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Crank-Nicolson difference scheme for the generalized Rosenau-Burgers equation

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Abstract—In this paper, numerical solution for the generalized Rosenau-Burgers equation is considered and Crank-Nicolson finite difference scheme is proposed. Existence of the solutions for the difference scheme has been shown. Stability, convergence and priori error estimate of the scheme are proved. Numerical results demonstrate that the scheme is efficient and reliable.

Keywords—Generalized Rosenau-Burgers equation, Difference scheme, Stability, Convergence.

I. Introduction

CONSIDER the following initial-boundary value problem for the generalized Rosenau-Burgers equation,

$$u_t+u_{xxxxt}-\alpha u_{xx}+\beta u_x+(\frac{u^{p+1}}{p+1})_x=0, x\in[0,L], t\in[0,T], \quad (1)$$

with an initial condition

$$u(x,0)=u_0(x), x\in[0,L], \quad (2)$$

and boundary conditions

$$u(0,t)=u(L,t)=0, u_{xx}(0,t)=u_{xx}(L,t)=0, t\in[0,T]. \quad (3)$$

where $\alpha>0, \beta\in R$ and $p\geq 1$ is a integer.

When $p=1$, equation (1) is called as usual Rosenau-Burgers equation arises in some natural phenomena, for example, in bore propagation and in water waves. The asymptotic behavior of the solution for the Cauchy problem to the Rosenau-Burgers equation, in particular, the stability of traveling waves and diffusion waves have been well studied in [7], [8], [9], [10]. Numerical scheme has been proposed such as finite difference method by Hu [6]. In this paper, an attempt has been here to discuss finite difference method for the generalized Rosenau-Burgers equation.

The problem (1)-(3) has the following results,

Lemma 1 Suppose $u_0\in H_0^2[0,L]$, then the solution of (1)-(3) satisfies:

$$||u||_{L_2}\leq C, ||u_x||_{L_2}\leq C, ||u||_\infty\leq C. \quad (4)$$

where C denotes a general positive constant, which may have different values in different occurrences.

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Proof. Let

$$\begin{aligned} E(t) &= ||u||_{L_2}^2+||u_{xx}||_{L_2}^2 \\ &= \int_0^L |u(x,t)|^2dx+\int_0^L |u_{xx}(x,t)|^2dx, \quad t\in[0,T]. \end{aligned}$$

Considering (3) and $u_t+u_{xxxxt}=\alpha u_{xx}-\beta u_x-(\frac{u^{p+1}}{p+1})_x$, we have

$$\begin{aligned} \frac{dE(t)}{dt} &= 2\int_0^L uu_tdx+2\int_0^L u_{xx}u_{xxt}dx \\ &= 2\int_0^L uu_tdx+2u_xu_{xxt}|_0^L-2\int_0^L u_xu_{xxxt}dx \\ &= 2\int_0^L uu_tdx-2\int_0^L u_{xxxt}dx \\ &= 2\int_0^L uu_tdx-2uu_{xxxt}|_0^L+2\int_0^L uu_{xxxxt}dx \\ &= 2\int_0^L u(u_t+u_{xxxxt})dx \\ &= 2\int_0^L u[\alpha u_{xx}-\beta u_x-(\frac{u^{p+1}}{p+1})_x]dx \\ &= 2\alpha\int_0^L uu_{xx}dx-\int_0^L(\beta uu_x+u^{p+1}u_x)dx \\ &= -2\alpha\int_0^L (u_x)^2dx\leq 0 \end{aligned}$$

So $E(t)$ decreases, that is

$$E(t)=||u||_{L_2}^2+||u_{xx}||_{L_2}^2\leq E(0), \quad t\in[0,T],$$

i.e.,

$$||u||_{L_2}\leq C, \quad ||u_{xx}||_{L_2}\leq C.$$

Using Hölder inequality and Cauchy-Schwartz inequality, we have

$$\begin{aligned} ||u_x||_{L_2}^2 &= \int_0^L (u_x)^2dx=-\int_0^L uu_{xx}dx \\ &\leq ||u||_{L_2}\cdot||u_{xx}||_{L_2}\leq\frac{1}{2}(||u||_{L_2}^2+||u_{xx}||_{L_2}^2), \end{aligned}$$

that is,

$$||u_x||_{L_2}\leq C.$$

Using Sobolev inequality, we get

$$||u_x||_\infty\leq C.$$

In this paper, a two-level Crank-Nicolson difference scheme is proposed. The outline of the paper is as follows. In section

2, difference scheme is presented and the estimate for the difference solution is derived. In section 3, we prove the existence and uniqueness of the scheme. In section 4, we prove the convergence and stability for the difference scheme. Finally some numerical experiments are given in Section 5 to verify our theoretical analysis.

II. Finite difference scheme

Let h and τ be the uniform step size in the spatial and temporal direction respectively. Denote $x_j = jh \quad (0 \le j \le J), t_n = n\tau \quad (0 \le n \le N, N = [\frac{T}{\tau}])$, $u_j^n \approx u(x_j, t_n)$ and $Z_h^0 = \{u = (u_j) | u_0 = u_J = 0, j = 0, 1, 2, \cdots, J\}$. Define

$$(u_j^n)_x = \frac{u_{j+1}^n - u_j^n}{h}, (u_j^n)_{\bar{x}} = \frac{u_j^n - u_{j-1}^n}{h},$$
$$(u_j^n)_{\hat{x}} = \frac{u_{j+1}^n - u_{j-1}^n}{2h}, (u_j^n)_t = \frac{u_j^{n+1} - u_j^n}{\tau},$$
$$(u_j^n)_{x\bar{x}} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}, u_j^{n+\frac{1}{2}} = \frac{u_j^{n+1} + u_j^n}{2},$$
$$(u^n, v^n) = h\sum_{j=0}^{J-1} u_j^n v_j^n, ||u^n||^2 = (u^n, u^n),$$
$$||u^n||_\infty = \max_{0\le j\le J-1} |u_j^n|.$$

Since $(\frac{u^{p+1}}{p+1})_x = \frac{1}{p+2}[u^p u_x + (u^{p+1})_x]$, then the following finite difference scheme is considered,

$$(u_j^n)_t + (u_j^n)_{xx\bar{x}\bar{x}t} - \alpha(u_j^{n+\frac{1}{2}})_{x\bar{x}} + \beta(u_j^{n+\frac{1}{2}})_{\hat{x}}$$
$$+\frac{1}{p+2}\{(u_j^{n+\frac{1}{2}})^p(u_j^{n+\frac{1}{2}})_{\hat{x}} + [(u_j^{n+\frac{1}{2}})^{p+1}]_{\hat{x}}\} = 0, \qquad (5)$$
$$u_j^0 = u_0(x_j), 0 \le j \le J-1, \qquad (6)$$
$$u_0^n = u_J^n = 0, (u_0^n)_{x\bar{x}} = (u_J^n)_{x\bar{x}} = 0.1 \le n \le N. \qquad (7)$$

Lemma 2 [6]. For any two mesh functions: $u, v \in Z_h^0$, we have

$$((u_j)_x, v_j) = -(u_j, (v_j)_{\bar{x}}), \quad (v_j, (u_j)_{x\bar{x}}) = -((v_j)_x, (u_j)_x),$$

and

$$(u_j, (u_j)_{x\bar{x}}) = -((u_j)_x, (u_j)_x) = -||u_x||^2.$$

Furthermore, if $(u_0^n)_{x\bar{x}} = (u_J^n)_{x\bar{x}} = 0$, then

$$(u_j, (u_j)_{xx\bar{x}\bar{x}}) = ||u_{xx}||^2.$$

Lemma 3(Discrete Sobolev's inequality[14]) There exist two constant C_1 and C_2 such that

$$||u^n||_\infty \le C_1||u^n|| + C_2||u_x^n||.$$

Theorem 1 Suppose $u_0 \in H_0^2[0, L]$, then the solution u^n of (5)-(7) satisfies:

$$||u^n|| \le C, \qquad ||u_x^n|| \le C$$

which yield $||u^n||_\infty \le C \quad (n = 1, 2, \cdots, N)$.

Proof. Computing the inner product of (5) with $2u^{n+\frac{1}{2}}$, according to boundary condition (7) and Lemma 2, we have

$$\frac{1}{\tau}(||u^{n+1}||^2 - ||u^n||^2) + \frac{1}{\tau}(||u_{xx}^{n+1}||^2 - ||u_{xx}^n||^2)$$
$$-\alpha((u_j^{n+\frac{1}{2}})_{x\bar{x}}, 2u_j^{n+\frac{1}{2}}) + \beta((u_j^{n+\frac{1}{2}})_{\hat{x}}, 2u_j^{n+\frac{1}{2}})$$
$$+(P, 2u_j^{n+\frac{1}{2}}) = 0, \qquad (8)$$

where

$$P = \frac{1}{p+2}\{(u_j^{n+\frac{1}{2}})^p(u_j^{n+\frac{1}{2}})_{\hat{x}} + [(u_j^{n+\frac{1}{2}})^{p+1}]_{\hat{x}}\}.$$

According to

$$((u^{n+\frac{1}{2}})_{x\bar{x}}, 2u^{n+\frac{1}{2}}) = -2||u_x^{n+\frac{1}{2}}||^2, \qquad (9)$$
$$((u_j^{n+\frac{1}{2}})_{\hat{x}}, 2u_j^{n+\frac{1}{2}}) = 0, \qquad (10)$$

and

$$(P, 2u_j^{n+\frac{1}{2}})$$
$$= \frac{2h}{p+2}\sum_{j=0}^{J-1}\{(u_j^{n+\frac{1}{2}})^p(u_j^{n+\frac{1}{2}})_{\hat{x}} + [(u_j^{n+\frac{1}{2}})^{p+1}]_{\hat{x}}\}u_j^{n+\frac{1}{2}}$$
$$= \frac{1}{p+2}\sum_{j=0}^{J-1}\{(u_j^{n+\frac{1}{2}})^{p+1}(u_{j+1}^{n+\frac{1}{2}} - u_{j-1}^{n+\frac{1}{2}})$$
$$+[(u_{j+1}^{n+\frac{1}{2}})^{p+1} - (u_{j-1}^{n+\frac{1}{2}})^{p+1}]u_j^{n+\frac{1}{2}}\}$$
$$= \frac{1}{p+2}\sum_{j=0}^{J-1}[(u_{j+1}^{n+\frac{1}{2}})^p u_j^{n+\frac{1}{2}} + (u_j^{n+\frac{1}{2}})^{p+1}]u_{j+1}^{n+\frac{1}{2}}$$
$$-\frac{1}{p+2}\sum_{j=0}^{J-1}[(u_j^{n+\frac{1}{2}})^p u_{j-1}^{n+\frac{1}{2}} + (u_{j-1}^{n+\frac{1}{2}})^{p+1}]u_j^{n+\frac{1}{2}}$$
$$= 0, \qquad (11)$$

we obtain

$$(||u^{n+1}||^2 - ||u^n||^2) + (||u_{xx}^{n+1}||^2 - ||u_{xx}^n||^2)$$
$$= -2\tau\alpha||u_x^{n+\frac{1}{2}}||^2 \le 0,$$

that is,

$$(||u^n||^2 + ||u_{xx}^n||^2) \le (||u^{n-1}||^2 + ||u_{xx}^{n-1}||^2)$$
$$\le \cdots \le (||u^0||^2 + ||u_{xx}^0||^2) = C.$$

Obviously,

$$||u^n|| \le C, ||u_{xx}^n|| \le C.$$

From Lemma 2 and using Cauchy-Schwartz inequality, we have

$$||u_x^n||^2 = -(u^n, u_{x\bar{x}}^n) \le ||u^n|| \cdot ||u_{xx}^n||$$
$$\le \frac{1}{2}(||u^n||^2 + ||u_{xx}^n||^2) \le C. \qquad (12)$$

By Lemma 3, we get $||u^n||_\infty \le C$.

III. Existence and Uniqueness

To prove the existence of solution for scheme (5)-(7), the following Browder fixed point Theorem should be introduced. For the proof, see[1].

Lemma 4(Browder fixed point Theorem) Let H be a finite dimensional inner product space. Suppose that $g : H \to H$ is continuous and there exist an $\alpha > 0$ such that $(g(x), x) > 0$ for all $x \in H$ with $||x|| = \alpha$. Then there exists $x^* \in H$ such that $g(x^*) = 0$ and $||x^*|| \le \alpha$.

Theorem 2 There exists $u^n \in Z_h^0$ satisfies the difference scheme (5)-(7).

Proof. By the mathematical induction, for $n \le N - 1$, assume that $u^0, u^1, \cdots, u^n$ satisfy (5)-(7). Next we prove that there exists u^{n+1} satisfied (5).

Define a operator g on Z_h^0 as follows,

$$\begin{aligned} g(v) &= 2v - 2u^n + 2v_{xx\bar{x}\bar{x}} - 2u^n_{xx\bar{x}\bar{x}} - \alpha\tau v_{x\bar{x}} + \beta\tau v_{\hat{x}} \\ &+ \frac{\tau}{p+2}\{(v_j)^p(v_j)_{\hat{x}} + [(v_j)^{p+1}]_{\hat{x}}\}. \end{aligned} \tag{13}$$

Taking the inner product of (13) with v, we get

$$(v_{\hat{x}}, v) = 0, \quad ((v_j)^p(v_j)_{\hat{x}} + [(v_j)^{p+1}]_{\hat{x}}, v) = 0,$$

and

$$\begin{aligned} &(g(v), v) \\ &= 2||v||^2 - 2(u^n, v) + 2||v_{xx}||^2 - 2(u^n_{xx}, v_{xx}) \\ &\quad -\alpha\tau(v_{x\bar{x}}, v) \\ &\ge 2||v||^2 - 2||u^n|| \cdot ||v|| + 2||v_{xx}||^2 - 2||u^n_{xx}|| \cdot ||v_{xx}|| \\ &\quad +\alpha\tau||v_x||^2 \\ &\ge 2||v||^2 - (||u||^2 + ||v||^2) + 2||v_{xx}||^2 \\ &\quad -(||u_{xx}||^2 + ||v_{xx}||^2) + \alpha\tau||v_x||^2 \\ &\ge ||v||^2 - (||u^n||^2 + ||u_{xx}||^2) + ||v_{xx}||^2 + \alpha\tau||v_x||^2 \\ &\ge ||v||^2 - (||u^n||^2 + ||u^n_{xx}||^2). \end{aligned}$$

Obviously, for $\forall v \in Z_h^0$, $(g(v), v) \ge 0$ with $||v||^2 = ||u^n||^2 + ||u^n_{xx}||^2 + 1$. It follows from Lemma 4 that there exists $v^* \in Z_h^0$ which satisfies $g(v^*) = 0$. Let $u^{n+1} = 2v^* - u^n$, it can be proved that u^{n+1} is the solution of the scheme (5)-(7).

IV. Convergence and Stability

Next, we discuss the convergence and stability of the scheme (5)-(7). Let $v(x, t)$ be the solution of problem (1)-(3), $v_j^n = v(x_j, t_n)$, then the truncation of the scheme (5)-(7) is

$$\begin{aligned} r_j^n &= (v_j^n)_t + (v_j^n)_{xx\bar{x}\bar{x}t} - \alpha(v_j^{n+\frac{1}{2}})_{x\bar{x}} + \beta(v_j^{n+\frac{1}{2}})_{\hat{x}} \\ &+ \frac{1}{p+2}\{(v_j^{n+\frac{1}{2}})^p(v_j^{n+\frac{1}{2}})_{\hat{x}} + [(v_j^{n+\frac{1}{2}})^{p+1}]_{\hat{x}}\}. \end{aligned} \tag{14}$$

Using Taylor expansion, we know that $r_j^n = O(\tau^2 + h^2)$ holds if $\tau, h \to 0$.

Lemma 5(Discrete Gronwall inequality[14]) Suppose $w(k), \rho(k)$ are nonnegative mesh functions and $\rho(k)$ is non-decreasing. If $C > 0$ and

$$w(k) \le \rho(k) + C\tau\sum_{l=0}^{k-1} w(l), \quad \forall k,$$

then

$$w(k) \le \rho(k)e^{C\tau k}, \quad \forall k.$$

Theorem 3 Suppose $u_0 \in H_0^2[0, L]$, then the solution u^n of the scheme (5)-(7) converges to the solution of problem (1)-(3) in the sense of $||\cdot||_\infty$ and the rate of convergence is $O(\tau^2 + h^2)$.

Proof. Subtracting (14) from (5) and letting $e_j^n = v_j^n - u_j^n$, we have

$$\begin{aligned} r_j^n &= (e_j^n)_t + (e_j^n)_{xx\bar{x}\bar{x}t} - \alpha(e_j^{n+\frac{1}{2}})_{x\bar{x}} + \beta(e_j^{n+\frac{1}{2}})_{\hat{x}} \\ &+ \frac{1}{p+2}\{(v_j^{n+\frac{1}{2}})^p(v_j^{n+\frac{1}{2}})_{\hat{x}} + [(v_j^{n+\frac{1}{2}})^{p+1}]_{\hat{x}}\} \\ &- \frac{1}{p+2}\{(u_j^{n+\frac{1}{2}})^p(u_j^{n+\frac{1}{2}})_{\hat{x}} + [(u_j^{n+\frac{1}{2}})^{p+1}]_{\hat{x}}\}, \end{aligned} \tag{15}$$

Computing the inner product of (15) with $2e^{n+\frac{1}{2}}$, and using $((e_j^{n+\frac{1}{2}})_{\hat{x}}, 2e_j^{n+\frac{1}{2}}) = 0$, we get

$$\begin{aligned} &(r_j^n, 2e^{n+\frac{1}{2}}) \\ &= \frac{1}{\tau}(||e^{n+1}||^2 - ||e^n||^2) + \frac{1}{\tau}(||e^{n+1}_{xx}||^2 - ||e^n_{xx}||^2) \\ &-\alpha((e_j^{n+\frac{1}{2}})_{x\bar{x}}, 2e^{n+\frac{1}{2}}) + (Q_1 + Q_2, 2e^{n+\frac{1}{2}}), \end{aligned} \tag{16}$$

where

$$Q_1 = \frac{1}{p+2}[(v_j^{n+\frac{1}{2}})^p(v_j^{n+\frac{1}{2}})_{\hat{x}} - (u_j^{n+\frac{1}{2}})^p(u_j^{n+\frac{1}{2}})_{\hat{x}}],$$

and

$$Q_2 = \frac{1}{p+2}\{[(v_j^{n+\frac{1}{2}})^{p+1}]_{\hat{x}} - [(u_j^{n+\frac{1}{2}})^{p+1}]_{\hat{x}}\}.$$

According to Lemma 2, Theorem 1 and Cauchy-Schwartz inequality, we have

$$\begin{aligned} &(Q_1, 2e^{n+\frac{1}{2}}) \\ &= \frac{2}{p+2}h\sum_{j=0}^{J-1}[(v_j^{n+\frac{1}{2}})^p(v_j^{n+\frac{1}{2}})_{\hat{x}} - (u_j^{n+\frac{1}{2}})^p(u_j^{n+\frac{1}{2}})_{\hat{x}}]e_j^{n+\frac{1}{2}} \\ &= \frac{2}{p+2}h\sum_{j=0}^{J-1}(v_j^{n+\frac{1}{2}})^p(e_j^{n+\frac{1}{2}})_{\hat{x}}e_j^{n+\frac{1}{2}} \\ &+ \frac{2}{p+2}h\sum_{j=0}^{J-1}[(v_j^{n+\frac{1}{2}})^p - (u_j^{n+\frac{1}{2}})^p](u_j^{n+\frac{1}{2}})_{\hat{x}}e_j^{n+\frac{1}{2}} \\ &= \frac{2}{p+2}h\sum_{j=0}^{J-1}(v_j^{n+\frac{1}{2}})^p(e_j^{n+\frac{1}{2}})_{\hat{x}}e_j^{n+\frac{1}{2}} \\ &+ \frac{2}{p+2}h\sum_{j=0}^{J-1}[e_j^{n+\frac{1}{2}}\sum_{k=0}^{p-1}(v_j^{n+\frac{1}{2}})^{p-1-k}(u_j^{n+\frac{1}{2}})^k](u_j^{n+\frac{1}{2}})_{\hat{x}}e_j^{n+\frac{1}{2}} \\ &\le Ch\sum_{j=0}^{J-1}|(e_j^{n+\frac{1}{2}})_{\hat{x}}||e_j^{n+\frac{1}{2}}| + Ch\sum_{j=0}^{J-1}|(e_j^{n+\frac{1}{2}})^2| \\ &\le C[||e_x^{n+\frac{1}{2}}||^2 + ||e^{n+\frac{1}{2}}||^2] \\ &\le C[||e_x^{n+1}||^2 + ||e_x^n||^2 + ||e^{n+1}||^2 + ||e^n||^2], \end{aligned} \tag{17}$$

and

$$\begin{aligned}&(Q_2, 2e^{n+\frac{1}{2}})\\&= \frac{2}{p+2}h\sum_{j=0}^{J-1}\{[(v_j^{n+\frac{1}{2}})^{p+1}]_{\hat{x}} - [(u_j^{n+\frac{1}{2}})^{p+1}]_{\hat{x}}\}e_j^{n+\frac{1}{2}}\\&= -\frac{2}{p+2}h\sum_{j=0}^{J-1}[(v_j^{n+\frac{1}{2}})^{p+1} - (u_j^{n+\frac{1}{2}})^{p+1}](e_j^{n+\frac{1}{2}})_{\hat{x}}\\&= -\frac{2}{p+2}h\sum_{j=0}^{J-1}e_j^{n+\frac{1}{2}}[\sum_{k=0}^{p}(v_j^{n+\frac{1}{2}})^{p-k}(u_j^{n+\frac{1}{2}})^k](e_j^{n+\frac{1}{2}})_{\hat{x}}\\&\le C[||e_x^{n+\frac{1}{2}}||^2 + ||e^{n+\frac{1}{2}}||^2]\\&\le C[||e_x^{n+1}||^2 + ||e_x^n||^2 + ||e^{n+1}||^2 + ||e^n||^2]. \quad (18)\end{aligned}$$

Furthermore,

$$((e_j^{n+\frac{1}{2}})_{x\bar{x}}, 2e^{n+\frac{1}{2}}) = -2||e_x^{n+\frac{1}{2}}||^2, \quad (19)$$

$$\begin{aligned}(r_j^n, 2e^{n+\frac{1}{2}}) &= (r_j^n, e^{n+1} + e^n)\\&\le ||r^n||^2 + \frac{1}{2}[||e^{n+1}||^2 + ||e^n||^2]. \quad (20)\end{aligned}$$

Substituting (17)-(20) into (16), we get

$$\begin{aligned}&(||e^{n+1}||^2 - ||e^n||^2) + (||e_{xx}^{n+1}||^2 - ||e_{xx}^n||^2)\\&\le C\tau[||e^{n+1}||^2 + ||e^n||^2 + ||e_x^{n+1}||^2 + ||e_x^n||^2]\\&+\tau||r^n||^2. \quad (21)\end{aligned}$$

Similarly to the proof of (12), we have

$$\begin{aligned}||e_x^{n+1}||^2 &\le \frac{1}{2}(||e^{n+1}||^2 + ||e_{xx}^{n+1}||^2),\\||e_x^n||^2 &\le \frac{1}{2}(||e^n||^2 + ||e_{xx}^n||^2). \quad (22)\end{aligned}$$

and (21) can be rewritten as

$$\begin{aligned}&(||e^{n+1}||^2 - ||e^n||^2) + (||e_{xx}^{n+1}||^2 - ||e_{xx}^n||^2)\\&\le C\tau[||e^{n+1}||^2 + ||e^n||^2 + ||e_{xx}^{n+1}||^2 + ||e_{xx}^n||^2]\\&+\tau||r^n||^2. \quad (23)\end{aligned}$$

Let $B^n = ||e^n||^2 + ||e_{xx}^n||^2$, then (23) is written as follows,

$$(1 - C\tau)[B^{n+1} - B^n] \le 2C\tau B^n + \tau||r^n||^2.$$

If τ is sufficiently small which satisfies $1 - C\tau > 0$, then

$$[B^{n+1} - B^n] \le C\tau B^n + C\tau||r^n||^2. \quad (24)$$

Summing up (24) from 0 to $n-1$, we have

$$B^n \le B^0 + C\tau\sum_{l=0}^{n-1}||r^l||^2 + C\tau\sum_{l=0}^{n-1}B^l.$$

Noticing

$$\tau\sum_{l=0}^{n-1}||r^l||^2 \le n\tau \max_{0\le l\le n-1}||r^l||^2 \le T\cdot O(\tau^2 + h^2)^2,$$

and $e^0 = 0$, we have $B^0 = O(\tau^2 + h^2)^2$. Hence

$$B^n \le O(\tau^2 + h^2)^2 + C\tau\sum_{l=0}^{n-1}B^l.$$

TABLE I
THE ERRORS ESTIMATES IN THE SENSE OF $||\cdot||_\infty$, WHEN $p = 2, \alpha = 0.1$ AND $\tau = 0.1$

	h=1/4	h=1/8	h=1/16	h=1/32
t=0.2	4.330014e-7	1.012584e-7	2.469951e-8	5.961687e-9
t=0.4	8.629880e-7	2.018789e-7	4.924355e-8	1.188390e-8
t=0.6	1.289969e-6	3.018655e-7	7.363520e-8	1.776952e-8
t=0.8	1.713953e-6	4.012209e-7	9.787626e-8	2.362015e-8
t=1.0	2.134945e-6	4.999438e-7	1.219645e-7	2.943340e-8

According to Lemma 5, we get $B^n \le O(\tau^2 + h^2)^2$, that is

$$||e^n|| \le O(\tau^2 + h^2), \qquad ||e_{xx}^n|| \le O(\tau^2 + h^2).$$

It follows from (22) that

$$||e_x^n|| \le O(\tau^2 + h^2).$$

By using Lemma 3, we have

$$||e^n||_\infty \le O(\tau^2 + h^2).$$

This completes the proof of Theorem 3.

Similarly, it can be proved that

Theorem 4 Under the conditions of Theorem 3, the solution of scheme (5)-(7) is stable by $||\cdot||_\infty$.

V. NUMERICAL EXPERIMENTS

Consider the generalized Rosenau-Burgers equation,

$$u_t + u_{xxxxt} - \alpha u_{xx} + u_x + (\frac{u^{p+1}}{p+1})_x = 0, (x,t) \in [0,1]\times[0,1], \quad (25)$$

with an initial condition

$$u(x,0) = x^4(1-x)^4, \qquad x \in [0,1], \quad (26)$$

and boundary conditions

$$u(0,t) = u(1,t) = 0, u_{xx}(0,t) = u_{xx}(1,t) = 0, t \in [0,1]. \quad (27)$$

We construct a scheme to (1)-(3) as nonlinear two-level scheme (5). Since we do not know the exact solution of (1)-(3), we consider the solution on mesh $h = \frac{1}{160}$ as reference solution and obtain the error estimates on mesh $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$ respectively for different choices of p and α, where we take $p = 2, 5, 8$ and $\alpha = 0.1, 0.5, 1$. The corresponding maximal errors e^n are listed on Table 1-3. On the other hand, from Figure 1 and Figure 2, it is observed that the height of the numerical approximation to u decreases in time because of the effect of $-\alpha u_{xx}$. This observation matches the theoretic property in Lemma 1, which states that the continuous energy $E(t)$ in problems (1)-(3) decreases in time.

From the numerical results, the finite difference schemes of this paper is efficient.

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TABLE II
THE ERRORS ESTIMATES IN THE SENSE OF $||\cdot||_\infty$, WHEN $p = 5, \alpha = 0.5$ AND $\tau = 0.1$

	h=1/4	h=1/8	h=1/16	h=1/32
t=0.2	2.155891e-6	5.039306e-7	1.229071e-7	2.965984e-8
t=0.4	4.278782e-6	1.000041e-6	2.438997e-7	5.885665e-8
t=0.6	6.369140e-6	1.488442e-6	3.630035e-7	8.759581e-8
t=0.8	8.427431e-6	1.969242e-6	4.802471e-7	1.158860e-7
t=1.0	1.045411e-5	2.442550e-6	5.956573e-7	1.437343e-7

TABLE III
THE ERRORS ESTIMATES IN THE SENSE OF $||\cdot||_\infty$, WHEN $p = 8, \alpha = 1$ AND $\tau = 0.1$

	h=1/4	h=1/8	h=1/16	h=1/32
t=0.2	4.281448e-6	1.000552e-6	2.440200e-7	5.888508e-8
t=0.4	8.437922e-6	1.971254e-6	4.807251e-7	1.160026e-7
t=0.6	1.247301e-5	2.912960e-6	7.103250e-7	1.714045e-7
t=0.8	1.639022e-5	3.826495e-6	9.330202e-7	2.251376e-7
t=1.0	2.019296e-5	4.712668e-6	1.149008e-6	2.772495e-7

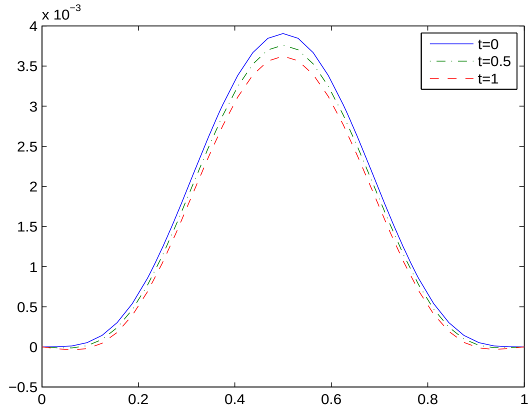

Fig. 1. Numerical solution of $u(x,t)$ with $h = \frac{1}{32}, \tau = 0.1, \alpha = 1, p = 2$

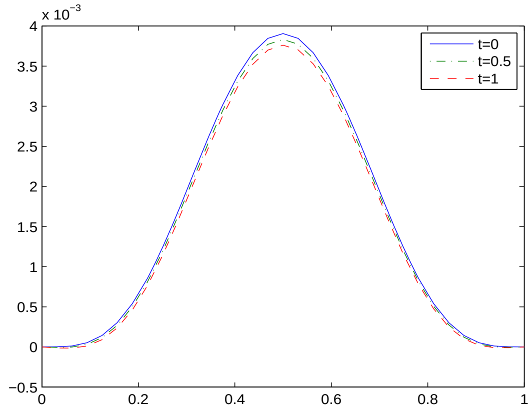

Fig. 2. Numerical solution of $u(x,t)$ with $h = \frac{1}{32}, \tau = 0.1, \alpha = 0.5, p = 5$

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