

# Decomposition of Graphs into Induced Paths and Cycles

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**Abstract**—A decomposition of a graph  $G$  is a collection  $\psi$  of subgraphs  $H_1, H_2, \dots, H_r$  of  $G$  such that every edge of  $G$  belongs to exactly one  $H_i$ . If each  $H_i$  is either an induced path or an induced cycle in  $G$ , then  $\psi$  is called an *induced path decomposition* of  $G$ . The minimum cardinality of an induced path decomposition of  $G$  is called the *induced path decomposition number* of  $G$  and is denoted by  $\pi_i(G)$ . In this paper we initiate a study of this parameter.

**Keywords**—Path decomposition, Induced path decomposition, Induced path decomposition number.

## I. INTRODUCTION

**B**Y a graph  $G = (V, E)$  we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of  $G$  are denoted by  $n$  and  $m$  respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [8]. All graphs in this paper are assumed to be connected and non-trivial. It is easy to see that

If  $P = (v_0, v_1, \dots, v_r)$  is a path in a graph  $G$ , then  $v_1, v_2, \dots, v_{r-1}$  are called *internal vertices* of  $P$  and  $v_0, v_r$  are called *external vertices* of  $P$ . If  $P = (v_0, v_1, \dots, v_r)$  and  $Q = (v_r = w_0, w_1, \dots, w_s)$  are two paths in  $G$ , then the walk obtained by concatenating  $P$  and  $Q$  at  $v_r$  is denoted by  $P \circ Q$  and the path  $(v_r, v_{r-1}, \dots, v_0)$  is denoted by  $P^{-1}$ . For a unicyclic graph  $G$  with cycle  $C$ , if  $w$  is a vertex of degree  $> 2$  on  $C$ , then the maximal subtree  $T$  of  $G$  such that  $V(T) \cap V(C) = \{w\}$  is called the *subtree rooted at  $w$* .

A *decomposition* of a graph  $G$  is a collection of subgraphs  $H_1, H_2, \dots, H_r$  of  $G$  such that every edge of  $G$  belongs to exactly one  $H_i$ . Various types of decompositions and corresponding parameters have been studied by several authors by imposing conditions on the members of the decomposition. Some such decomposition parameters are path decomposition number, acyclic path decomposition number and simple acyclic path decomposition number which are defined as follows.

Let  $\psi = \{H_1, H_2, \dots, H_r\}$  be a decomposition of a graph  $G$ . If each  $H_i$  is either a path or a cycle, then  $\psi$  is called a *path decomposition* of  $G$ . If each  $H_i$  is a path, then  $\psi$  is called an *acyclic path decomposition* of  $G$ . Further, an acyclic path decomposition in which any two paths have at most one vertex in common is called a *simple acyclic path decomposition* of  $G$ . The minimum cardinality of a path decomposition (acyclic path decomposition, simple acyclic path decomposition) of  $G$  is called the *path decomposition number* (acyclic path

*decomposition number, simple acyclic path decomposition number*) of  $G$  and is denoted by  $\pi(G)$  ( $\pi_a(G)$ ,  $\pi_{as}(G)$ ).

The parameter  $\pi_a$  was introduced by Harary [9] and further studied by Harary and Schwenk [10], Peroche [11], Stanton *et al.* [12] and Arumugam and Suresh Suseela [7] who used the notation  $\pi$  for the acyclic path decomposition number of  $G$  and called an acyclic path decomposition as a path cover. The parameter  $\pi_{as}$  was introduced by Arumugam and Sahul Hamid [5] who used  $\pi_s$  for simple acyclic path decomposition number and called a simple acyclic path decomposition as a simple path cover and the parameter  $\pi$  was introduced by Arumugam *et al.* [6].

Further, by imposing on each of the decomposition defined above the condition that every vertex of  $G$  is an internal vertex of at most one member of the decomposition, we get another set of path covering parameters namely graphoidal covering number  $\eta(G)$ , acyclic graphoidal covering number  $\eta_a(G)$ , simple graphoidal covering number  $\eta_s(G)$  and simple acyclic graphoidal covering number  $\eta_{as}(G)$  and all these parameters can be found respectively in [1], [7], [4] and [3].

Arumugam and Sahul Hamid [5] observed that every member of a simple acyclic path decomposition of a graph  $G$  is an induced path in  $G$ . However, a collection  $\psi$  of induced paths such that every edge of  $G$  is in exactly one path in  $\psi$  need not be a simple acyclic path decomposition of  $G$ . Motivated by this observation, Arumugam [2] introduced the concept of induced path decomposition and induced path decomposition number of a graph.

In this paper we initiate a study of this parameter and determine the value of the parameter for several families of graphs. Also, we obtain some bounds and characterize the graphs attaining the bounds.

## II. MAIN RESULTS

**Definition 2.1.** An *induced path decomposition* of a graph  $G$  is a path decomposition  $\psi$  of  $G$  such that every member of  $\psi$  is either an induced path or an induced cycle in  $G$ . The minimum cardinality of an induced path decomposition of  $G$  is called the *induced path decomposition number* of  $G$  and is denoted by  $\pi_i(G)$ . An induced path decomposition  $\psi$  of  $G$  with  $|\psi| = \pi_i(G)$  is called a *minimum induced path decomposition* of  $G$ .

Obviously, in a tree every path decomposition is an induced path decomposition. The following theorem says, in fact, that trees are the only graphs in which every path decomposition is induced.

**Theorem 2.2.** Every path decomposition of  $G$  is an induced path decomposition of  $G$  if and only if  $G$  is a tree.

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**Proof.** Since every path in a tree is an induced path, every path decomposition of  $G$  is an induced path decomposition.

Conversely, suppose  $G$  is not a tree. Let  $C$  be a cycle in  $G$ . Let  $u$  and  $v$  be two adjacent vertices on  $C$ . Let  $P$  be the  $u - v$  path of length greater than 1. Then  $\psi = \{(u, v), P\} \cup [E(G) - E(C)]$  is a path decomposition of  $G$  which is not an induced path decomposition.

The following theorem is useful in determining the value of  $\pi_i$  for several families of graphs.

**Theorem 2.3.** For any induced path decomposition  $\psi$  of a graph  $G$ , let  $t_\psi = \sum_{P \in \psi} t(P)$ , where  $t(P)$  denotes the number of internal vertices of  $P$  and let  $t = \max t_\psi$  where the maximum is taken over all induced path decomposition  $\psi$  of  $G$ . Then  $\pi_i(G) = m - t$ .

**Proof.** Let  $\psi$  be an induced path decomposition of  $G$ . Then

$$\begin{aligned} m &= \sum_{P \in \psi} |E(P)| \\ &= \sum_{P \in \psi} (t(P) + 1) \\ &= \sum_{P \in \psi} t(P) + |\psi| \\ &= t_\psi + |\psi| \end{aligned}$$

Hence  $|\psi| = m - t_\psi$  so that  $\pi = m - t$ .

**Corollary 2.4.** Let  $G$  be a graph with  $k$  vertices of odd degree. Then  $\pi_i(G) = \frac{k}{2} + \sum \lfloor \frac{\deg v}{2} \rfloor - t$ .

**Proof.** Since  $m = \frac{k}{2} + \sum_{v \in V} \lfloor \frac{\deg v}{2} \rfloor$ , the result follows from Theorem 2.3.

**Corollary 2.5.** For any graph  $G$ ,  $\pi_i(G) \geq \frac{k}{2}$ . Further, equality holds if and only if there exists an induced path decomposition  $\psi$  of  $G$  such that every vertex  $v$  of  $G$  is an internal vertex of  $\lfloor \frac{\deg v}{2} \rfloor$  paths in  $\psi$ .

In the following theorems we determine the value of  $\pi_i$  for some families of graphs such as complete bipartite graphs, trees, unicyclic graphs and wheels.

**Theorem 2.6.** Let  $r$  and  $s$  be positive integers with  $r \leq s$ . Then, for the complete bipartite graph  $K_{r,s}$ , we have

$$\pi_i(K_{r,s}) = \begin{cases} \lfloor \frac{r}{2} \rfloor \lfloor \frac{s}{2} \rfloor & \text{if } rs \text{ is even} \\ \frac{(r+1)(s+1)}{2} & \text{if } rs \text{ is odd} \end{cases}$$

**Proof.** Let  $X = \{x_1, x_2, \dots, x_r\}$  and  $Y = \{y_1, y_2, \dots, y_s\}$  be the bipartition of  $K_{r,s}$ . We observe that every member of an induced path decomposition of  $K_{r,s}$  is a cycle  $C_4$  on four vertices or a path  $P_3$  of length 2 or an edge. Now, let

$$P_{ij} = (x_{2i-1}, y_{2j-1}, x_{2i}, y_{2j}, x_{2i-1}), i = 1, 2, \dots, \left\lfloor \frac{r}{2} \right\rfloor, \\ j = 1, 2, \dots, \left\lfloor \frac{s}{2} \right\rfloor$$

**Case 1.**  $r$  and  $s$  are even.

Then

$$\psi = \bigcup_{i=1}^{\frac{r}{2}} \left( \bigcup_{j=1}^{\frac{s}{2}} P_{ij} \right)$$

is an induced path decomposition of  $K_{r,s}$  so that  $\pi_i(K_{r,s}) \leq \frac{r}{2} \cdot \frac{s}{2}$ . Further, for any induced path decomposition  $\psi$  of  $K_{r,s}$ , every member of  $\psi$  covers at most four edges of  $K_{r,s}$  so that  $|\psi| \geq \frac{rs}{4}$  and hence  $\pi_i(K_{r,s}) \geq \frac{rs}{4}$ . Thus  $\pi_i(K_{r,s}) = \frac{rs}{4}$ .

**Case 2.**  $r$  is even and  $s$  is odd.

Let  $Q_i = (x_{2i-1}, y_s, x_{2i}), i = 1, 2, \dots, \frac{r}{2}$ . Then

$$\psi = \left\{ \bigcup_{i=1}^{\frac{r}{2}} \left( \bigcup_{j=1}^{\lfloor \frac{s}{2} \rfloor} P_{ij} \right) \right\} \cup \{Q_1, Q_2, \dots, Q_{\frac{r}{2}}\}$$

is an induced path decomposition of  $K_{r,s}$  so that  $\pi_i(K_{r,s}) \leq |\psi| = \frac{r}{2} \cdot \lfloor \frac{s}{2} \rfloor$ . Further, any induced path decomposition of  $K_{r,s}$  can have at most  $\frac{r}{2} \cdot \lfloor \frac{s}{2} \rfloor$  cycles of length 4 and  $\frac{r}{2}$  paths of length 2 and hence we have  $|\psi| \geq \frac{r}{2} \lfloor \frac{s}{2} \rfloor + \frac{r}{2}$  so that  $\pi_i(K_{r,s}) \geq \frac{r}{2} \lfloor \frac{s}{2} \rfloor$ . Thus  $\pi_i(K_{r,s}) = \frac{r}{2} \lfloor \frac{s}{2} \rfloor$ .

Similarly, we can prove that  $\pi_i(K_{r,s}) = \lfloor \frac{r}{2} \rfloor \frac{s}{2}$  if  $r$  is odd and  $s$  is even.

**Case 3.**  $r$  and  $s$  are odd. Let  $Q_i = (y_{2i-1}, x_r, y_{2i}), i = 1, 2, \dots, \frac{s-1}{2}$

$$R_i = (x_{2i-1}, y_s, x_{2i}), i = 1, 2, \dots, \frac{r-1}{2}$$

Then

$$\psi = \left\{ \bigcup_{i=1}^{\frac{r-1}{2}} \left( \bigcup_{j=1}^{\frac{s-1}{2}} P_{ij} \right) \right\} \cup \{Q_1, Q_2, \dots, Q_{\frac{r-1}{2}}\} \\ \cup \{R_1, R_2, \dots, R_{\frac{r-1}{2}}\} \cup \{(x_r, y_r)\}$$

is an induced path decomposition of  $K_{r,s}$  and hence  $\pi_i(K_{r,s}) \leq \frac{(r+1)(s+1)}{4}$ . Further, any induced path decomposition  $\psi$  of  $K_{r,s}$  can have at most  $\lfloor \frac{r}{2} \rfloor \lfloor \frac{s}{2} \rfloor$  cycles of length four and hence  $|\psi| \geq \lfloor \frac{r}{2} \rfloor \lfloor \frac{s}{2} \rfloor + \lceil \frac{r+s-1}{2} \rceil = \frac{(r+1)(s+1)}{4}$ . Thus  $\pi_i(K_{r,s}) = \frac{(r+1)(s+1)}{4}$ .

**Theorem 2.7.** For the wheel  $W_n$  on  $n$  vertices, we have

$$\pi_i(W_n) = \begin{cases} 4 & \text{if } n = 4 \\ \lceil \frac{n-1}{2} \rceil + 1 & \text{if } n \geq 5 \end{cases}$$

**Proof.** Let  $V(W_n) = \{v_0, v_1, \dots, v_{n-1}\}$  and  $E(W_n) = \{v_0 v_i : 1 \leq i \leq n-1\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-2\} \cup \{v_{n-1} v_1\}$ .

If  $n = 4$ , then  $\psi = \{(v_0, v_1, v_2, v_0), (v_0, v_3), (v_1, v_3), (v_2, v_3)\}$  is an induced path decomposition of  $W_4$  and since a member of any induced path decomposition of  $W_4$  is either an edge or a triangle, it follows that  $\pi_i(W_4) = 4$ .

Suppose  $n \geq 5$ . Then for  $i = 1, 2, \dots, \lceil \frac{n-1}{2} \rceil - 1$ , let

$$P_i = \begin{cases} (v_i, v_0, v_{\frac{n-1}{2}+i}) & \text{if } n \text{ is odd} \\ (v_i, v_0, v_{\frac{n-2}{2}+i}) & \text{if } n \text{ is even.} \end{cases}$$

$$P_{\lceil \frac{n-1}{2} \rceil} = \begin{cases} (v_{\frac{n-1}{2}}, v_0, v_{n-1}) & \text{if } n \text{ is odd} \\ (v_0, v_{n-1}) & \text{if } n \text{ is even.} \end{cases}$$

Then  $\psi = \{P_1, P_2, \dots, P_{\lceil \frac{n-1}{2} \rceil}, (v_1, v_2, \dots, v_{n-1}, v_1)\}$  is an induced path decomposition of  $W_n$  so that  $\pi_i(W_n) \leq |\psi| = \lceil \frac{n-1}{2} \rceil + 1$ . Further, since the minimum number of paths required to cover the set of edges  $\{v_0 v_i : 1 \leq i \leq n-1\}$  is  $\lceil \frac{n-1}{2} \rceil$ , it follows that for any induced path decomposition  $\psi$  of  $W_n$ , we have  $|\psi| \geq \lceil \frac{n-1}{2} \rceil + 1$  and hence  $\pi_i(W_n) = \lceil \frac{n-1}{2} \rceil + 1$ .

**Remark.** Since every path in a tree  $T$  is induced, every acyclic path decomposition of  $T$  is an induced path decomposition and hence  $\pi_i(T) = \pi_a(T)$ . Also it has been proved in [12] that  $\pi_a(T) = \frac{k}{2}$ , where  $k$  is the number of vertices of odd degree so that  $\pi_i(T) = \frac{k}{2}$ .

**Theorem 2.8.** Let  $G$  be a unicyclic graph with cycle  $C$ . Let  $r$  denote the number of vertices of degree greater than two on  $C$ . Let  $k$  be the number of vertices of odd degree. Then

$$\pi_i(G) = \begin{cases} 1 & \text{if } r = 0 \\ \frac{k}{2} + 1 & \text{if } r = 1 \text{ or } r = 2 \text{ and the vertices} \\ & \text{of degree } > 2 \text{ on } C \text{ are adjacent} \\ \frac{k}{2} & \text{otherwise.} \end{cases}$$

**Proof.** Let  $C = (v_1, v_2, \dots, v_n, v_1)$ .

**Case 1.**  $r = 0$ .

Then  $G = C$  so that  $\pi_i(G) = 1$ .

**Case 2.**  $r = 1$ .

Let  $v_1$  be the unique vertex of degree greater than 2 on  $C$ . Let  $T$  be the subtree rooted at  $v_1$ . Then  $\pi_i(T) = \frac{k}{2}$ . Let  $\psi_1$  be a minimum induced path decomposition of  $T$ . Then  $\psi = \psi_1 \cup \{C\}$  is an induced path decomposition of  $G$  so that  $\pi_i(G) \leq |\psi| = |\psi_1| + 1 = \frac{k}{2} + 1$ . Further, for any induced path decomposition  $\psi$  of  $G$ , there exists a vertex  $v_i$  on  $C$  such that  $v_i$  is an internal vertex in at most  $\lfloor \frac{\deg v_i}{2} \rfloor - 1$  paths and hence  $\pi_i(G) \geq \frac{k}{2} + 1$ . Thus  $\pi_i(G) = \frac{k}{2} + 1$ .

**Case 3.**  $r = 2$ .

Let  $v_1$  and  $v_i$  be the vertices of degree greater than 2 on  $C$ . Let  $T_1$  and  $T_2$  respectively denote the subtrees rooted at  $v_1$  and  $v_i$ . Let  $\psi_1$  and  $\psi_2$  be minimum induced path decompositions of  $T_1$  and  $T_2$  respectively.

**Subcase 3.1.** The vertices  $v_1$  and  $v_i$  are adjacent.

Then  $\psi = \psi_1 \cup \psi_2 \cup \{C\}$  is an induced path decomposition of  $G$  so that  $\pi_i(G) \leq |\psi| = |\psi_1| + |\psi_2| + 1 = \frac{k}{2} + 1$ . Further, for any induced path decomposition  $\psi$  of  $G$ , there exists a vertex  $v_j$  on  $C$  such that  $v_j$  is an internal vertex in at most  $\lfloor \frac{\deg v_j}{2} \rfloor - 1$  paths in  $\psi$  and hence  $\pi_i(G) \geq \frac{k}{2} + 1$ . Thus  $\pi_i(G) = \frac{k}{2} + 1$ .

**Subcase 3.2.** The vertices  $v_1$  and  $v_i$  are not adjacent.

Suppose  $\deg v_1$  and  $\deg v_i$  are odd. Let  $P_1$  and  $P_2$  denote respectively the paths in  $\psi_1$  and  $\psi_2$  having  $v_1$  and  $v_i$  as its

terminal vertices. Let

$$\begin{aligned} Q_1 &= P_1 \circ (v_1, v_2, \dots, v_i) \circ P_2^{-1} \text{ and} \\ Q_2 &= (v_i, v_{i+1}, \dots, v_n, v_i). \end{aligned}$$

Since  $v_1$  and  $v_i$  are not adjacent, the paths  $Q_1$  and  $Q_2$  are induced. Hence  $\psi = (\psi_1 - \{P_1\}) \cup (\psi_2 - \{P_2\}) \cup \{Q_1, Q_2\}$  is an induced path decomposition of  $G$  so that  $\pi_i(G) \leq |\psi| = |\psi_1| + |\psi_2| = \frac{k}{2}$ .

Suppose  $\deg v_1$  and  $\deg v_i$  are even. Let  $P_1$  be an  $u_1 - w_1$  path in  $\psi_1$  having  $v_1$  as an internal vertex. Let  $P_2$  be an  $u_2 - w_2$  path in  $\psi_2$  having  $v_i$  as an internal vertex. Let  $R_1$  and  $R_2$  denote the  $(u_1, v_1)$ -section and  $(w_1, v_1)$ -section of  $P_1$  respectively. Let  $R'_1$  and  $R'_2$  be the  $(v_i, u_2)$ -section and  $(v_i, w_2)$ -section of  $P_2$  respectively. Now, let

$$\begin{aligned} Q_1 &= R_1 \circ (v_1, v_2, \dots, v_i) \circ R'_1 \text{ and} \\ Q_2 &= R_2 \circ (v_1, v_n, v_{n-1}, \dots, v_i) \circ R'_2. \end{aligned}$$

Since  $v_1$  and  $v_i$  are not adjacent, the paths  $Q_1$  and  $Q_2$  are induced. Hence  $\psi = (\psi_1 - \{P_1\}) \cup (\psi_2 - \{P_2\}) \cup \{Q_1, Q_2\}$  is an induced path decomposition of  $G$  so that  $\pi_i(G) \leq |\psi| = \frac{k}{2}$ .

Suppose  $\deg v_1$  is odd and  $\deg v_i$  is even. Let  $P_1$  be the path in  $\psi_1$  having  $v_1$  as a terminal vertex and let  $P_2 = (u_1, u_2, \dots, u_r, v_i, u_{r+1}, \dots, u_s)$  be an  $u_1 - u_s$  path having  $v_i$  as an internal vertex. Now, let

$$\begin{aligned} Q_1 &= P_1 \circ (v_1, v_2, \dots, v_i, u_r, u_{r-1}, \dots, u_1) \text{ and} \\ Q_2 &= (v_1, v_n, \dots, v_i, u_{r+1}, u_{r+2}, \dots, u_s). \end{aligned}$$

Then  $\psi = (\psi_1 - \{P_1\}) \cup (\psi_2 - \{P_2\}) \cup \{Q_1, Q_2\}$  is an induced path decomposition of  $G$  so that  $\pi_i(G) \leq |\psi| = \frac{k}{2}$ .

Further,  $\pi_i(G) \geq \frac{k}{2}$  and hence  $\pi_i(G) = \frac{k}{2}$ .

**Case 4.**  $r > 2$ .

Let  $v_{i_1}, v_{i_2}, \dots, v_{i_r}$ , where  $1 \leq i_1 \leq i_2 \leq \dots \leq i_r$ , be the vertices of degree greater than 2 on  $C$ . Let  $T_{i_j}$ ,  $1 \leq j \leq r$ , be the subtree rooted at the vertex  $v_{i_j}$  and let  $\psi_{i_j}$  be a minimum induced path decomposition of  $T_{i_j}$ . Consider the vertices  $v_{i_1}, v_{i_2}$  and  $v_{i_3}$ . Let  $C_1, C_2$  and  $C_3$  denote the  $(v_{i_1}, v_{i_2})$ -section,  $(v_{i_2}, v_{i_3})$ -section and  $(v_{i_3}, v_{i_1})$ -section of  $C$  respectively. For  $j = 1, 2, 3$ , let  $P_j$  be either an  $u_j - w_j$  path in  $\psi_{i_j}$  having  $v_{i_j}$  as an internal vertex or a path in  $\psi_{i_j}$  having  $v_{i_j}$  as a terminal vertex. Then  $\psi = \left( \left( \bigcup_{j=1}^r \psi_{i_j} \right) - \{P_1, P_2, P_3\} \right) \cup \{Q_1, Q_2, Q_3\}$  is an induced path decomposition of  $G$  such that every vertex  $v$  of  $G$  is an internal vertex of  $\lfloor \frac{\deg v}{2} \rfloor$  paths in  $\psi$  and hence  $\pi_i(G) = \frac{k}{2}$ .

We now proceed to obtain some bounds for  $\pi_i$  and characterize graphs attaining the bounds.

**Remark.** For any graph  $G$ ,  $\pi_i(G) \leq m$ . Further equality holds if and only if  $G = K_2$ , for if  $G \neq K_2$ , then  $G$  contains an induced path of length greater than one or a triangle and hence  $\pi_i(G) < m$ .

**Theorem 2.9.** For any graph  $G$  with girth  $g$ , we have  $\pi_i(G) \leq m - g + 1$ . Further equality holds if and only if

$G$  is either a cycle or  $K_4$  or  $K_4 - e$  or one of the graphs  $G_1$  and  $G_2$  which are described as follows.

- (i)  $G_1$  is the graph obtained from a cycle by attaching exactly one pendant edge at a vertex of the cycle.
- (ii)  $G_2$  is the graph obtained from a cycle by attaching exactly one pendant edge at two adjacent vertices of the cycle.

**Proof.** Let  $C$  be a cycle of length  $g$  in  $G$ . Then  $C$  is induced so that  $\psi = \{C\} \cup (E(G) - E(C))$  is an induced path decomposition of  $G$  and hence  $\pi_i(G) \leq |\psi| = m - g + 1$ .

Now, suppose  $G$  is a graph with  $\pi_i(G) = m - g + 1$ . Let  $C = (v_1, v_2, \dots, v_g, v_1)$  be a cycle of length  $g$  in  $G$ . If  $G$  has an induced path  $P$  with length greater than one and  $|V(P) \cap V(C)| = 1$ , then  $\{C, P\} \cup S$ , where  $S$  is the set of edges of  $G$  not covered by  $C$  and  $P$  is an induced path decomposition of  $G$  with  $|\psi| < m - g + 1$ , which is a contradiction. Hence every vertex not on  $C$  is adjacent to a vertex on  $C$ , no two vertices not on  $C$  are adjacent and every vertex on  $C$  has degree at most 3.

**Claim 1.** Any two vertices of degree 3 on  $C$  are adjacent.

Let  $v_{i_1}$  and  $v_{i_2}$ , where  $i_1 < i_2$ , be two vertices of degree 3 on  $C$  and let  $x$  and  $y$  be the vertices (not on  $C$ ) adjacent to  $v_{i_1}$  and  $v_{i_2}$  respectively. Suppose  $v_{i_1}$  and  $v_{i_2}$  are not adjacent. Then  $g \geq 4$ . Consider an  $(v_{i_1}, v_{i_2})$ -section of  $C$ , say  $C_1$ . Suppose either  $x$  or  $y$ , say  $x$  is adjacent to a vertex of  $C_1$ . Let  $i_3$  be the least positive integer with  $i_1 < i_2 < i_3$  such that  $x$  is adjacent to  $v_{i_3}$ . Then  $v_{i_1}$  and  $v_{i_3}$  are not adjacent because  $g \geq 4$ . Now, let  $P_1$  be the  $(v_{i_1}, v_{i_3})$ -section of  $C$  containing  $v_{i_2}$  and let  $C_1$  be the cycle consisting of the  $(v_{i_1}, v_{i_3})$ -section of  $C$  not containing  $v_{i_2}$  followed by the path  $(v_{i_3}, x, v_{i_1})$ . Then  $\psi = \{P_1, C_1\} \cup [E(G) - (E(P_1) \cup E(C_1))]$  is an induced path decomposition of  $G$  with  $|\psi| < m - g + 1$ , which is a contradiction. Thus neither  $x$  nor  $y$  is adjacent to any vertex of  $C_1$  and hence  $\psi_1 = \{P = C - C_1, P' = (x, v_{i_1}) \cup C_1 \cup (v_{i_2}, y)\} \cup [E(G) - (E(P) \cup E(P'))]$  is an induced path decomposition of  $G$  with  $|\psi| < m - g + 1$ , which is again a contradiction. Thus any two vertices of degree 3 on  $C$  are adjacent.

Now, one can observe that the following are immediate consequences of the above claim.

- (i) Every vertex not on  $C$  is of degree at most 3 so that  $\Delta(G) \leq 3$ .
- (ii) There exist at most two vertices not on  $C$ .
- (iii) If there are two vertices not on  $C$ , then they are pendant.

Now, if there is no vertex outside  $C$ , then  $G$  is a cycle. If there are exactly two vertices not on  $C$ , then it follows from Claim 1 and the above observation (iii) that  $G$  is isomorphic to  $G_2$ . Now, suppose there is exactly one vertex not on  $C$ , say  $v$ . If  $\deg v = 3$ , then the neighbours of  $v$  lie on  $C$  so that it follows from claim 1 that they are adjacent and consequently  $G$  is isomorphic to  $K_4$ . Similarly, if  $\deg v = 2$ , then the neighbours of  $v$  lie on  $C$  and they are adjacent so that  $g = 3$  and hence  $G$  is isomorphic to  $K_4 - e$ . If  $\deg v = 1$ , then  $G$  is isomorphic to  $G_1$ .

The converse is just a simple verification.

Obviously one can observe that  $\pi_i(G) \geq \lceil \frac{\Delta}{2} \rceil$ . Further, we observe that a tree attains this bound if and only if it has at

most one vertex with  $\deg v \geq 3$ . Apart from trees, an infinite family of unicyclic graphs too attain this bound which we now characterize in the following theorem.

**Theorem 2.10.** Let  $G$  be a unicyclic graph with cycle  $C$ . Let  $r$  denote the number of vertices on  $C$  with degree greater than 2. Then  $\pi_i(G) = \lceil \frac{\Delta}{2} \rceil$  if and only if the following are satisfied.

- (i)  $r \leq 2$
- (ii) Every vertex not on  $C$  has degree 1 or 2.
- (iii) If  $r = 2$ , then the two vertices on  $C$  with degree greater than 2 are not adjacent and one of these vertices is of degree either 3 or 4.

**Proof.** Let  $u$  be a vertex with  $\deg u = \Delta$ . Let  $\psi$  be a minimum induced path decomposition of  $G$ . Then every member of  $\psi$  passes through  $u$ . If  $u$  does not lie on  $C$ , then  $\psi$  will not cover at least one edge of  $C$  and hence  $u$  lies on  $C$ . Since  $\psi$  is a collection of edge-disjoint induced paths covering all the edges of  $G$ , it follows that every vertex not on  $C$  is of degree either 1 or 2 and  $r \leq 2$ .

Suppose  $r = 2$ . Let  $x$  and  $y$  be the vertices of degree greater than 2 on  $C$ . Suppose  $x$  and  $y$  are adjacent. Now, it is clear that not both  $x$  and  $y$  are of degree  $\Delta$ . Assume without loss of generality that  $\deg x = \Delta$ . Let  $P$  be the path containing the edge not on  $C$  which is incident at  $y$ . Then  $P$  contains the edge  $xy$  so that there exist two paths  $P_1$  and  $P_2$  in  $\psi$  which cover the edges of the  $(x, y)$ -section of  $C$  of length at least 2. Hence one of these paths does not pass through the vertex  $x$ , which is a contradiction. Thus  $x$  and  $y$  are not adjacent. Further, since every member of  $\psi$  passes through the vertices of maximum degree it follows that either  $x$  or  $y$  has degree  $\leq 4$ .

Conversely, suppose conditions (i)–(iii) of the theorem hold. If  $r = 0$ , obviously  $\pi_i(G) = \frac{\Delta}{2}$ . Suppose  $r = 1$ . If  $\Delta$  is even, then  $G$  has  $\Delta - 2$  vertices of odd degree and if  $\Delta$  is odd, then  $G$  has  $\Delta - 1$  vertices of odd degree, so that in either of the cases it follows from Theorem 2.9 that  $\pi_i(G) = \lceil \frac{\Delta}{2} \rceil$ . Now, suppose  $r = 2$ . Let  $x$  and  $y$  be the vertices of degree greater than 2 on  $C$ , with  $\deg x = \Delta$  and  $\deg y = 3$  or 4. Then  $G$  has  $\Delta$  or  $\Delta + 1$  vertices of odd degree according as  $\Delta$  is even or odd and since  $x$  and  $y$  are not adjacent it follows from Theorem 2.9 that  $\pi_i(G) = \lceil \frac{\Delta}{2} \rceil$ .

**Theorem 2.11.** If  $G$  is a regular graph, then  $\pi_i(G) = \lceil \frac{\Delta}{2} \rceil$  if and only if  $G = K_2$  or  $G$  is a cycle.

**Proof.** Suppose  $G$  is regular with  $\pi_i(G) = \lceil \frac{\Delta}{2} \rceil$ . Then every member of any minimum induced path decomposition  $\psi$  of  $G$  passes through all the vertices of  $G$ . Hence a minimum induced path decomposition of  $G$  consist of hamiltonian paths and hamiltonian cycles and since  $\psi$  is induced it follows that  $|\psi| = 1$  and consequently  $G = K_2$  or  $G$  is a cycle. The converse is obvious.

So far we have determined the value of  $\pi_i$  for several families of graphs and have obtained bounds for  $\pi_i$  with characterization of graphs attaining the bounds. Now, it is of some interest discussing the relation of  $\pi_i$  with some existing path covering parameters such as path decomposition number  $\pi$ , simple acyclic path decomposition number  $\pi_{as}$  and simple

graphoidal covering number  $\eta_s$ .

It follows immediately from definitions that  $\pi \leq \pi_i \leq \pi_{as}$ . Of course, the difference between  $\pi_i$  and  $\pi$  can be made arbitrarily large. For the graph  $G$  obtained from a path  $(v_1, v_2, \dots, v_{n+3})$  by introducing  $n+2$  new vertices, namely,  $w_1, w_2, \dots, w_{n+2}$ , and joining  $w_i, 1 \leq i \leq n+2$ , to both  $v_i$  and  $v_{i+1}$ , we have  $\pi(G) = 2$  and  $\pi_i(G) = n+2$  and hence  $\pi_i - \pi = n$ . Further, if  $G'$  is the graph obtained from  $G$  by subdividing the edges  $v_i v_{i+1}, 1 \leq i \leq n+2$ , by a vertex, then  $\pi_i(G') = 2$  and  $\pi_{as}(G') = 2n+4$  and hence the difference between parameters  $\pi_{as}$  and  $\pi_i$  also can be made as large as possible.

Also, obviously for a cycle with at least four vertices  $\pi = 1, \pi_i = 2$  and  $\pi_{as} = 3$  and for any tree  $\pi = \pi_i = \pi_{as} = \frac{k}{2}$ , where  $k$  is the number of vertices of odd degree. Further, not only for a tree but also for the unicyclic graphs with at least three vertices of degree greater than two on the unique cycle, the three parameters coincide. Thus these parameters may coincide or may assume distinct values and so the following problem naturally arises.

**Problem.** Characterize graphs for which  $\pi = \pi_i = \pi_{as}$ .

As a direct application of the definitions, we have  $\pi_i \leq \eta_s$  and one can observe that these parameters coincide for the graphs for which  $\Delta \leq 3$ . Of course, as one would expect, graphs assuming same value for these parameters may have vertices of higher degree (consider the graph with exactly one cut-vertex and each of whose blocks is a cycle), and however trees are not of this kind. For, if  $T$  is a tree having a vertex  $v$  with degree more than three, then every minimum simple graphoidal cover  $\psi$  contains two paths  $P_1$  and  $P_2$  having  $v$  as a terminal vertex and hence  $(\psi - \{P_1, P_2\}) \cup \{P_1 \circ P_2^{-1}\}$  is an induced path decomposition of  $T$  with cardinality  $|\psi| - 1$  so that  $\pi_i(T) \leq \eta_s(T) - 1$ . Thus, we have

**Theorem 2.12** If  $T$  is tree, then  $\eta_s(T) = \pi_i(T)$  if and only if  $\Delta \leq 3$ .

#### CONCLUSION AND SCOPE

A decomposition of a graph  $G$  is a collection of edge-disjoint subgraphs of  $G$  whose union is  $G$ . Various types of decomposition and corresponding parameters have been studied by imposing certain condition on the members of the decomposition. The key condition that we impose here is “inducedness” and arrived at the concept of induced path decomposition and the induced path decomposition number  $\pi_i(G)$ . Here, we first determined  $\pi_i(G)$  for several families of graphs and obtained some bounds for  $\pi_i$  together with the characterization of graphs attaining these bounds and finally discuss the relation of  $\pi_i$  with some well-known related parameters.

Even if this paper is just an initiation of the concept of induced path decomposition, numerous problems can be identified for further investigation and here are some interesting problems.

- (a) Characterize graphs for which  $\pi_i = \frac{k}{2}$ , where  $k$  is the number of vertices of odd degree.
- (b) Characterize graphs for which  $\pi_i = \lceil \frac{\Delta}{2} \rceil$ .

- (c) Characterize graphs for which (i)  $\pi_i = \pi$ , (ii)  $\pi_i = \pi_{as}$  and (iii)  $\pi_i = \eta_s$ .

Also, as we did, one can impose the condition “inducedness” on any kind of path decomposition and can arrive at a number of new path covering parameters.

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