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Curvature of Almost Split Quaternion Kaehler Manifolds

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***Abstract*—In this work some characterizations of semi Riemannian curvature tensor on almost split quaternion Kaehler manifolds and some characterizations of Ricci tensor on almost split quaternion Kaehler manifolds are given.**

***Keywords*—Almost split quaternion Kaehler manifold, Riemann curvature, Ricci curvature.**

I. Introduction

QUATERNION Kaehler Manifolds are frequent studied a subject. It is important study some characterizations of Riemann curvature and Ricci curvature on quaternion Kaehler manifolds. Split quaternions are a new developing topic. Inoguchi, J. studied on this topic. Now in this article we obtain these characterizations of Riemann curvature and Ricci curvature on split quaternion Kaehler manifolds.

II. Preliminaries

Definition 1. Let M be a semi Riemannian manifold and $\chi(M)$ be vector field's space on M with levi-civita connection D. The function

$$R:\chi(M)^3 \to \chi(M)$$

given by

$$R_{XY}Z = D_{[X,Y]}Z - [D_X, D_Y]Z$$

is a (1.3) tensor field on M called the semi Riemannian curvature tensor of M , where X and Y arbitrary elements of $\chi(M)$ [2]. The alternative notation $R(X,Y)Z$ for $R_{XY}Z$ is convenient when X and Y are replaced by more complicated expressions.

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Definition 2. Let R be the Riemannian curvature of a semi Riemannian manifold M . The Ricci curvature tensor Ric of M is the contraction $C_3^1(R) \in I_2^0(M)$, whose components relative to a coordinate system are $R_{ij} = \sum R_{ijm}^m$ [2].

Lemma 3. The Ricci curvature tensor Ric is symmetric and is given relatives to a frame field by

$$Ric(X,Y) = \sum_m \varepsilon_m \left\langle R_{Xe_m}Y, e_m \right\rangle$$

where as usual $\varepsilon_m = \langle e_m, e_m \rangle$ [2].

Definition 4. The algebra H of quaternions is defined as the 4-dimensional vector space over R having a basis $\{1, i, j, k\}$ with the following properties

$$i^2 = j^2 = k^2 = -1$$
$$ij = -ji = k, ki = -ik = j, jk = -kj = i. \qquad (1)$$

From (1) it is clear that H is not commutative and 1 is the identity element of H . It also H is an associative algebra. For $q = a.1 + b.i + c.j + d.k \in H$ $(a,b,c,d \in R)$, we define the conjugate $\overline{q}$ of q as $\overline{q} = a.1 - b.i - c.j - d.k.$ For every $q = a.1 + b.i + c.j + d.k \in H$ we have $q.\overline{q} = (a^2 + b^2 + c^2 + d^2)$. We define the norm N_q of the quaternion q to be the nonnegative real number $a^2 + b^2 + c^2 + d^2$. $N_q = 0$ if and only if $q = 0$ [4].

Definition 5. We may construct a 1:1 correspondence between $H^n = H \times H \times ... \times H$ (n-times product) and R^{4n} by setting $(x_1, x_2, ..., x_{4n}) \in R^{4n}$ and $(q_1, q_2, ..., q_n) \in H^n$ are the corresponding elements iff $q_\alpha = x_\alpha + x_{n+\alpha}i + x_{2n+\alpha}j + x_{3n+\alpha}k (1 \le \alpha \le n)$, so a vector in R^{4n} or H^n may be considered as a vector in the other. By this correspondence we define the linear mappings F_0,G_0 and

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H_0 of R^{4n} onto itself as multiplication by i,j and k respectively from right in H^n. We have

$$\begin{aligned} &F_0^{\,2} = G_0^{\,2} = H_0^{\,2} = -I \\ &F_0.G_0 = -G_0.F_0 = H_0, \\ &H_0.F_0 = -F_0.H_0 = G_0, \\ &G_0.H_0 = -H_0G_0 = F_0 \end{aligned} \tag{2}$$

and call F_0, G_0, H_0 standard quaternion structure in R^{4n}. R^{4n} can be made into a quaternionic vector space by defining the multiplication of vectors of R^{4n} by a $q \in H$ as follows. For $x \in R^{4n}$ and $q = a.1 + b.i + c.j + d.k \in H$ define xq as follow [5].

$$\begin{aligned} &x(a.1 + b.i + c.j + d.k) = \\ &ax + bF_0x + cG_0x + dH_0x. \end{aligned}$$

Definition 6. An almost quaternion manifold (M,V) is defined to be a Riemannian manifold M of dimension n, together with a 3-dimensional bundle V of tensors of type (1,1) over M, there is a local base $\{F,G,H\}$ of V such that

$$\begin{aligned} &F^2 = G^2 = H^2 = -I \\ &FG = -GF = H, HF \\ &\quad = -FH = G, GH = -HG = F \end{aligned} \tag{3}$$

where I is the identity tensor field of type (1,1) in M. We call $\{F,G,H\}$ a canonical local base of the bundle V in U and V an almost quaternion structure on M. If there is a global base $\{F,G,H\}$ of the bundle V which satisfies (3), then $\{F,G,H\}$ is called a canonical global base of V [3].

Proposition 7. An almost quaternion manifold is of dimension $n = 4m$ [3].

Proposition 8. In any almost quaternion manifold (M,V) there is a Riemannian metric g such that $g(\phi X, Y) + g(X, \phi Y) = 0$ holds for any cross section ϕ of V where $X, Y \in \chi(M)$ [3].

Definition 9. Let (M,V) be an almost quaternion manifold and g be the Riemannian metric as defined in proposition 3. The pair (g,V) is called an almost quaternion metric structure and (M,g,V) an almost quaternion metric manifold [3].

Definition 10. If for every (local or global) cross-section ϕ of the bundle $V, D_X\phi$ is also a (respectively local or global) cross-section of V, where D is the Riemannian connection of (M,g,V) and X an arbitrary vector field in M, then (M,g,V) (or M) is called a quaternion Kaehlerian manifold and (g,V) a quaternion Kaehlerian structure [3].

Theorem 11. For an almost quaternion metric manifold (M,g,V) the following are equivalent :

i) (M,g,V) is a quaternion Kaehler manifold

ii) $\{F,G,H\}$ is a canonical local base of V in a coordinate neighbourhood U of M, then

$$\begin{aligned} &D_X F = r(X)G - q(X)H \\ &D_X G = -r(X)F + p(X)H \\ &D_X H = q(X)F - p(X)G \end{aligned} \tag{4}$$

for any $X \in \chi(M)$, where p, q and r are certain local 1-forms defined in U [3].

Lemma 13. Let (M,V) be an almost quaternion manifold with $\dim M = 4m (m \geq 1)$. If $\{F,G,H\}$ is a canonical local base of V in a coordinate neighbourhood U of M, then there exists elements $X_1, X_2, ..., X_m$ of T_xM for $x \in U$ such that

$$\begin{aligned} &\{X_1, X_2, ..., X_m, FX_1, FX_2, ..., FX_m, \\ &GX_1, GX_2, ..., GX_m, HX_1, HX_2, ..., HX_m\} \end{aligned}$$

is a base of T_xM [5].

Definition 13. Let $\{F,G,H\}$ be a canonical local base of the bundle V of an almost quaternion metric manifold (M,g,V) in a coordinate neighbourhood U. Each of F, G and H is almost Hermitian with respect to g. We define the local 2-forms Φ, Ψ and θ in U as follows.

$$\begin{aligned} &\Phi(X,Y) = g(FX,Y), \Psi(X,Y) = g(GX,Y), \\ &\theta(X,Y) = g(HX,Y). \end{aligned}$$

Now we define the 4-form Ω as

$$\Omega=\Phi\wedge\Phi+\Psi\wedge\Psi+\theta\wedge\theta$$

and the tensor field $\wedge$ of type (2.2) by

$$\wedge=F\otimes F+G\otimes G+H\otimes H \text{ in } U \text{ [3].}$$

Theorem 14. An almost quaternion metric manifold is a quaternion Kaehlerian manifold if and only if $\nabla\Omega=0$ or $\nabla\wedge=0$ [3].

Definition 15. The algebra H' of split quaternions is defined as the 4-dimensional vector space over R having a basis $\{1,i,j,k\}$ with the following properties

$$\begin{aligned} &i^2=-1,\ j^2=k^2=1 \\ &ij=-ji=k, kj=-jk=i, ki=-ik=j \end{aligned} \qquad (5)$$

from (5) it is clear that H' is not commutative and 1 is the identity element of H'.It is easy to verify that H' is an associative algebra. For

$$q=a.1+b.i+c.j+d.k\in H'$$

$(a,b,c,d\in R)$, we define the conjugate $\overline{q}$ of q as

$$\overline{q}=a.1-b.i-c.j-d.k\in H'.$$

For every $q=a.1+b.i+c.j+d.k\in H'$ we have $q\overline{q}=a^2+b^2-c^2-d^2$. We define the norm N_q and the inverse q^{-1} of the quaternion respectively the real number $N_q=a^2+b^2-c^2-d^2$ and $q^{-1}=\dfrac{\overline{q}}{N_q}, N_q\neq 0$ [1]. Thus the space H' correspondence with semi Euclidean space

$$E_2^4=\{q=(a,b,c,d):a,b,c,d\in R, \\ g(q,q)=-a^2-b^2+c^2+d^2\}$$

[1].

III. On the Almost Split Quaternion Kaehlerian Manifolds Curvatures

Definition 16. An almost split quaternion Lorenz manifold (M,V) is defined to be a semi Riemannian manifold M of dimension n, together with a 3-dimensional bundle V of tensors of type (1,1) over M satisfying the following condition. In any coordinate neighbourhood U of M, there is local base $\{F,G,H\}$ of V such that

$$\begin{aligned} &F^2=-I, G^2=H^2=I \\ &FG=-GF=H, \\ &HF=-FH=G, HG=-GH=F \end{aligned} \qquad (6)$$

where I is the identity tensor field of type (1,1) in M. We call $\{F,G,H\}$ a canonical local base of the bundle V in U and V an almost split quaternion structure on M. If there is a global base $\{F,G,H\}$ of the bundle V which satisfies (6) then $\{F,G,H\}$ is called a canonical global base of V.

Proposition 17. In any almost split quaternion manifold (M,V) there is a metric tensor g such that $g(\phi X,Y)+g(X,\phi Y)=0$ holds for any cross-section ϕ of V where $X,Y\in\chi(M)$.

Proof. Let g′ be a Lorentz metric on the manifold M. Define a tensor field g of degree 2 on M as follows

$$g(X,Y)=g'(X,Y)+g'(FX,FY) \\ +g'(GX,GY)+g'(HX,HY)$$

where $\{F,G,H\}$ is a canonical local basis of V in a coordinate neighbourhood U of M and X,Y arbitrary vector fields on M. g satisfies all the conditions for a Lorentz metric, because g′ is a Lorentz metric. We now prove that g satisfies

$$g(\phi X,Y)+g(X,\phi Y)=0$$

for any cross-section ϕ of V where $X,Y\in\chi(M)$.Since $\{F,G,H\}$ is a local base for V, any cross-section ϕ of V may be written uniquely as a linear combination of F,G and H, so H is sufficient to show e.g. $g(FX,Y)+g(X,FY)=0$.

$$g(FX,Y)+g(X,FY)=$$
$$g'(FX,Y)+g'(F^2X,FY)+g'(GFX,GY)$$
$$+g'(HFX,HY)+g'(X,FY)+g'(FX,F^2Y)$$
$$+g'(FX,F^2Y)+g'(GX,GFY)+g'(HX,HFY)$$
$$g(FX,Y)+g(X,FY)=g'(FX,Y)$$
$$-g'(X,FY)-g'(HX,GY)$$
$$+g'(GX,HY)+g'(X,FY)-g'(FX,Y)$$
$$-g'(GX,HY)+g'(HX,GY)$$
$$g(FX,Y)+g(X,FY)=0$$

Definition 18. Let (M,V) be an almost manifold and g be the metric tensor as defined in proposition 8. The pair (V,g) is called an almost split quaternion metric structure and (M,g,V) an almost split quaternion metric manifold.

Definition 19. If for every (local or global) cross-section ϕ of the bundle V, $D_X\phi$ is also a cross-section of V, where D is the semi-Riemannian connection of (M,g,V) and X an arbitrary vector field in M, then ((M,g,V) (or M) is called a almost split quaternion Kaehlerian structure.

Theorem 20. For an almost split quaternion metric manifold (M,g,V) the following statements are equivalent

i) (M,g,V) is a split quaternion Kaehler manifold

ii) $\{F,G,H\}$ is a canonical local base of V in a coordinate neighbourhood U of M, then

$$\begin{aligned} D_X F &= r(X)G + q(X)H \\ D_X G &= r(X)F + p(X)H \\ D_X H &= q(X)F - p(X)G \end{aligned} \tag{7}$$

for any $X \in \chi(M)$, where p,q and r are certain local 1-forms defined in U.

Proof. (i)⇒(ii) Let (M,g,V) be a split quaternion Kaehlerian manifold and $\{F,G,H\}$ be a canonical local base of V in a coordinate neighbourhood U of M, then

$$\begin{aligned} D_X F &= a_{11}F + a_{12}G + a_{13}H \\ D_X G &= a_{21}F + a_{22}G + a_{23}H \\ D_X H &= a_{31}F + a_{32}G + a_{33}H \end{aligned} \tag{8}$$

for some local 1-forms $a_{ij}\,(i,j=1,2,3)$ defined in U. But $D_X G$ can be written as

$$D_X(HF) = (D_X H)F + H(D_X F)$$

so

$$\begin{aligned} a_{12}F + a_{22}G + a_{23}H &= (D_X H)F + H(D_X F) \\ &= (a_{31} - a_{13}) + a_{12}F \\ &\quad + (a_{22} + 2a_{33})G - a_{32}H \end{aligned}$$

and we obtain $a_{31} = a_{13}, a_{12} = a_{21}, a_{33} = 0$ and $a_{23} = -a_{32}$. So the system (8) reduces to the form

$$\begin{aligned} D_X F &= a_{22}F + a_{12}G + a_{13}H \\ D_X G &= a_{12}F + a_{22}G + a_{23}H \\ D_X H &= a_{31}F - a_{23}G. \end{aligned}$$

Using

$$D_X H = D_X(FG) = (D_X F)G + F(D_X G)$$

we see that

$$\begin{aligned} a_{31}F - a_{23}G &= (D_X F)G + F(D_X G) \\ a_{31}F - a_{23}G &= a_{22} + a_{31}F - a_{23}G + a_{22}H \end{aligned}$$

we obtain $a_{22} = 0$ so

$$\begin{aligned} D_X F &= a_{12}G + a_{31}H \\ D_X G &= a_{12}F + a_{23}H \\ D_X H &= a_{31}F - a_{32}G \end{aligned}$$

and finally letting

$$p(X) = a_{23}, q(X) = a_{13}, r(X) = a_{12}$$

we reach the system (7).

(ii)⇒(i) Suppose (ii) holds and let $\phi = aF + bG + cH$ is a cross section of bundle V where a,b,c are locally defined functions on U, then

$$\begin{aligned} D_X\phi &= D_X(aF+bG+cH) \\ &= X(a)F + aD_X F + X(b)G + bD_X G \\ &\quad + X(c)H + cD_X H \\ &= \left(X(a)+br(X)+cq(X)\right)F \\ &\quad + \left(ar(X)+X(b)-cp(X)\right)G \\ &\quad + \left(aq(X)+bp(X)+X(c)\right)H \end{aligned}$$

which shows that $D_X\phi$ φ is also a cross section of V .

Lemma 21. Let (M,g,V) be an almost split quaternion Kaehler manifold. Let $\{F,G,H\}$ be a base of V and semi-Riemannian curvature tensor R. Then for arbitrarily $X,Y \in \chi(M)$

$$\begin{aligned} \left(R(X,Y)F\right) &= C(X,Y)G + B(X,Y)H \\ &= R(X,Y)F - FR(X,Y) \\ &= \left[R(X,Y),F\right] \end{aligned} \tag{9}$$

$$\begin{aligned} \left(R(X,Y)G\right) &= C(X,Y)F + A(X,Y)H \\ &= \left[R(X,Y),G\right] \end{aligned} \tag{10}$$

$$\begin{aligned} \left(R(X,Y)H\right) &= B(X,Y)F - A(X,Y)G \\ &= \left[R(X,Y),H\right] \end{aligned} \tag{11}$$

where

$A = dp + q\wedge r, B = dq - r\wedge p, C = dr - p\wedge q.$

Proof. Firstly, we proof of the first equation for any $Z \in \chi(M)$:

$$\left(R(X,Y)F\right)(Z) = (D_X D_Y F - D_Y D_X F - D_{[X,Y]}F)(Z)$$

using (ii) of Theorem 20

$$\begin{aligned} \left(R(X,Y)F\right)(Z) &= \{D_X(r(Y)G+q(Y)H) \\ &\quad - D_Y(r(X)G+q(X)H) \\ &\quad - r\left([X,Y]\right)G - q\left([X,Y]\right)H\}(Z) \\ &= \{D_X(r(Y)G) + D_X\left(q(Y)H\right) \\ &\quad - D_Y\left(r(X)G\right) - D_Y\left(q(X)H\right) \\ &\quad - r[X,Y]G + q\left([X,Y]\right)H\}(Z) \\ &= \{X\left(r(Y)\right)G + r(Y)(r(X)F \\ &\quad + p(X)H) + Xq(Y)H \\ &\quad + q(Y)\left(q(X)F - p(X)G\right) \\ &\quad - Y\left(r(X)\right)G + r(X)\left(r(Y)F + p(Y)H\right) \\ &\quad - Y\left(q(X)\right)H - q(X)\left(q(Y)F - p(Y)G\right) \\ &\quad - r\left([X,Y]\right)G - q\left([X,Y]\right)H(Z) \\ &= \{X\left(r(Y)\right) - Y\left(r(X)\right) - r\left([X,Y]\right) \\ &\quad - \left(p(X)q(Y) - q(X)p(Y)\right)(GZ) \\ &\quad + X\left(q(Y)\right) - Y\left(q(X)\right) \\ &\quad - (r(X)p(Y) - r(Y)p(X))\}(HZ) \end{aligned}$$

where

$$dr(X,Y) = X(r(Y)) - Y(r(X)) - r\left([X,Y]\right)$$
$$dq\left(X,Y\right) = X\left(q(Y)\right) - Y(q(X)) - q\left([X,Y]\right)$$

then

$$\begin{aligned} (R(X,Y)F)(Z) &= \left(dr(X,Y) - (p\wedge q)(X,Y)\right)GZ \\ &\quad + \left(dq(X,Y) - (r\wedge p)(X,Y)\right)HZ \text{ so} \\ &= C(X,Y)GZ + B(X,Y)HZ \end{aligned}$$

we obtain

$$\left(R(X,Y)F\right) = C(X,Y)G + B(X,Y)H.$$

Now, we show that the following equality is valid.

$$\begin{aligned} \left(R(X,Y)F\right) &= R(X,Y)F - F(R(X,Y))H \\ &= \left[R(X,Y),F\right] \end{aligned}$$

for any $Z \in \chi(M)$

$$
\begin{aligned}
R(X,Y)(FZ) &= D_X D_Y (FZ) \\
&\quad - D_Y D_X (FZ) - D_{[X,Y]}(FZ) \\
&= D_X \left(\left(D_Y F \right) Z + F \left(D_Y Z \right) \right) \\
&\quad - D_Y \left(\left(D_X F \right) Z + F \left(D_X Z \right) \right) \\
&\quad - \left(D_{[X,Y]} F \right) Z - F D_{[X,Y]} Z \\
&= D_X \left(D_Y F \right) Z + \left(D_Y F \right)\left(D_X Z \right) \\
&\quad + \left(D_X F \right)\left(D_Y Z \right) + F D_X \left(D_Y Z \right) \\
&\quad - D_Y \left(D_X F \right) Z - \left(D_X F \right)\left(D_Y Z \right) \\
&\quad - \left(D_Y F \right)\left(D_X Z \right) - F D_Y D_X Z \\
&\quad - \left(D_{[X,Y]} F \right) Z - F D_{[X,Y]} Z \\
&= D_X \left(D_Y F \right) Z - D_Y \left(D_X F \right) Z \\
&\quad - \left(D_{[X,Y]} F \right) Z \\
&\quad + F \left(D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z \right) \text{ so} \\
&= \left(R\left(X,Y \right) F \right) Z + F \left(R\left(X,Y \right) Z \right)
\end{aligned}
$$

$$\left(R\left(X,Y \right) F \right)(Z) = R\left(X,Y \right)\left(FZ \right) - F\left(R\left(X,Y \right) Z \right)$$

then we have

$$\left(R\left(X,Y \right) F \right) = R(X,Y)F - FR(X,Y) = \left[R(X,Y), F \right]$$

Similarly, we can show (10) and (11) are also true.

Lemma 22. Let $\left(M, g, V \right)$ be an almost split quaternion Kaehlerian manifold. Let $\left\{ F, G, H \right\}$ be a base of V and semi-Riemannian curvature tensor R. Then for arbitrarily $x, Y, Z \in \chi\left(M \right)$

$$g\left(R\left(X,Y \right) FZ, FW \right) - g\left(R\left(X,Y \right) Z, W \right) = -C(X,Y) g(Z,HW) - B(X,Y) g(Z,GW) \qquad (12)$$

$$g\left(R\left(X,Y \right) GZ, GW \right) - g\left(R\left(X,Y \right) Z, W \right) = -C(X,Y) g(Z,HW) - A(X,Y) g(Z,FW) \qquad (13)$$

$$g\left(R\left(X,Y \right) HZ, HW \right) - g\left(R\left(X,Y \right) Z, W \right) = B(X,Y) g(Z,GW) - A(X,Y) g(Z,FW) \qquad (14)$$

Proof. Since Lemma 9, we have

$$\left(R\left(X,Y \right) FZ \right) = \left(R\left(X,Y \right) F \right)(Z) + FR(X,Y)(Z)$$

then we obtain

$$
\begin{aligned}
& g\left(R\left(X,Y \right) FZ, FW \right) - g\left(R\left(X,Y \right) Z, W \right) = \\
& g\left[\left(R\left(X,Y \right) F \right) Z + F\left(R\left(X,Y \right) Z \right) \right]\left(FW \right) \\
& - g\left(R\left(X,Y \right) Z, W \right) \\
& = g\left(\left(R\left(X,Y \right) F \right) Z, FW \right) \\
& = g\left(\left(C\left(X,Y \right) G + B\left(X,Y \right) H \right) Z, FW \right) \\
& = -C\left(X,Y \right) g\left(Z, HW \right) - B\left(X,Y \right) g\left(Z, GW \right)
\end{aligned}
$$

and similar calculations we can obtain the other two equations (13) and (14).

Lemma 23. Let $\left(M, g, V \right)$ be a split quaternion Kaehler manifold of dimension 4m with canonical local base $\left\{ F, G, H \right\}$ of V. Let R be the semi-Riemannian curvature tensor of M such that $\left\{ e_i \right\}, 1 \le i \le 4m$ be an orthonormal base of M for any $X, Y \in \chi\left(M \right)$

$$A(X,Y) = \frac{-1}{2m} \sum_{i=1}^{4m} \varepsilon_i g\left(R\left(X,Y \right) e_i, Fe_i \right) \qquad (15)$$

$$B(X,Y) = \frac{1}{2m} \sum_{i=1}^{4m} \varepsilon_i g\left(R\left(X,Y \right) e_i, Ge_i \right) \qquad (16)$$

$$C(X,Y) = \frac{-1}{2m} \sum_{i=1}^{4m} \varepsilon_i g\left(R\left(X,Y \right) e_i, He_i \right) \qquad (17)$$

Proof. Substituting $Z = e_i, W = Fe_i$ into the equation (13) of Lemma 22, we get

$$g\left(R\left(X,Y\right)Ge_i,GFe_i\right)-g\left(R\left(X,Y\right)e_i,Fe_i\right)=$$
$$-A\left(X,Y\right)g\left(e_i,F^2e_i\right)-C(X,Y)g\left(e_i,HFe_i\right)$$
$$g\left(R\left(X,Y\right)Ge_i,-He_i\right)-g\left(R\left(X,Y\right)e_i,Fe_i\right)$$
$$=A\left(X,Y\right)g\left(e_i,e_i\right)-C(X,Y)g\left(e_i,Ge_i\right)$$
$$-g\left(HR\left(X,Y\right)Ge_i,e_i\right)-g\left(R\left(X,Y\right)e_i,Fe_i\right)$$
$$=A(X,Y)\varepsilon_i.4m-g\left(R\left(X,Y\right)e_i,HGe_i\right)$$
$$-g\left(R\left(X,Y\right)e_i,Fe_i\right)$$
$$=A(X,Y)\varepsilon_i.4m$$
$$-g\left(R\left(X,Y\right)e_i,Fe_i\right)-g\left(R\left(X,Y\right)e_i,Fe_i\right)$$
$$=A(X,Y)\varepsilon_i.4m$$

which gives

$$A(X,Y)=\frac{1}{2}\sum_{i=1}^{4m}\varepsilon_i g\left(R\left(X,Y\right)e_i,Fe_i\right).$$

By use similar calculations we can also prove (16) and (17).

Lemma 24. Let (M,g,V) be a split quaternion Kaehler manifold of dimension 4m with canonical local basis $\{F,G,H\}$ of V. Let S be the semi-Riemannian Ricci curvature tensor of M. Then for any $X,Y\in\chi\left(M\right)$

$$S(X,Y)=-mA(X,FY)+B(X,GY)-C(X,HY) \tag{18}$$

$$S(X,Y)=A(X,FY)-mB(X,GY)+C(X,HY) \tag{19}$$

$$S(X,Y)=A(X,FY)-B(X,GY)-mC(X,HY) \tag{20}$$

and if $m>1$, then

$$S(X,Y)=-\frac{2}{3}\left(B\left(X,GY\right)+C(X,HY)\right) \tag{21}$$

Proof. From the equation (15) of Lemma 23 and the Bianchi's 1st identity

$$\sum_{i=1}^{4m}\varepsilon_i g\left(R(X,e_i\right)Fe_i,Y)=\frac{1}{2}\sum_{i=1}^{4m}\varepsilon_i$$
$$\left[g\left(R(X,e_i\right)Fe_i,Y)-g(R(X,Fe_i)e_i,Y)\right]$$
$$=\frac{1}{2}\sum_{i=1}^{4m}\varepsilon_i[g\left(R(X,e_i\right)Fe_i,Y)$$
$$+g(R(Fe_i,X)e_i,Y)]$$
$$=\frac{1}{2}\sum_{i=1}^{4m}\varepsilon_i g\left(R(Fe_i,e_i\right)X,Y)$$
$$=\frac{1}{2}\sum_{i=1}^{4m}\varepsilon_i g\left(R(X,Y\right)Fe_i,e_i)$$
$$=-mA(X,Y).$$

Hence, we have

$$-mA(X,FY)=\sum_{i=1}^{4m}\varepsilon_i g\left(R(X,e_i\right)Fe_i,FY).$$

Similarly, we have

$$mB(X,GY)=\sum_{i=1}^{4m}\varepsilon_i g\left(R(X,e_i\right)Ge_i,GY)$$

$$mA(X,HY)=\sum_{i=1}^{4m}\varepsilon_i g\left(R(X,e_i\right)He_i,HY)$$

We denote by the Ricci tensor

$$S(X,Y)=\sum_{i=!}^{4m}\varepsilon_i g\left(R\left(e_i,Y\right)Z,e_i\right).$$

Substituting $Y=Z=e_i$ and $W=Y$ into the equation (12) of Lemma 22 we get

$$g\left(R\left(X,e_i\right)Fe_i,FY\right)-g(R(X,e_i)e_i,Y)$$
$$=-B(X,e_i)g(e_i,GY)+C(X,e_i)g(e_i,HY)$$
$$-mA(X,FY)-S(X,Y)$$
$$=-B(X,GY)+C(X,HY)$$

$$S(X,Y)=-mA(X,FY)+B(X,GY)-C(X,HY).$$

With similar calculations we can obtain (19), (20) and (21).

Lemma 25. With the same assumptions and notations of Lemma 12, we have

$$dA+r\wedge B-q\wedge C=0 \tag{22}$$
$$dB+p\wedge C-r\wedge A=0 \tag{23}$$
$$dC+p\wedge B-q\wedge A=0 \tag{24}$$.

Proof. From Lemma 9, we get

$$A = dp + q \wedge r$$
$$B = dq - r \wedge p$$
$$C = dr - p \wedge q$$

or

$$dA = d^2 p + dq \wedge r + q \wedge dr = dq \wedge r + q \wedge dr.$$

So we have

$$dA = \left(B + r \wedge p\right) \wedge r + q \wedge \left(C + p \wedge q\right)$$
$$= B \wedge r + q \wedge C = 0$$

or

$$dA + r \wedge B - q \wedge C = 0.$$

Similarly we can show (23) and (24).

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