Coverage probability of confidence intervals for the normal mean and variance with restricted parameter space

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Abstract—Recent articles have addressed the problem to construct the confidence intervals for the mean of a normal distribution where the parameter space is restricted, see for example Wang [Confidence intervals for the mean of a normal distribution with restricted parameter space. Journal of Statistical Computation and Simulation, Vol. 78, No. 9, 2008, 829–841.], we derived, in this paper, analytic expressions of the coverage probability and the expected length of confidence interval for the normal mean when the whole parameter space is bounded. We also construct the confidence interval for the normal variance with restricted parameter for the first time and its coverage probability and expected length are also mathematically derived. As a result, one can use these criteria to assess the confidence interval for the normal mean and variance when the parameter space is restricted without the back up from simulation experiments.

Keywords—confidence interval, coverage probability, expected length, restricted parameter space.

I. INTRODUCTION

IT is widely known from the recent paper of Mandelkern [6] that the statistical estimation from the classical Neyman's method performs unsatisfactory when the parameter space is restricted since the Neyman's method based mainly on the statistical inference established in a natural parameter space. Mandelkern [6] gave an example in Physics shown that the standard method to construct a confidence interval for mean is not satisfactory when the parameter space is restricted. Feldman and Cousins [3] and Roe and Woodroofe [1] are also interested setting the confidence intervals in the restricted parameter space, i.e. $\mu > 0$. A recent paper of Wang [4] also proposed a new confidence interval for the normal mean when parameter space is bounded, $\mu \in (a, b), 0 < a < b$. Wang proposed a method call "rp-interval", based on his paper [5], to construct the confidence interval for the normal mean compared to other confidence intervals constructed from the standard interval, minimax interval, likelihood ratio interval and Bayesian credible interval. Wang's confidence interval performs well but that confidence interval assumed that the variance of normal distribution is known and Wang set the variance equals one in his proposed confidence interval. In practice, however, the variance is usually an unknown parameter and it must be estimated from the data. This paper emphasized with this problem and also construct the new confidence interval for the normal variance with also involved the restricted parameter space. We derived analytic expressions for

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coverage probabilities and expected lengths of our proposed confidence intervals.

The paper is organized as follows. Section 2 presents the confidence intervals for the normal mean. The confidence interval for the normal mean with restricted parameter space is investigated in section 3. Coverage probabilities and expected lengths of each interval in section 3 is mathematically outlined in section 4. Section 5 gives coverage probability and expected lengths of confidence interval for the normal mean with restricted parameter space. The confidence interval for the normal variance is outlined in section 6. Coverage probabilities and expected lengths of each interval in section 6 is mathematically outlined in section 7. Section 8 gives coverage probability and expected lengths of confidence interval for the normal varaince with restricted parameter space. Section 9 contains a discussion of the results and conclusions.

II. CONFIDENCE INTERVALS FOR NORMAL POPULATION MEAN

Let $X = [X_1, \ldots, X_n]$ be a random sample from the normal distribution with mean μ and standard deviation σ. The sample mean and variance for X are, respectively, denoted as $\bar{X}$ and S^2 when $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$, and $S^2 = (n-1)^{-1} \sum_{i=1}^{n} \left(X_i - \bar{X} \right)^2$. We are interested in $100(1-\alpha)\%$ confidence interval for μ.

1) Standard confidence interval for μ: In practice, σ^2 is an unknown parameter and S^2 is an unbiased estimator for σ^2. In this case, a well-known $100(1 - \alpha)\%$ confidence for μ is

$$CI_\mu = \left[\bar{X} - c\frac{S}{\sqrt{n}}, \bar{X} + c\frac{S}{\sqrt{n}} \right] \qquad (1)$$

where c is $t_{1-\alpha/2}$, an upper $1 - \alpha/2$ percentiles of the t-distribution with n-1 degrees of freedom. This confidence interval is optimal see e.g. Casella and Berker [2]. But in case a restricted parameter space, i.e., a bounded mean, $a < \mu < b$, it is doubtful that whether or not this confidence interval is optimal and satisfactory, see Wang [4]. In the next section we review a method to construct a confidence interval for μ when $a < \mu < b$.

III. CONFIDENCE INTERVALS FOR NORMAL POPULATION MEAN WITH RESTRICTED PARAMETER SPACE

It is widely accepted that a confidence interval for μ when $a < \mu < b$ is the confidence interval of the intersection

between CI_μ and $a < \mu < b$ that is given by

$$CI_a = \left[\max\left(a, \bar{X} - c\frac{S}{\sqrt{n}}\right), \min\left(b, \bar{X} + c\frac{S}{\sqrt{n}}\right) \right] \quad (2)$$

There are four possible results for confidence interval (2) as follows:

Case 1a), if $a > \bar{X} - c\frac{S}{\sqrt{n}}$ and $b > \bar{X} + c\frac{S}{\sqrt{n}}$ then CI_a is reduced to

$$CI_{a1} = \left[a, \bar{X} + c\frac{S}{\sqrt{n}} \right]. \quad (3)$$

Case 1b), if $a > \bar{X} - c\frac{S}{\sqrt{n}}$ and $b < \bar{X} + c\frac{S}{\sqrt{n}}$ then CI_a is reduced to

$$CI_{a2} = \left[a, b \right]. \quad (4)$$

Case 1c), if $a < \bar{X} - c\frac{S}{\sqrt{n}}$ and $b > \bar{X} + c\frac{S}{\sqrt{n}}$ then CI_a is reduced to

$$CI_{a3} = \left[\bar{X} - c\frac{S}{\sqrt{n}}, \bar{X} + c\frac{S}{\sqrt{n}} \right]. \quad (5)$$

Case 1d), if $a < \bar{X} - c\frac{S}{\sqrt{n}}$ and $b < \bar{X} + c\frac{S}{\sqrt{n}}$ then CI_a is reduced to

$$CI_{a3} = \left[\bar{X} - c\frac{S}{\sqrt{n}}, b \right]. \quad (6)$$

In the next section, we will mathematically investigate the coverage probability and the expected length of the intervals (1) and (3)-(6).

IV. COVERAGE PROBABILITIES AND EXPECTED LENGTH OF EACH CONFIDENCE INTERVAL FOR MEAN

2) Coverage probability and Expected length of CI_μ:
Following Niwitpong and Niwitpong [7], we now derive an analytic expression for the coverage probability for CI_μ. Let $P(\mu \in CI_\mu)$ be the coverage probability of confidence interval CI_μ then an analytic expression for this confidence coverage is given by

$$
\begin{aligned}
P(\mu &\in CI_\mu) \\
&= P\left[\bar{X} - c\frac{S}{\sqrt{n}} < \mu < \bar{X} + c\frac{S}{\sqrt{n}} \right] \\
&= P\left[-c\frac{S}{\sqrt{n}} < \mu - \bar{X} < c\frac{S}{\sqrt{n}} \right] \\
&= P\left[\frac{-c\frac{S}{\sqrt{n}}}{\frac{\sigma}{\sqrt{n}}} < \frac{\mu - \bar{X}}{\frac{\sigma}{\sqrt{n}}} < \frac{c\frac{S}{\sqrt{n}}}{\frac{\sigma}{\sqrt{n}}} \right] \\
&= P\left[\frac{-cS}{\sigma} < Z < \frac{cS}{\sigma} \right] \\
&= P[A_1 < Z < A_2] \\
&= E[I_{\{A_1 < Z < A_2\}}(\tau)] \\
&= E[E[I_{\{A_1 < Z < A_2\}}(\tau)]|S^2] \\
&= E[\Phi(A_2) - \Phi(A_1)]
\end{aligned}
$$

where

$$I_{\{A_1 < Z < A_2\}}(\tau) = \begin{cases} 1, & \text{if } \tau \in \{A_1 < Z < A_2\} \\ 0, & \text{otherwise} \end{cases}$$

$A_1 = \frac{-cS}{\sigma}$, $A_2 = \frac{cS}{\sigma}$, Z is the standard normal distribution and $\Phi(\cdot)$ is a standard normal function. The expected length of CI_μ is therefore

$$E\left(2c\frac{S}{\sqrt{n}}\right) = \frac{2c}{\sqrt{n}}E(s) = \frac{2c}{\sqrt{n}}\frac{\sqrt{n-1}\Gamma((n-1)/2)}{\sqrt{2}\Gamma(n/2)}\sigma$$

3) Coverage probability and Expected length of CI_{a1}:
Similarly to CI_μ, we now derive an analytic expression for the coverage probability for CI_{a1}. Let $P(\mu \in CI_{a1})$ be the coverage probability of confidence interval CI_{a1} then an analytic expression for this confidence coverage is given by

$$
\begin{aligned}
P(\mu &\in CI_{a1}) \\
&= P\left[a < \mu < \bar{X} + c\frac{S}{\sqrt{n}} \right] \\
&= P\left[a - \bar{X} < \mu - \bar{X} < c\frac{S}{\sqrt{n}} \right] \\
&= P\left[\frac{a - \bar{X}}{\frac{\sigma}{\sqrt{n}}} < \frac{\mu - \bar{X}}{\frac{\sigma}{\sqrt{n}}} < \frac{c\frac{S}{\sqrt{n}}}{\frac{\sigma}{\sqrt{n}}} \right] \\
&= P\left[\frac{-cS}{\sigma} < Z < \frac{-a + \bar{X}}{\frac{\sigma}{\sqrt{n}}} \right] \\
&= P[B_1 < Z < B_2] \\
&= E[I_{\{B_1 < Z < B_2\}}(\tau)] \\
&= E[E[I_{\{B_1 < Z < B_2\}}(\tau)]|S^2] \\
&= E[\Phi(B_2) - \Phi(B_1)]
\end{aligned}
$$

where

$$I_{\{B_1 < Z < B_2\}}(\tau) = \begin{cases} 1, & \text{if } \tau \in \{B_1 < Z < B_2\} \\ 0, & \text{otherwise} \end{cases}$$

$B_1 = \frac{-cS}{\sigma}$, $B_2 = \frac{-a + \bar{X}}{\frac{\sigma}{\sqrt{n}}}$. The expected length of CI_{a1} is therefore

$$
\begin{aligned}
E(CI_\mu) &= E\left(\bar{X} - a - c\frac{S}{\sqrt{n}} \right) \\
&= \mu - a + \frac{c}{\sqrt{n}}E(S) \\
&= \mu - a + \frac{c}{\sqrt{n}}\frac{\sqrt{n-1}\Gamma((n-1)/2)}{\sqrt{2}\Gamma(n/2)}\sigma
\end{aligned}
$$

4) Coverage probability and Expected length of CI_{a2}:
Similarly to CI_μ, we now derive an analytic expression for the coverage probability for CI_{a2}. Let $P(\mu \in CI_{a2})$ be the coverage probability of confidence interval CI_{a2} then an analytic expression for this confidence coverage is given by

$$P(\mu \in CI_{a2})$$
$$= P\left[a < \mu < b\right]$$
$$= P\left[a - \bar{X} < \mu - \bar{X} < a - \bar{X}\right]$$
$$= P\left[\frac{a - \bar{X}}{\frac{\sigma}{\sqrt{n}}} < \frac{\mu - \bar{X}}{\frac{\sigma}{\sqrt{n}}} < \frac{b - \bar{X}}{\frac{\sigma}{\sqrt{n}}}\right]$$
$$= P\left[\frac{-b + \bar{X}}{\frac{\sigma}{\sqrt{n}}} < Z < \frac{-a + \bar{X}}{\frac{\sigma}{\sqrt{n}}}\right]$$
$$= P[C_1 < Z < C_2]$$
$$= E[I_{\{C_1 < Z < C_2\}}(\tau)]$$
$$= E[E[I_{\{C_1 < Z < C_2\}}(\tau)]|S^2]$$
$$= E[\Phi(C_2) - \Phi(C_1)]$$

where

$$I_{\{C_1 < Z < C_2\}}(\tau) = \begin{cases} 1, & \text{if } \tau \in \{C_1 < Z < C_2\} \\ 0, & \text{otherwise} \end{cases}$$

$C_1 = \frac{-b + \bar{X}}{\frac{\sigma}{\sqrt{n}}}$, $C_2 = \frac{-a + \bar{X}}{\frac{\sigma}{\sqrt{n}}}$. The expected length of CI_{a2} is therefore $E(CI_{a2} = E(b - c) = b - c$.

5) Coverage probability and Expected length of CI_{a3}: It is easily seen that the coverage probability and expected length of CI_{a3} is the same as CI_μ.

6) Coverage probability and Expected length of CI_{a4}: Similarly to CI_μ, we now derive an analytic expression for the coverage probability for CI_{a4}. Let $P(\mu \in CI_{a4})$ be the coverage probability of confidence interval CI_{a4} then an analytic expression for this confidence coverage is given by

$$P(\mu \in CI_{a4})$$
$$= P\left[\bar{X} - c\frac{S}{\sqrt{n}} < \mu < b\right]$$
$$= P\left[-c\frac{S}{\sqrt{n}} < \mu - \bar{X} < b - \bar{X}\right]$$
$$= P\left[\frac{\frac{-b + \bar{X}}{\sqrt{n}}}{\frac{\sigma}{\sqrt{n}}} < \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} < \frac{c\frac{S}{\sqrt{n}}}{\frac{\sigma}{\sqrt{n}}}\right]$$
$$= P[D_1 < Z < D_2]$$
$$= E[I_{\{D_1 < Z < D_2\}}(\tau)]$$
$$= E[E[I_{\{D_1 < Z < D_2\}}(\tau)]|S^2]$$
$$= E[\Phi(D_2) - \Phi(D_1)]$$

where

$$I_{\{D_1 < Z < D_2\}}(\tau) = \begin{cases} 1, & \text{if } \tau \in \{D_1 < Z < D_2\} \\ 0, & \text{otherwise} \end{cases}$$

$D_1 = \frac{\frac{-b + \bar{X}}{\sqrt{n}}}{\frac{\sigma}{\sqrt{n}}}$, $D_2 = \frac{c\frac{S}{\sqrt{n}}}{\frac{\sigma}{\sqrt{n}}}$.

The expected length of CI_{a4} is therefore

$$E(CI_{a4}) = E\left(b - \bar{X} + c\frac{S}{\sqrt{n}}\right)$$
$$= b - \mu + \frac{c}{\sqrt{n}}E(S)$$
$$= b - \mu + \frac{c}{\sqrt{n}}\frac{\sqrt{n - 1}\Gamma((n - 1)/2)}{\sqrt{2}\Gamma(n/2)}\sigma.$$

V. COVERAGE PROBABILITY AND EXPECTED LENGTH OF CI_a

From subsections 3-6, we have derived the coverage probability and expected length of each interval. We now propose the coverage probability for CI_a, that is the coverage probability of the confidence interval for μ with a bounded mean, as follow;

$$P(\mu \in CI_a) = P(\mu \in CI_a|Case1a)$$
$$+ P(\mu \in CI_a|Case1b)$$
$$+ P(\mu \in CI_a|Case1c)$$
$$+ P(\mu \in CI_a|Case1d)$$
$$= P(\mu \in CI_{a1}) + P(\mu \in CI_{a2})$$
$$+ P(\mu \in CI_{a3}) + P(\mu \in CI_{a4})$$
$$= 1 - \alpha.$$

Also the expected length of CI_a is given by

$$E(\mu \in CI_a) = E(\mu \in CI_a|Case1a)$$
$$+ E(\mu \in CI_a|Case1b)$$
$$+ E(\mu \in CI_a|Case1c)$$
$$+ E(\mu \in CI_a|Case1d)$$
$$= E(\mu \in CI_{a1}) + E(\mu \in CI_{a2})$$
$$+ E(\mu \in CI_{a3}) + E(\mu \in CI_{a4})$$
$$= E(CI_{a1}) + E(CI_{a2}) + E(CI_{a3})$$
$$+ E(CI_{a4}).$$

VI. CONFIDENCE INTERVALS FOR NORMAL POPULATION VARIANCE

As in the previous section, in this section we are concerning mainly on constructing $100(1-\alpha)\%$ confidence interval for σ^2.

7) Standard confidence interval for σ^2: It is well known that a standard $100(1-\alpha)\%$ confidence interval for σ^2 is (see e.g. Casella and Berger [2])

$$CI_{\sigma^2} = \left[\frac{(n - 1)S_x^2}{\chi_{1-\alpha/2, n-1}^2}, \frac{(n - 1)S_x^2}{\chi_{\alpha/2, n-1}^2}\right] \quad (7)$$

where $\chi_{\alpha/2, n-1}^2$ is an $\alpha/2$ percentiles of the Chi-square distribution with $n-1$ degrees of freedom. Confidence interval CI_{σ^2} is doubtful whether it works or not when there is a bounded mean say $\mu \in (a, b), 0 < a < b.$, see the same argument for the standard interval for μ see e.g. Wang [4], CI_{σ^2} is derived from the classical procedure for the natural parameter space but this interval (7) is not suitable for the domain of the bounded parameter space. In the next section we carefully describe this situation.

8) Confidence interval for σ^2 with a bounded mean :
We begin this section by considering the relation between a bounded mean and a bounded variance. In other words, we will show now if there is a bounded mean then also the variance is bounded. Consider,

$$
\begin{aligned}
a < \mu < b \quad &\rightarrow \quad a^2 < \mu^2 < b^2 \\
&\rightarrow \quad -b^2 < -\mu^2 < -a^2 \\
&\rightarrow \quad \frac{\sum X_i^2}{N} - b^2 < \frac{\sum X_i^2}{N} - \mu^2 < \frac{\sum X_i^2}{N} - a^2 \\
&\rightarrow \quad \sigma_b^2 < \sigma^2 < \sigma_a^2
\end{aligned}
$$

where $\sigma_a^2 = \frac{\sum X_i^2}{N} - a^2$ and $\sigma_b^2 = \frac{\sum X_i^2}{N} - b^2$. Hence the variance of X is also bounded, i.e. $\sigma_b^2 < \sigma^2 < \sigma_a^2$.

According to Wang [4], the confidence interval for σ^2 with bounded mean is

$$
CI_b = \left[\max\left(\sigma_b^2, \frac{(n-1)S_x^2}{\chi^2_{1-\alpha/2,n-1}} \right), \min\left(\sigma_a^2, \frac{(n-1)S_x^2}{\chi^2_{\alpha/2,n-1}} \right) \right] \quad (8)
$$

which is the intersection of (σ_b^2, σ_a^2) and confidence interval CI_{σ^2}, a confidence interval for σ^2. There are four possible results for confidence interval (8) as follows:

Case 1, if $\sigma_b^2 > \frac{(n-1)S_x^2}{\chi^2_{1-\alpha/2,n-1}}$ and $\sigma_a^2 > \frac{(n-1)S_x^2}{\chi^2_{\alpha/2,n-1}}$ then CI_b is reduced to

$$
CI_{b1} = \left[\sigma_b^2, \frac{(n-1)S_x^2}{\chi^2_{\alpha/2,n-1}} \right]. \quad (9)
$$

Case 2, if $\sigma_b^2 > \frac{(n-1)S_x^2}{\chi^2_{1-\alpha/2,n-1}}$ and $\sigma_a^2 < \frac{(n-1)S_x^2}{\chi^2_{\alpha/2,n-1}}$ then CI_b is reduced to

$$
CI_{b2} = \left[\sigma_b^2, \sigma_a^2 \right]. \quad (10)
$$

Case 3, if $\sigma_b^2 < \frac{(n-1)S_x^2}{\chi^2_{1-\alpha/2,n-1}}$ and $\sigma_a^2 > \frac{(n-1)S_x^2}{\chi^2_{\alpha/2,n-1}}$ then CI_b is reduced to

$$
CI_{b3} = \left[\frac{(n-1)S_x^2}{\chi^2_{1-\alpha/2,n-1}}, \frac{(n-1)S_x^2}{\chi^2_{\alpha/2,n-1}} \right]. \quad (11)
$$

Case 4, if $\sigma_b^2 < \frac{(n-1)S_x^2}{\chi^2_{1-\alpha/2,n-1}}$ and $\sigma_a^2 < \frac{(n-1)S_x^2}{\chi^2_{\alpha/2,n-1}}$ then CI_b is reduced to

$$
CI_{b4} = \left[\frac{(n-1)S_x^2}{\chi^2_{1-\alpha/2,n-1}}, \sigma_a^2 \right]. \quad (12)
$$

In the next section, we will mathematically investigate the coverage probability and expected length of the intervals (7) and (9)-(12).

VII. Coverage probabilities and Expected length of each confidence interval for variance

We begin with the well-known χ^2_{n-1} random variable whose the expected value and the variance are respectively $E(\chi^2_{n-1}) = n - 1$ and $Var(\chi^2_{n-1}) = 2(n-1)$. Define

$$
Z = \frac{(n-1)S^2/\sigma^2 - (n-1)}{\sqrt{2(n-1)}} = \frac{S^2 - \sigma^2}{\sqrt{Var(S^2)}}
$$

when $Var(S^2) = \sigma^4/2(n-1)$ and the test statistic, Z is the standard normal distribution.

9) Coverage probability and Expected length of CI_{σ^2}:
Following Niwitpong and Niwitpong [7], we now derive an analytic expression for the coverage probability for CI_{σ^2}. Let $P(\sigma^2 \in CI_{\sigma^2})$ be the coverage probability of confidence interval CI_{σ^2}, $c1 = \chi^2_{1-\alpha/2,n-1}$ and $c2 = \chi^2_{\alpha/2,n-1}$ then an analytic expression for this confidence coverage is given by

$$
\begin{aligned}
&P(\sigma^2 \in CI_{\sigma^2}) \\
=\ &P\left[\frac{(n-1)S^2}{c1} < \sigma^2 < \frac{(n-1)S^2}{c2} \right] \\
=\ &P\left[-\frac{(n-1)S^2}{c1} > -\sigma^2 > -\frac{(n-1)S^2}{c2} \right] \\
=\ &P\left[S^2 - \frac{(n-1)S^2}{c1} > S^2 - \sigma^2 > S^2 - \frac{(n-1)S^2}{c2} \right] \\
=\ &P\left[\frac{c1S^2 - (n-1)S^2}{c1\sqrt{Var(S^2)}} > \frac{S^2 - \sigma^2}{\sqrt{Var(S^2)}} > \frac{c2S^2 - (n-1)S^2}{c2\sqrt{Var(S^2)}} \right] \\
=\ &P[W_1 < Z < W_2] \\
=\ &E[I_{\{W_1 < Z < W_2\}}(\tau)] \\
=\ &E[E[I_{\{W_1 < Z < W_2\}}(\tau)]|S^2] \\
=\ &E[\Phi(W_2) - \Phi(W_1)]
\end{aligned}
$$

where

$$
I_{\{W_1 < Z < W_2\}}(\tau) = \begin{cases} 1, & \text{if } \tau \in \{W_1 < Z < W_2\} \\ 0, & \text{otherwise} \end{cases}
$$

$W_1 = \frac{c2S^2 - (n-1)S^2}{c2\sqrt{Var(S^2)}}$, $W_2 = \frac{c1S^2 - (n-1)S^2}{c1\sqrt{Var(S^2)}}$ and $\Phi(\cdot)$ is a standard normal function.

The expected length of CI_{σ^2} is therefore

$$
E(CI_{\sigma^2}) = E\left(\frac{(n-1)S^2}{c2} - \frac{(n-1)S^2}{c1} \right) \quad (13)
$$

$$
= (n-1)\sigma^2 \left(\frac{1}{c2} - \frac{1}{c1} \right) \quad (14)
$$

where $E(S^2) = \sigma^2$.

10) Coverage probability and Expected length of CI_{b1}:
Similarly to CI_{σ^2}, we now derive an analytic expression for the coverage probability for CI_{b1}. Let $P(\sigma^2 \in CI_{b1})$ be the coverage probability of confidence interval CI_{b1} then an analytic expression for this confidence coverage is given by

$$
\begin{aligned}
&P[\sigma^2 \in CI_{b1}] \\
=\ &P\left[\sigma_b^2 < \sigma^2 < \frac{(n-1)S^2}{c2} \right] \\
=\ &P\left[-\sigma_b^2 > -\sigma^2 > -\frac{(n-1)S^2}{c2} \right] \\
=\ &P\left[S^2 - \sigma_b^2 > S^2 - \sigma^2 > S^2 - \frac{(n-1)S^2}{c2} \right]
\end{aligned}
$$

$$= P\left[\frac{S^2 - \sigma_b^2}{\sqrt{Var(S^2)}} > \frac{S^2 - \sigma^2}{\sqrt{Var(S^2)}} > \frac{c2S^2 - (n-1)S^2}{c2\sqrt{Var(S^2)}}\right]$$

$$= P[R_1 < Z < R_2]$$

$$= E[I_{\{R_1 < Z < R_2\}}(\tau)]$$

$$= E[E[I_{\{R_1 < Z < R_2\}}(\tau)]|S^2]$$

$$= E[\Phi(R_2) - \Phi(R_1)]$$

where

$$I_{\{R_1 < Z < R_2\}}(\tau) = \begin{cases} 1, & \text{if } \tau \in \{R_1 < Z < R_2\} \\ 0, & \text{otherwise} \end{cases}$$

$R_1 = \frac{c2S^2 - (n-1)S^2}{c2\sqrt{Var(S^2)}}$ and $R_2 = \frac{S^2 - \sigma_b^2}{\sqrt{Var(S^2)}}$.

The expected length of CI_{b1} is therefore

$$E(CI_{\sigma^2}) = E\left(\frac{(n-1)S^2}{c2} - \sigma_b^2\right) \tag{15}$$

$$= \frac{(n-1)\sigma^2}{c2} - \sigma_b^2 \tag{16}$$

11) Coverage probability and Expected length of CI_{b2}:
Similarly to CI_{σ^2}, we now derive an analytic expression for the coverage probability for CI_{b1}. Let $P(\sigma^2 \in CI_{b2})$ be the coverage probability of confidence interval CI_{b2} then an analytic expression for this confidence coverage is given by

$$P[\sigma^2 \in CI_{b2}]$$

$$= P\left[\sigma_b^2 < \sigma^2 < \sigma_a^2\right]$$

$$= P\left[-\sigma_b^2 > -\sigma^2 > -\sigma_a^2\right]$$

$$= P\left[S^2 - \sigma_b^2 > S^2 - \sigma^2 > S^2 - \sigma_a^2\right]$$

$$= P\left[\frac{S^2 - \sigma_b^2}{\sqrt{Var(S^2)}} > \frac{S^2 - \sigma^2}{\sqrt{Var(S^2)}} > \frac{S^2 - \sigma_a^2}{\sqrt{Var(S^2)}}\right]$$

$$= P[Q_1 < Z < Q_2]$$

$$= E[I_{\{Q_1 < Z < Q_2\}}(\tau)]$$

$$= E[E[I_{\{Q_1 < Z < Q_2\}}(\tau)]|S^2]$$

$$= E[\Phi(Q_2) - \Phi(Q_1)]$$

where

$$I_{\{Q_1 < Z < Q_2\}}(\tau) = \begin{cases} 1, & \text{if } \tau \in \{Q_1 < Z < Q_2\} \\ 0, & \text{otherwise} \end{cases}$$

$Q_1 = \frac{S^2 - \sigma_a^2}{\sqrt{Var(S^2)}}$ and $Q_2 = \frac{S^2 - \sigma_b^2}{\sqrt{Var(S^2)}}$.

The expected length of CI_{b2} is therefore

$$E(CI_{b2}) = E(\sigma_a^2 - \sigma_b^2) \tag{17}$$

$$= \sigma_a^2 - \sigma_b^2 \tag{18}$$

12) Coverage probability and Expected length of CI_{b3}:
It is easily seen that the coverage probability and expected length of CI_{b3} is the same as CI_{σ^2}.

13) Coverage probability and Expected length of CI_{b4}:
Similarly to CI_{σ^2}, we now derive an analytic expression for the coverage probability for CI_{b4}. Let $P(\sigma^2 \in CI_{b4})$ be the coverage probability of confidence interval CI_{b4} then an analytic expression for this confidence coverage is given by

$$P[\sigma^2 \in CI_{b4}]$$

$$= P\left[\frac{(n-1)S^2}{c1} < \sigma^2 < \sigma_a^2\right]$$

$$= P\left[-\frac{(n-1)S^2}{c1} > -\sigma^2 > -\sigma_a^2\right]$$

$$= P\left[S^2 - \frac{(n-1)S^2}{c1} > S^2 - \sigma^2 > S^2 - \sigma_a^2\right]$$

$$= P\left[\frac{c1S^2 - (n-1)S^2}{c1\sqrt{Var(S^2)}} > \frac{S^2 - \sigma^2}{\sqrt{Var(S^2)}} > \frac{S^2 - \sigma_a^2}{\sqrt{Var(S^2)}}\right]$$

$$= P[K_1 < Z < K_2]$$

$$= E[I_{\{K_1 < Z < K_2\}}(\tau)]$$

$$= E[E[I_{\{K_1 < Z < K_2\}}(\tau)]|S^2]$$

$$= E[\Phi(K_2) - \Phi(K_1)]$$

where

$$I_{\{K_1 < Z < K_2\}}(\tau) = \begin{cases} 1, & \text{if } \tau \in \{K_1 < Z < K_2\} \\ 0, & \text{otherwise} \end{cases}$$

$K_1 = \frac{S^2 - \sigma_a^2}{\sqrt{Var(S^2)}}$ and $K_2 = \frac{c1S^2 - (n-1)S^2}{c1\sqrt{Var(S^2)}}$.

The expected length of CI_{b4} is therefore

$$E(CI_{b4}) = E\left(\sigma_a^2 - \frac{(n-1)S^2}{c1}\right) \tag{19}$$

$$= \sigma_a^2 - \frac{(n-1)\sigma^2}{c1} \tag{20}$$

VIII. COVERAGE PROBABILITY AND EXPECTED LENGTH OF CI_b

From subsections (10)-(13), we have derived the coverage probability and expected length of each interval. We now propose the coverage probability for CI_b, that is the coverage probability of the confidence interval for σ^2 with a bounded mean, as follow;

$$\begin{aligned} P(\sigma^2 \in CI_b) &= P(\sigma^2 \in CI_b | Case1) \\ &+ P(\sigma^2 \in CI_b | Case2) \\ &+ P(\sigma^2 \in CI_b | Case3) \\ &+ P(\sigma^2 \in CI_b | Case1) \\ &= P(\sigma^2 \in CI_{b1}) + P(\sigma^2 \in CI_{b2}) \\ &+ P(\sigma^2 \in CI_{b3}) + P(\sigma^2 \in CI_{b4}) \\ &= 1 - \alpha \end{aligned}$$

Also the expected length of CI_b is given by

$$
\begin{aligned}
E(\sigma^2 \in CI_b) &= E(\sigma^2 \in CI_b | Case1) \\
&+ E(\sigma^2 \in CI_b | Case2) \\
&+ E(\sigma^2 \in CI_b | Case3) \\
&+ E(\sigma^2 \in CI_b | Case1) \\
&= E(\sigma^2 \in CI_{b1}) + E(\sigma^2 \in CI_{b2}) \\
&+ E(\sigma^2 \in CI_{b3}) + E(\sigma^2 \in CI_{b4}) \\
&= E(CI_{b1}) + E(CI_{b2}) + E(CI_{b3}) \\
&+ E(CI_{b4}).
\end{aligned}
$$

IX. Conclusion

In this paper we proposed new confidence intervals for the normal population mean and variance with restricted parameter space. We derived, mathematically, coverage probabilities and expected lengths of these intervals. It is shown in sections 5 and 8 that the coverage probabilities of CI_μ and CI_{σ^2} are equal to the nominal value $1 - \alpha$. To compare confidence intervals between CI_μ and CI_a and CI_{σ^2} and CI_b, one can easily find a ratio of expected lengths of each interval derived in sections IV and V and sections VII and VIII respectively. Further research may consider statistical estimation for the difference between normal means and normal variances with restricted parameter spaces. Also one might consider to derive their coverage probabilities and expected lengths as shown in this paper.

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