of G is an induced path decomposition.

Conversely, suppose G is not a tree. Let C be a cycle in G. Let u and v be two adjacent vertices on C. Let P be the $u-v$ path of length greater than 1. Then $\psi = \{(u,v), P\} \cup [E(G) - E(C)]$ is a path decomposition of G which is not an induced path decomposition.

The following theorem is useful in determining the value of π_i for several families of graphs.

Theorem 2.3. For any induced path decomposition ψ of a graph G, let $t_\psi = \sum_{P\in\psi} t(P)$, where $t(P)$ denotes the number of internal vertices of P and let $t = \max t_\psi$ where the maximum is taken over all induced path decomposition ψ of G. Then $\pi_i(G) = m - t$.

Proof. Let ψ be an induced path decomposition of G. Then

$$\begin{aligned} m &= \sum_{P\in\psi} |E(P)| \\ &= \sum_{P\in\psi} (t(P)+1) \\ &= \sum_{P\in\psi} t(P) + |\psi| \\ &= t_\psi + |\psi| \end{aligned}$$

Hence $|\psi| = m - t_\psi$ so that $\pi = m - t$.

Corollary 2.4. Let G be a graph with k vertices of odd degree. Then $\pi_i(G) = \frac{k}{2} + \sum_{v\in V} \lfloor \frac{\deg v}{2} \rfloor - t$.

Proof. Since $m = \frac{k}{2} + \sum_{v\in V} \lfloor \frac{\deg v}{2} \rfloor$, the result follows from Theorem 2.3.

Corollary 2.5. For any graph $G, \pi_i(G) \geq \frac{k}{2}$. Further, equality holds if and only if there exists an induced path decomposition ψ of G such that every vertex v of G is an internal vertex of $\lfloor \frac{\deg v}{2} \rfloor$ paths in ψ.

In the following theorems we determine the value of π_i for some families of graphs such as complete bipartite graphs, trees, unicyclic graphs and wheels.

Theorem 2.6. Let r and s be positive integers with $r \leq s$. Then, for the complete bipartite graph $K_{r,s}$, we have

$$\pi_i(K_{r,s}) = \begin{cases} \lceil \frac{r}{2} \rceil \lceil \frac{s}{2} \rceil & \text{if } rs \text{ is even} \\ \frac{(r+1)(s+1)}{2} & \text{if } rs \text{ is odd} \end{cases}$$

Proof. Let $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$ be the bipartition of $K_{r,s}$. We observe that every member of an induced path decomposition of $K_{r,s}$ is a cycle C_4 on four vertices or a path P_3 of length 2 or an edge. Now, let

$$P_{ij} = (x_{2i-1}, y_{2j-1}, x_{2i}, y_{2j}, x_{2i-1}), i = 1, 2, \ldots, \left\lceil \frac{r}{2} \right\rceil, \quad j = 1, 2, \ldots, \left\lceil \frac{s}{2} \right\rceil$$

Case 1. r and s are even.

Then

$$\psi = \bigcup_{i=1}^{\frac{r}{2}} \Big(\bigcup_{i=1}^{\frac{s}{2}} P_{ij} \Big)$$

is an induced path decomposition of $K_{r,s}$ so that $\pi_i(K_{r,s}) \leq \frac{r}{2} \cdot \frac{s}{2}$. Further, for any induced path decomposition ψ of $K_{r,s}$, every member of ψ covers at most four edges of $K_{r,s}$ so that $|\psi| \geq \frac{rs}{4}$ and hence $\pi_i(K_{r,s}) \geq \frac{rs}{4}$. Thus $\pi_i(K_{r,s}) = \frac{rs}{4}$.

Case 2. r is even and s is odd.

Let $Q_i = (x_{2i-1}, y_s, x_{2i}), i = 1, 2, \ldots, \frac{r}{2}$. Then

$$\psi = \left\{ \bigcup_{i=1}^{\frac{r}{2}} \Big(\bigcup_{j=1}^{\lceil \frac{s}{2} \rceil} P_{ij} \Big) \right\} \cup \{Q_1, Q_2, \ldots, Q_{\frac{r}{2}}\}$$

is an induced path decomposition of $K_{r,s}$ so that $\pi_i(K_{r,s}) \leq |\psi| = \frac{r}{2} \cdot \lceil \frac{s}{2} \rceil$. Further, any induced path decomposition of $K_{r,s}$ can have at most $\frac{r}{2} \cdot \lfloor \frac{s}{2} \rfloor$ cycles of length 4 and $\frac{r}{2}$ paths of length 2 and hence we have $|\psi| \geq \frac{r}{2} \lfloor \frac{s}{2} \rfloor + \frac{r}{2}$ so that $\pi_i(K_{r,s}) \geq \frac{r}{2} \lceil \frac{s}{2} \rceil$. Thus $\pi_i(K_{r,s}) = \frac{r}{2} \lceil \frac{s}{2} \rceil$.

Similarly, we can prove that $\pi_i(K_{r,s}) = \lceil \frac{r}{2} \rceil \frac{s}{2}$ if r is odd and s is even.

Case 3. r and s are odd. Let $Q_i = (y_{2i-1}, x_r, y_{2i}), i = 1, 2, \ldots, \frac{s-1}{2}$
$R_i = (x_{2i-1}, y_s, x_{2i}), i = 1, 2, \ldots, \frac{r-1}{2}$
Then

$$\psi = \left\{ \bigcup_{i=1}^{\frac{r-1}{2}} \Big(\bigcup_{j=1}^{\frac{s-1}{2}} P_{ij} \Big) \right\} \cup \{Q_1, Q_2, \ldots, Q_{\frac{r-1}{2}}\}$$
$$\cup \{R_1, R_2, \ldots, R_{\frac{r-1}{2}}\} \cup \{(x_r, y_r)\}$$

is an induced path decomposition of $K_{r,s}$ and hence $\pi_i(K_{r,s}) \leq \frac{(r+1)(s+1)}{4}$. Further, any induced path decomposition ψ of $K_{r,s}$ can have at most $\lfloor \frac{r}{2} \rfloor \lfloor \frac{s}{2} \rfloor$ cycles of length four and hence $|\psi| \geq \lfloor \frac{r}{2} \rfloor \lfloor \frac{s}{2} \rfloor + \lceil \frac{r+s-1}{2} \rceil = \frac{(r+1)(s+1)}{4}$. Thus $\pi_i(K_{r,s}) = \frac{(r+1)(s+1)}{4}$.

Theorem 2.7. For the wheel W_n on n vertices, we have

$$\pi_i(W_n) = \begin{cases} 4 & \text{if } n = 4 \\ \lceil \frac{n-1}{2} \rceil + 1 & \text{if } n \geq 5 \end{cases}$$

Proof. Let $V(W_n) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(W_n) = \{v_0 v_i : 1 \leq i \leq n-1\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-2\} \cup \{v_{n-1} v_1\}$.

If $n = 4$, then $\psi = \{(v_0, v_1, v_2, v_0), (v_0, v_3), (v_1, v_3), (v_2, v_3)\}$ is an induced path decomposition of W_4 and since a member of any induced path decomposition of W_4 is either an edge or a triangle, it follows that $\pi_i(W_4) = 4$.

Suppose $n \geq 5$. Then for $i = 1, 2, \ldots, \lceil \frac{n-1}{2} \rceil - 1$, let

$$P_i = \begin{cases} (v_i, v_0, v_{\frac{n-1}{2}+i}) & \text{if } n \text{ is odd} \\ (v_i, v_0, v_{\frac{n-2}{2}+i}) & \text{if } n \text{ is even.} \end{cases}$$

$$P_{\lceil \frac{n-1}{2} \rceil} = \begin{cases} (v_{\frac{n-1}{2}}, v_0, v_{n-1}) & \text{if } n \text{ is odd} \\ (v_0, v_{n-1}) & \text{if } n \text{ is even.} \end{cases}$$

Then $\psi = \{P_1, P_2, \ldots, P_{\lceil \frac{n-1}{2} \rceil}, (v_1, v_2, \ldots, v_{n-1}, v_1)\}$ is an induced path decomposition of W_n so that $\pi_i(W_n) \leq |\psi| = \lceil \frac{n-1}{2} \rceil + 1$. Further, since the minimum number of paths required to cover the set of edges $\{v_0v_i : 1 \leq i \leq n-1\}$ is $\lceil \frac{n-1}{2} \rceil$, it follows that for any induced path decomposition ψ of W_n, we have $|\psi| \geq \lceil \frac{n-1}{2} \rceil + 1$ and hence $\pi_i(W_n) = \lceil \frac{n-1}{2} \rceil + 1$.

Remark. Since every path in a tree T is induced, every acyclic path decomposition of T is an induced path decomposition and hence $\pi_i(T) = \pi_a(T)$. Also it has been proved in [12] that $\pi_a(T) = \frac{k}{2}$, where k is the number of vertices of odd degree so that $\pi_i(T) = \frac{k}{2}$.

Theorem 2.8. Let G be a unicyclic graph with cycle C. Let r denote the number of vertices of degree greater than two on C. Let k be the number of vertices of odd degree. Then

$$\pi_i(G) = \begin{cases} 1 & \text{if } r = 0 \\ \frac{k}{2} + 1 & \text{if } r = 1 \text{ or } r = 2 \text{ and the vertices} \\ & \text{of degree } > 2 \text{ on } C \text{ are adjacent} \\ \frac{k}{2} & \text{otherwise.} \end{cases}$$

Proof. Let $C = (v_1, v_2, \ldots, v_n, v_1)$.

Case 1. $r = 0$.

Then $G = C$ so that $\pi_i(G) = 1$.

Case 2. $r = 1$.

Let v_1 be the unique vertex of degree greater than 2 on C. Let T be the subtree rooted at v_1. Then $\pi_i(T) = \frac{k}{2}$. Let ψ_1 be a minimum induced path decomposition of T. Then $\psi = \psi_1 \cup \{C\}$ is an induced path decomposition of G so that $\pi_i(G) \leq |\psi| = |\psi_1| + 1 = \frac{k}{2} + 1$. Further, for any induced path decomposition ψ of G, there exists a vertex v_i on C such that v_i is an internal vertex in at most $\lfloor \frac{\deg v_i}{2} \rfloor - 1$ paths and hence $\pi_i(G) \geq \frac{k}{2} + 1$. Thus $\pi_i(G) = \frac{k}{2} + 1$.

Case 3. $r = 2$.

Let v_1 and v_i be the vertices of degree greater than 2 on C. Let T_1 and T_2 respectively denote the subtrees rooted at v_1 and v_i. Let ψ_1 and ψ_2 be minimum induced path decompositions of T_1 and T_2 respectively.

Subcase 3.1. The vertices v_1 and v_i are adjacent.

Then $\psi = \psi_1 \cup \psi_2 \cup \{C\}$ is an induced path decompostion of G so that $\pi_i(G) \leq |\psi| = |\psi_1| + |\psi_2| + 1 = \frac{k}{2} + 1$. Further, for any induced path decomposition ψ of G, there exists a vertex v_j on C such that v_j is an internal vertex in at most $\lfloor \frac{\deg v_j}{2} \rfloor - 1$ paths in ψ and hence $\pi_i(G) \geq \frac{k}{2} + 1$. Thus $\pi_i(G) = \frac{k}{2} + 1$.

Subcase 3.2. The vertices v_1 and v_i are not adjacent.

Suppose $\deg v_1$ and $\deg v_i$ are odd. Let P_1 and P_2 denote respectively the paths in ψ_1 and ψ_2 having v_1 and v_i as its terminal vertices. Let

$$\begin{aligned} Q_1 &= P_1 \circ (v_1, v_2, \ldots, v_i) \circ P_2^{-1} \text{ and} \\ Q_2 &= (v_i, v_{i+1}, \ldots, v_n, v_i). \end{aligned}$$

Since v_1 and v_i are not adjacent, the paths Q_1 and Q_2 are induced. Hence $\psi = (\psi_1 - \{P_1\}) \cup (\psi_2 - \{P_2\}) \cup \{Q_1, Q_2\}$ is an induced path decomposition of G so that $\pi_i(G) \leq |\psi| = |\psi_1| + |\psi_2| = \frac{k}{2}$.

Suppose $\deg v_1$ and $\deg v_i$ are even. Let P_1 be an $u_1 - w_1$ path in ψ_1 having v_1 as an internal vertex. Let P_2 be an $u_2 - w_2$ path in ψ_2 having v_i as an internal vertex. Let R_1 and R_2 denote the (u_1, v_1)-section and (w_1, v_1)-section of P_1 respectively. Let R_1' and R_2' be the (v_i, u_2)-section and (v_i, w_2)-section of P_2 respectively. Now, let

$$\begin{aligned} Q_1 &= R_1 \circ (v_1, v_2, \ldots, v_i) \circ R_1' \text{ and} \\ Q_2 &= R_2 \circ (v_1, v_n, v_{n-1}, \ldots, v_i) \circ R_2'. \end{aligned}$$

Since v_1 and v_i are not adjacent, the paths Q_1 and Q_2 are induced. Hence $\psi = (\psi_1 - \{P_1\}) \cup (\psi_2 - \{P_2\}) \cup \{Q_1, Q_2\}$ is an induced path decomposition of G so that $\pi_i(G) \leq |\psi| = \frac{k}{2}$.

Suppose $\deg v_1$ is odd and $\deg v_i$ is even. Let P_1 be the path in ψ_1 having v_1 as a terminal vertex and let $P_2 = (u_1, u_2, \ldots, u_r, v_i, u_{r+1}, \ldots, u_s)$ be an $u_1 - u_s$ path having v_i as an internal vertex. Now, let

$$\begin{aligned} Q_1 &= P_1 \circ (v_1, v_2, \ldots, v_i, u_r, u_{r-1}, \ldots, u_1) \text{ and} \\ Q_2 &= (v_1, v_n, \ldots, v_i, u_{r+1}, u_{r+2}, \ldots, u_s). \end{aligned}$$

Then $\psi = (\psi_1 - \{P_1\}) \cup (\psi_2 - \{P_2\}) \cup \{Q_1, Q_2\}$ is an induced path decomposition of G so that $\pi_i(G) \leq |\psi| = \frac{k}{2}$.

Further, $\pi_i(G) \geq \frac{k}{2}$ and hence $\pi_i(G) = \frac{k}{2}$.

Case 4. $r > 2$.

Let $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$, where $1 \leq i_1 \leq i_2 \leq \cdots < i_r$, be the vertices of degree greater than 2 on C. Let $T_{i_j}, 1 \leq j \leq r$, be the subtree rooted at the vertex v_{i_j} and let ψ_{i_j} be a minimum induced path decomposition of T_{i_j}. Consider the vertices v_{i_1}, v_{i_2} and v_{i_3}. Let C_1, C_2 and C_3 denote the (v_{i_1}, v_{i_2})-section, (v_{i_2}, v_{i_3})-section and (v_{i_3}, v_{i_1})-section of C respectively. For $j = 1, 2, 3$, let P_j be either an $u_j - w_j$ path in ψ_{i_j} having v_{i_j} as an internal vertex or a path in ψ_{i_j} having v_{i_j} as a terminal vertex. Then $\psi = \left(\left(\bigcup_{j=1}^{r} \psi_{i_j}\right) - \{P_1, P_2, P_3\}\right) \cup \{Q_1, Q_2, Q_3\}$ is an induced path decomposition of G such that every vertex v of G is an internal vertex of $\lfloor \frac{\deg v}{2} \rfloor$ paths in ψ and hence $\pi_i(G) = \frac{k}{2}$.

We now proceed to obtain some bounds for π_i and characterize graphs attaining the bounds.

Remark. For any graph $G, \pi_i(G) \leq m$. Further equality holds if and only if $G = K_2$, for if $G \neq K_2$, then G contains an induced path of length greater than one or a triangle and hence $\pi_i(G) < m$.

Theorem 2.9. For any graph G with girth g, we have $\pi_i(G) \leq m - g + 1$. Further equality holds if and only if

G is either a cycle or K_4 or $K_4 - e$ or one of the graphs G_1 and G_2 which are described as follows.

(i) G_1 is the graph obtained from a cycle by attaching exactly one pendant edge at a vertex of the cycle.
(ii) G_2 is the graph obtained from a cycle by attaching exactly one pendant edge at two adjacent vertices of the cycle.

Proof. Let C be a cycle of length g in G. Then C is induced so that $\psi = \{C\} \cup (E(G) - E(C))$ is an induced path decomposition of G and hence $\pi_i(G) \le |\psi| = m - g + 1$.

Now, suppose G is a graph with $\pi_i(G) = m - g + 1$. Let $C = (v_1, v_2, \ldots, v_g, v_1)$ be a cycle of length g in G. If G has an induced path P with length greater than one and $|V(P) \cap V(C)| = 1$, then $\{C, P\} \cup S$, where S is the set of edges of G not covered by C and P is an induced path decomposition of G with $|\psi| < m - g + 1$, which is a contradiction. Hence every vertex not on C is adjacent to a vertex on C, no two vertices not an C are adjacent and every vertex on C has degree at most 3.

Claim 1. Any two vertices of degree 3 on C are adjacent.

Let v_{i_1} and v_{i_2}, where $i_1 < i_2$, be two vertices of degree 3 on C and let x and y be the vertices (not on C) adjacent to v_{i_1} and v_{i_2} respectively. Suppose v_{i_1} and v_{i_2} are not adjacent. Then $g \ge 4$. Consider an (v_{i_1}, v_{i_2})-section of C, say C_1. Suppose either x or y, say x is adjacent to a vertex of C_1. Let i_3 be the least positive integer with $i_1 < i_2 < i_3$ such that x is adjacent to v_{i_3}. Then v_{i_1} and v_{i_3} are not adjacent because $g \ge 4$. Now, let P_1 be the (v_{i_1}, v_{i_3})-section of C containing v_{i_2} and let C_1 be the cycle consisting of the (v_{i_1}, v_{i_3})-section of C not containing v_{i_2} followed by the path (v_{i_3}, x, v_{i_1}). Then $\psi = \{P_1, C_1\} \cup [E(G) - (E(P_1) \cup E(C_1))]$ is an induced path decomposition of G with $|\psi| < m - g + 1$, which is a contradiction. Thus neither x nor y is adjacent to any vertex of C_1 and hence $\psi_1 = \{P = C - C_1, P' = (x, v_{i_1}) \circ C_1 \circ (v_{i_2}, y)\} \cup [E(G) - (E(P) \cup E(P'))]$ is an induced path decomposition of G with $|\psi| < m - g + 1$, which is again a contradiction. Thus any two vertices of degree 3 on C are adjacent.

Now, one can observe that the following are immediate consequences of the above claim.

(i) Every vertex not on C is of degree at most 3 so that $\Delta(G) \le 3$.
(ii) There exist at most two vertices not on C.
(iii) If there are two vertices not on C, then they are pendant.

Now, if there is no vertex outside C, then G is a cycle. If there are exactly two vertices not on C, then it follows from Claim 1 and the above observation (iii) that G is isomorphic to G_2. Now, suppose there is exactly one vertex not on C, say v. If $\deg v = 3$, then the neighbours of v lie on C so that it follows from claim 1 that they are adjacent and consequently G is isomorphic to K_4. Similarly, if $\deg v = 2$, then the neighbours of v lie on C and they are adjacent so that $g = 3$ and hence G is isomorphic to $K_4 - e$. If $\deg v = 1$, then G is isomorphic to G_1.

The converse is just a simple verification.

Obviously one can observe that $\pi_i(G) \ge \lceil \frac{\Delta}{2} \rceil$. Further, we observe that a tree attains this bound if and only if it has at most one vertex with $\deg v \ge 3$. Apart from trees, an infinite family of unicyclic graphs too attain this bound which we now characterize in the following theorem.

Theorem 2.10. Let G be a unicyclic graph with cycle C. Let r denote the number of vertices on C with degree greater than 2. Then $\pi_i(G) = \lceil \frac{\Delta}{2} \rceil$ if and only if the following are satisfied.

(i) $r \le 2$
(ii) Every vertex not on C has degree 1 or 2.
(iii) If $r = 2$, then the two vertices on C with degree greater than 2 are not adjacent and one of these vertices is of degree either 3 or 4.

Proof. Let u be a vertex with $\deg u = \Delta$. Let ψ be a minimum induced path decomposition of G. Then every member of ψ passes through u. If u does not lie on C, then ψ will not cover at least one edge of C and hence u lies on C. Since ψ is a collection of edge-disjoint induced paths covering all the edges of G, it follows that every vertex not on C is of degree either 1 or 2 and $r \le 2$.

Suppose $r = 2$. Let x and y be the vertices of degree greater than 2 on C. Suppose x and y are adjacent. Now, it is clear that not both x and y are of degree Δ. Assume without loss of generality that $\deg x = \Delta$. Let P be the path containing the edge not on C which is incident at y. Then P contains the edge xy so that there exist two paths P_1 and P_2 in ψ which cover the edges of the (x, y)-section of C of length at least 2. Hence one of these paths does not pass through the vertex x, which is a contradiction. Thus x and y are not adjacent. Further, since every member of ψ passes through the vertices of maximum degree it follows that either x or y has degree ≤ 4.

Conversely, suppose conditions (i)–(iii) of the theorem hold. If $r = 0$, obviously $\pi_i(G) = \frac{\Delta}{2}$. Suppose $r = 1$. If Δ is even, then G has $\Delta - 2$ vertices of odd degree and if Δ is odd, then G has $\Delta - 1$ vertices of odd degree, so that in either of the cases it follows from Theorem 2.9 that $\pi_i(G) = \lceil \frac{\Delta}{2} \rceil$. Now, suppose $r = 2$. Let x and y be the vertices of degree greater than 2 on C, with $\deg x = \Delta$ and $\deg y = 3$ or 4. Then G has Δ or $\Delta + 1$ vertices of odd degree according as Δ is even or odd and since x and y are not adjacent it follows from Theorem 2.9 that $\pi_i(G) = \lceil \frac{\Delta}{2} \rceil$.

Theorem 2.11. If G is a regular graph, then $\pi_i(G) = \lceil \frac{\Delta}{2} \rceil$ if and only if $G = K_2$ or G is a cycle.

Proof. Suppose G is regular with $\pi_i(G) = \lceil \frac{\Delta}{2} \rceil$. Then every member of any minimum induced path decomposition ψ of G passes through all the vertices of G. Hence a minimum induced path decomposition of G consist of hamiltonian paths and hamiltonian cycles and since ψ is induced it follows that $|\psi| = 1$ and consequently $G = K_2$ or G is a cycle. The converse is obvious.

So far we have determined the value of π_i for several families of graphs and have obtained bounds for π_i with characterization of graphs attaining the bounds. Now, it is of some interest discussing the relation of π_i with some existing path covering parameters such as path decomposition number π, simple acyclic path decomposition number π_{as} and simple graphoidal covering number η_s.

It follows immediately from definitions that $\pi \leq \pi_i \leq \pi_{as}$. Of course, the difference between π_i and π can be made arbitrarily large. For the graph G obtained from a path $(v_1, v_2, \ldots, v_{n+3})$ by introducing $n+2$ new vertices, namely, $w_1, w_2, \ldots, w_{n+2}$, and joining $w_i, 1 \leq i \leq n+2$, to both v_i and v_{i+1}, we have $\pi(G) = 2$ and $\pi_i(G) = n+2$ and hence $\pi_i - \pi = n$. Further, if G' is the graph obtained from G by subdividing the edges $v_i v_{i+1}, 1 \leq i \leq n+2$, by a vertex, then $\pi_i(G') = 2$ and $\pi_{as}(G') = 2n+4$ and hence the difference between parameters π_{as} and π_i also can be made as large as possible.

Also, obviously for a cycle with at least four vertices $\pi = 1, \pi_i = 2$ and $\pi_{as} = 3$ and for any tree $\pi = \pi_i = \pi_{as} = \frac{k}{2}$, where k is the number of vertices of odd degree. Further, not only for a tree but also for the unicyclic graphs with at least three vertices of degree greater than two on the unique cycle, the three parameters coincide. Thus these parameters may coincide or may assume distinct values and so the following problem naturally arises.

Problem. Characterize graphs for which $\pi = \pi_i = \pi_{as}$.

As a direct application of the definitions, we have $\pi_i \leq \eta_s$ and one can observe that these parameters coincide for the graphs for which $\Delta \leq 3$. Of course, as one would expect, graphs assuming same value for these parameters may have vertices of higher degree (consider the graph with exactly one cut-vertex and each of whose blocks is a cycle), and however trees are not of this kind. For, if T is a tree having a vertex v with degree more than three, then every minimum simple graphoidal cover ψ contains two paths P_1 and P_2 having v as a terminal vertex and hence $(\psi - \{P_1, P_2\}) \cup \{P_1 \circ P_2^{-1}\}$ is an induced path decomposition of T with cardinality $|\psi| - 1$ so that $\pi_i(T) \leq \eta_s(T) - 1$. Thus, we have

Theorem 2.12 If T is tree, then $\eta_s(T) = \pi_i(T)$ if and only if $\Delta \leq 3$.

Conclusion and Scope

A decomposition of a graph G is a collection of edge-disjoint subgraphs of G whose union is G. Various types of decomposition and corresponding parameters have been studied by imposing certain condition on the members of the decomposition. The key condition that we impose here is "inducedness" and arrived at the concept of induced path decomposition and the induced path decomposition number $\pi_i(G)$. Here, we first determined $\pi_i(G)$ for several families of graphs and obtained some bounds for π_i together with the characterization of graphs attaining these bounds and finally discuss the relation of π_i with some well-known related parameters.

Even if this paper is just an initiation of the concept of induced path decomposition, numerous problems can be identified for further investigation and here are some interesting problems.

(a) Characterize graphs for which $\pi_i = \frac{k}{2}$, where k is the number of vertices of odd degree.

(b) Characterize graphs for which $\pi_i = \lceil \frac{\Delta}{2} \rceil$.

(c) Characterize graphs for which (i) $\pi_i = \pi$, (ii) $\pi_i = \pi_{as}$ and (iii) $\pi_i = \eta_s$.

Also, as we did, one can impose the condition "inducedness" on any kind of path decomposition and can arrive at a number of new path covering parameters.

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