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Coupled Lateral-Torsional Free Vibrations Analysis of Laminated Composite Beam using Differential Quadrature Method

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Abstract**—**In this paper the Differential Quadrature Method (DQM) is employed to study the coupled lateral-torsional free vibration behavior of the laminated composite beams. In such structures due to the fiber orientations in various layers, the lateral displacement leads to a twisting moment. The coupling of lateral and torsional vibrations is modeled by the bending-twisting material coupling rigidity. In the present study, in addition to the material coupling, the effects of shear deformation and rotary inertia are taken into account in the definition of the potential and kinetic energies of the beam. The governing differential equations of motion which form a system of three coupled PDEs are solved numerically using DQ procedure under different boundary conditions consist of the combinations of simply, clamped, free and other end conditions. The resulting natural frequencies and mode shapes for cantilever beam are compared with similar results in the literature and good agreement is achieved.

Keywords**—**Differential Quadrature Method, Free vibration, Laminated composite beam, Material coupling.

I. INTRODUCTION

COMPOSITE structures like beams, panels and plates are extensively used in various fields of engineering such as aerospace, mechanical, civil and mining engineering. Mechanical properties like as high strength/stiffness to weight ratio and excellent fatigue strength of composite materials have increased their applications in the construction of several structures. Therefore, the vibrational behavior of composite structures has been studied by many researchers in recent years. Free vibration analysis of the simple laminated composite beams started by Abarcar [1], Mansfield [2] and Teh [3] in 1970's. They neglected the effects of shear deformation and rotary inertia in their studies. It is known that when the cross-sectional dimensions of the beam are large or higher frequencies of the beam are studied, the effects of shear deformation and rotary inertia must be taken into account similar to the Timoshenko beam theory. Furthermore, low shear moduli of fibrous composites that results in low shear stiffness of the beam, intensifies this requirement. Also in composite beams, because of the ply orientation and stacking sequence of the fibers imbedded in continuous resin media, the effect of bending-torsion material coupling should be considered [1,4,5].

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This effect adds some additional terms and an additional equation to the equations of motion of the metallic Timoshenko beam and so they become more complicated to solve. Using some numerical approaches, the vibrational behavior of composite beams based on the Timoshenko beam theory was studied [6, 7]. Bank and Kao [8] studied the free and forced vibrations of the thin-walled fiber-reinforced composite material beams using the Timoshenko beam theory. Williams and Banerjee [9] and Banerjee [10] developed the dynamic stiffness matrix method for the problems of free vibration of composite Timoshenko beam and axially loaded composite Timoshenko beam, respectively. Moreover the later is analyzed by Kaya and Ozdemir Ozgumus [11] using the differential transform method (DTM).

Among the various numerical methods the differential quadrature method (DQM) which was introduced by Bellman and Casti [12] is a powerful method for solving initial and boundary value problems. This method needs less computational efforts as compared with the other numerical methods such as finite element method and finite difference method. One of the advantages of this method is the use of less grid points with acceptable accurate solutions of the differential equations. The method firstly used by Bert et al [13] for solving problems in structural mechanics and then has been widely used for static and free vibration analysis of beams and plates in various problems. Its early development and some of its application can be found in review papers by Bert and Malik [14, 15].

In this paper the DQ procedure is developed for free vibration analysis of coupled lateral-torsional vibrations of laminated composite beam according to the Timoshenko beam theory. This paper is organized as follows. Firstly the equations of motion and the boundary conditions are derived. Then the differential quadrature method is used to discretize the equations of motion as well as the boundary conditions. Using this procedure an eigenvalue problem is obtained that its solution represents the natural frequencies and mode shapes of the beam under the corresponding boundary conditions. Then the acquired results for cantilever beam are compared with the presenting results in literature where they are in good agreement. Also the convergence of the solutions is studied in order to examine the accuracy of the method where it can be seen that the results rapidly converge together. The free vibration analysis of the beam under the other boundary conditions is investigated and the numerical results are presented.

II. FORMULATION

The differential equations of motion for free vibrations of a laminated composite beam can be easily derived using Hamilton's principle. According to this principle the integration of the Lagrangian of a dynamical system on any arbitrary interval of time is stationary, i.e.

$$\delta\int_{t_1}^{t_2}(U_k-U_p)\,dt=0 \tag{1}$$

where U_k and U_p are the kinetic and the potential energies, respectively. Considering w as transverse deflection, θ as bending rotation and ψ as twist angle of the beam, the total kinetic energy U_k of the beam is given by

$$U_k=\tfrac{1}{2}\int_0^L\left[\rho A(w_t)^2+I_\alpha(\psi_t)^2+\rho I(\theta_t)^2\right]dx \tag{2}$$

and the total potential energy of the beam is

$$\begin{aligned}U_p=\tfrac{1}{2}\int_0^L&[\kappa AG(w_x-\theta)^2+GJ(\psi_x)^2\\&+2K\theta_x\psi_x+EI(\theta_x)^2]\,dx\end{aligned} \tag{3}$$

where, ρ is the density of the material, A is the cross-sectional area, I_α is the polar mass moment of inertia per unit length, I is the second moment of area of the beam cross-section, EI is the bending rigidity, GJ is torsional rigidity, K is bending-torsion coupling rigidity, L is the length of the beam and κAG is shear rigidity of the material (includes shear correction factor) and differentiation with respect to space and time are shown by indices x and t respectively. Substituting Eqs. (2,3) into Eq. (1), using integration by parts and simplifying the results, the equations of motion for the laminated composite beam are derived in the following form

$$EI\theta_{xx}+\kappa AG(w_x-\theta)+K\psi_{xx}-\rho I\theta_{tt}=0 \tag{4}$$

$$\kappa AG(w_{xx}-\theta_x)-\rho Aw_{tt}=0 \tag{5}$$

$$GJ\psi_{xx}+K\theta_{xx}-I_\alpha\psi_{tt}=0 \tag{6}$$

Beside the above coupled PDEs, geometric and natural BCs must be taken into account. Notice that natural BCs include the values of shear force S, bending moment M and twisting torque T at the boundaries. These quantities are represented by the following expressions

$$\begin{aligned}S&=\kappa AG(w_x-\theta)\\M&=EI\theta_x+K\psi_x\\T&=K\theta_x+GJ\psi_x\end{aligned} \tag{7}$$

Now, assuming synchronous motion in which the general shape of the beam does not change with time. Mathematically, this implies that the unknown functions w, θ and ψ are separable in space and time

$$\begin{aligned}w(x,t)&=W(x)\Phi(t)\\\theta(x,t)&=\Theta(x)\Phi(t)\\\psi(x,t)&=\Psi(x)\Phi(t)\end{aligned} \tag{8}$$

Substituting these functions into Eqs. (4-6) leads to

$$\Phi_{tt}+\omega^2\Phi=0 \tag{9}$$

which shows a harmonic motion and the following coupled ODEs

$$EI\Theta_{xx}+\kappa AG(W_x-\Theta)+K\Psi_{xx}+\rho I\omega^2\Theta=0 \tag{10}$$

$$\kappa AG(W_{xx}-\Theta_x)+\rho A\omega^2W=0 \tag{11}$$

$$GJ\Psi_{xx}+K\Theta_{xx}+I_\alpha\omega^2\Psi=0 \tag{12}$$

These ODEs have the following non-dimensional form

$$s\tilde{\theta}_{\tilde{x}\tilde{x}}+\tilde{w}_{\tilde{x}}+(brs-1)\tilde{\theta}+k_bs\tilde{\psi}_{\tilde{x}\tilde{x}}=0 \tag{13}$$

$$\tilde{w}_{\tilde{x}\tilde{x}}-\tilde{\theta}_{\tilde{x}}+bs\tilde{w}=0 \tag{14}$$

$$\tilde{\psi}_{\tilde{x}\tilde{x}}+k_t\tilde{\theta}_{\tilde{x}\tilde{x}}+a\tilde{\psi}=0 \tag{15}$$

where the non-dimensional parameters are defined as

$$\begin{aligned}&\tilde{x}=\frac{x}{L}\,,\quad \tilde{w}=\frac{W}{L}\,,\quad \tilde{\theta}=\Theta\,,\quad \tilde{\psi}=\Psi\\&k_t=\frac{K}{GJ}\,,\quad k_b=\frac{K}{EI}\,,\quad a=\frac{I_\alpha\omega^2L^2}{GJ}\,,\ b=\frac{\rho A\omega^2L^4}{EI}\\&r=\frac{I}{AL^2}\,,\quad s=\frac{EI}{\kappa AGL^2}\end{aligned} \tag{16}$$

Consequently, the non-dimensional shear force, bending moment and twisting torque are

$$\tilde{S}=\tilde{w}_{\tilde{x}}-\tilde{\theta}\,,\quad \tilde{M}=\tilde{\theta}_{\tilde{x}}+k_b\tilde{\psi}_{\tilde{x}}\,,\ \tilde{T}=\tilde{\theta}_{\tilde{x}}+\tilde{\psi}_{\tilde{x}} \tag{17}$$

III. DQ-DISCRETIZATION

According to the differential quadrature method, the length of the beam is discretized into a set of N discrete grid points. In DQ method, the partial derivative of a function with respect to the space variable at a given discrete point is approximately expressed by a weighted linear sum of the function values at all discrete points. Consider a onedimensional function *f(x)*, the approximate value of the *n-th* derivative of *f(x)* at the *i-th* discrete point is given by [16]

$$\frac{\partial^n f(x_i)}{\partial x^n} = \sum_{j=1}^{N} A_{ij}^{(n)} f(x_j) \tag{18}$$

where n is the order of derivative, x is the independent variable, x_j are the positions of the discrete grid points, $f(x_j)$ are the values of the function at the grid points and $A_{ij}^{(n)}$ are the elements of the weighting coefficient matrix attached to these function values.

In order to determine the weighting coefficients, a set of test functions should be used in equation (18). For the polynomial basis function of DQ, a set of Lagrange polynomials are employed as the test function. The weighting coefficients for the first-order derivative in x-direction are thus determined as

$$A_{ij}^{(1)} = \frac{\coprod(x_i)}{(x_i - x_j)\coprod(x_j)}\ ,\ i,j = 1,2,\ldots,N\ ;\ i \neq j \tag{19}$$

where

$$\coprod(x_i) = \prod_{k=1,k\neq i}^{N}(x_i - x_k) \tag{20}$$

$$\coprod(x_j) = \prod_{k=1,k\neq j}^{N}(x_j - x_k) \tag{21}$$

The off-diagonal elements of the weighting coefficient matrix for the second and higher order derivatives are obtained through the following recurrence relation

$$A_{ij}^{(n)} = n\left[A_{ij}^{(n-1)}A_{ij}^{(1)} - \frac{A_{ij}^{(n-1)}}{x_i - x_j}\right], i,j = 1,\ldots,N; i \neq j \tag{22}$$

and their diagonal elements are given by

$$A_{ii}^{(n)} = -\sum_{k=1,k\neq i}^{N} A_{ik}^{(n)}\quad ,\quad i = 1,2,\ldots,N \tag{23}$$

In the present study the Chebyshev-Gauss-Lobotto quadrature points in x-direction are used as

$$\frac{x_i}{L} = \frac{1}{2}[1 - \cos(\frac{(i-1)\pi}{N-1})]\quad ,\quad i = 1,2,\ldots,N \tag{24}$$

where L is the length of the beam. Substituting Eq.18 into Eqs.13, 14 and 15 leads to these discrete domain equations

$$s\sum_{j=1}^{N} A_{ij}^{(2)}\tilde{\theta}_j + \sum_{j=1}^{N} A_{ij}^{(1)}\tilde{w}_j + (brs-1)\tilde{\theta}_i + k_b s\sum_{j=1}^{N} A_{ij}^{(2)}\tilde{\psi}_j = 0 \tag{25}$$

$$\sum_{j=1}^{N} A_{ij}^{(2)}\tilde{w}_j - \sum_{j=1}^{N} A_{ij}^{(1)}\tilde{\theta}_j + bs\tilde{w}_i = 0 \tag{26}$$

$$\sum_{j=1}^{N} A_{ij}^{(2)}\tilde{\psi}_j + k_t\sum_{j=1}^{N} A_{ij}^{(2)}\tilde{\theta}_j + a\tilde{\psi}_i = 0 \tag{27}$$

In a similar manner, the DQ-discretization of shear force S, bending moment M and twisting torque T at each point on the beam can be stated as

$$\begin{aligned}
\tilde{S}_i &= \sum_{j=1}^{N} A_{ij}^{(1)}\tilde{w}_j - \tilde{\theta}_i \quad , \quad i = 1,2,\ldots,N \\
\tilde{M}_i &= \sum_{j=1}^{N} A_{ij}^{(1)}\tilde{\theta}_j + k_b\sum_{j=1}^{N} A_{ij}^{(1)}\tilde{\psi}_j\ ,\ i = 1,2,\ldots,N \\
\tilde{T}_i &= \sum_{j=1}^{N} A_{ij}^{(1)}\tilde{\theta}_j + \sum_{j=1}^{N} A_{ij}^{(1)}\tilde{\psi}_j\ ,\ i = 1,2,\ldots,N
\end{aligned} \tag{28}$$

In order to create the eigenvalue system of equations, the degrees of freedom are separated into the domain and boundary the degrees of freedom as

$$\begin{aligned}
\{\tilde{w}_d\} &= [\tilde{w}_2\ \tilde{w}_3 \ldots \tilde{w}_{N-1}]^T, \quad \{\tilde{w}_b\} = [\tilde{w}_1\ \tilde{w}_N]^T \\
\{\tilde{\theta}_d\} &= [\tilde{\theta}_2\ \tilde{\theta}_3 \ldots \tilde{\theta}_{N-1}]^T, \quad \{\tilde{\theta}_b\} = [\tilde{\theta}_1\ \tilde{\theta}_N]^T \\
\{\tilde{\psi}_d\} &= [\tilde{\psi}_2\ \tilde{\psi}_3 \ldots \tilde{\psi}_{N-1}]^T\ , \quad \{\tilde{\psi}_b\} = [\tilde{\psi}_1\ \tilde{\psi}_N]^T
\end{aligned} \tag{29}$$

where the subscripts d and b denotes the values at domain and boundary grid points, respectively. The discretized form of equations of motion and boundary conditions can be rearranged in an assembled form as follows

$$[A_1]\{U_b\} + [A_2]\{U_d\} = \omega^2\{U_d\} \tag{30}$$

$$[B_1]\{U_b\} + [B_2]\{U_d\} = \mathbf{0} \tag{31}$$

where the components of the coefficient matrices $[A_1]$ and $[A_2]$ are obtained from Eqs.(25-27) and the components of the coefficient matrices $[B_1]$ and $[B_2]$ are obtained from the discretized form of the boundary conditions. The vectors $\{U_b\}$ and $\{U_d\}$ are defined as

$$\{U_b\} = \{\{\tilde{w}_b\}\ \ \{\tilde{\theta}_b\}\ \ \{\tilde{\psi}_b\}\}^T \tag{32}$$

$$\{U_d\} = \{\{\tilde{w}_d\}\ \ \{\tilde{\theta}_d\}\ \ \{\tilde{\psi}_d\}\}^T \tag{33}$$

For eliminating $\{U_b\}$ from Eq. (30), first it should be obtained from Eq. (31), that is

$$\{U_b\} = -[B_1]^{-1}[B_2]\{U_d\} \tag{34}$$

Substituting Eq. (34) into Eq. (30), the eigenvalue system of equations is obtained as

$$[C]\{U_d\} = \omega^2\{U_d\} \tag{35}$$

in which

$$[C] = -[A_1][B_1]^{-1}[B_2] + [A_2] \tag{36}$$

Solving the eigenvalue system of equations (35) leads to the natural frequencies as well as mode shapes of the beam under consideration.

IV. Cantilever Beam

There are three boundary conditions at each end of the beam. These BCs are any triple proper-combinations of the geometric or natural end conditions. By a simple calculation the total number of possible BCs is thirty six. In this paper, among these possible BCs only a few cases are studied and the complete procedure to find the natural frequencies and mode shapes is briefly explained for a cantilever beam (the same route is applicable for other cases). In the case of the cantilever beam the boundary conditions at the clamped edge are given by

$$At \ \ \tilde{x} = 0: \quad \begin{aligned} \tilde{w}_1 &= 0 \\ \tilde{\theta}_1 &= 0 \\ \tilde{\psi}_1 &= 0 \end{aligned} \tag{38}$$

and the boundary conditions at the free end are as

$$At \ \ \tilde{x} = 1: \quad \begin{aligned} \tilde{S}_N &= \sum_{j=1}^{N} A_{Nj}^{(1)} \tilde{w}_j - \tilde{\theta}_N = 0 \\ \tilde{M}_N &= \sum_{j=1}^{N} A_{Nj}^{(1)} \tilde{\theta}_j + k_b \sum_{j=1}^{N} A_{Nj}^{(1)} \tilde{\psi}_j = 0 \\ \tilde{T}_N &= \sum_{j=1}^{N} A_{Nj}^{(1)} \tilde{\theta}_j + \sum_{j=1}^{N} A_{Nj}^{(1)} \tilde{\psi}_j = 0 \end{aligned} \tag{38}$$

Based on these equations the coefficient matrices in the Eq. (31) are determined easily and the eigenvalue problem is constructed as discussed above.

V. Numerical Results

The coupled lateral-torsional free vibration of the laminated composite cantilever beam was analyzed by Banerjee [17]. To validate and confirm the accuracy of solution procedure, the numerical results are calculated for the glass-epoxy composite beam with the data used in Ref. [17] where its physical and geometric properties are represented in *TABLE I*. The natural frequencies of the cantilever beam are presented in *TABLE II* where their comparison with the results of the Ref. [17] shows a good agreement. Also the convergence of the solutions is shown in *TABLE III* for various boundary conditions. It can be seen that the numerical results have converged rapidly and this table truly illustrates the effectiveness of the method. The first four natural frequencies of the beam under various boundary conditions are presented in *TABLE IV*. Also the first three normalized mode shapes of the cantilever beam are shown in figure (1). Furthermore the same figures are presented for the beam with different types of boundary conditions include free-free, clamped-clamped and pseudo-simply supported.

TABLE I
PHYSICAL PROPERTIES OF GLASS-EPOXY COMPOSITE BEAM WITH ALL FIBER ANGLES SET TO +15° AND CROSS-SECTIONAL DIMENSIONS: THICKNESS (h=3.18 mm) & WIDTH (b=12.7 mm).

$EI(Nm^2)$	$GJ(Nm^2)$	$K(Nm^2)$	$\rho A(kg/m^3)$	$I_\alpha(kgm)$	$\kappa AG(N)$	$L(mm)$
0.2865	0.1891	0.1143	0.0544	0.777×10^{-6}	6343.3	190.5

TABLE II
COMPARISON OF THE FIRST FOUR NATURAL FREQUENCIES OF THE CANTILEVER BEAM WITH THE ANALYTICAL SOLUTION [17]

BC's	ω_1 (rad/s)	ω_2 (rad/s)	ω_3 (rad/s)	ω_4 (rad/s)
Exact([17])	193.19	1192.42	3259.65	4073.21
DQM	193.1900	1192.4178	3259.6591	4073.1966

For semi-definite system only non-zero mode shapes are shown (e.g. in the case of Free-Free ends).

VI. Conclusion

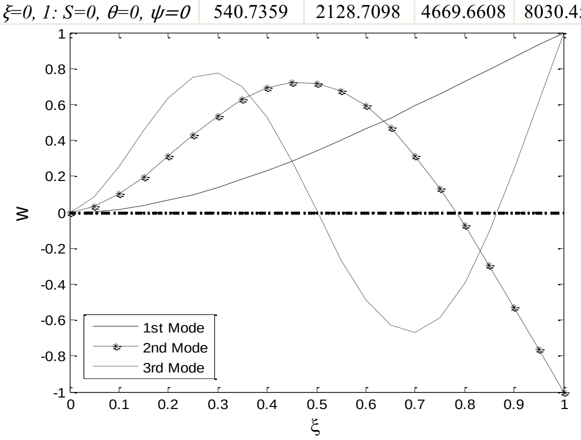
The coupled lateral-torsional vibrations of laminated composite beam were studied using differential quadrature method. In the formulation of the problem, the bending-twisting material coupling, the effects of shear deformation and rotary inertia were taken into account. The natural frequencies and mode shapes for cantilever beam were compared with similar results in the literature and good agreement was achieved. Also the convergence of the solutions was studied in order to accuracy estimation of the solution procedure. The effect of boundary conditions on the vibrational phenomena was investigated. The results showed that how the torsional motion is affected by the lateral vibration. Also, the results for some other BCs were presented. Focusing on the data represented here indicates that the increase/decrease of the natural frequencies is compatible with the nature of BCs. Based on the solution used in this study for the beam with rectangular cross-section, the numerical results for the composite beams with the other cross-sections such as box or airfoil can be obtained easily.

TABLE III
CONVERGENCE OF THE NATURAL FREQUENCIES OF THE BEAM WITH VARIOUS BOUNDARY CONDITIONS

BCs	Number of Nodes	ω_1(rad/s)	ω_2(rad/s)
C-F	8	193.1784	1198.2143
	9	193.1886	1192.1616
	10	193.1899	1192.2605
	11	193.1900	1192.4157
	12	193.1900	1192.4203
	13	193.1900	1192.4178
	14	193.1900	1192.4178
C-C	9	1203.5037	3224.1802
	10	1203.5021	3224.0408
	11	1203.4998	3224.0012
	12	1203.4998	3223.9472
	13	1203.4998	3223.9485
	14	1203.4998	3223.9495
	15	1203.4998	3223.9495
S-S	9	540.7336	2131.8335
	10	540.7344	2128.5449
	11	540.7359	2128.6151
	12	540.7359	2128.7137
	13	540.7359	2128.7117
	14	540.7359	2128.7098
	15	540.7359	2128.7098

TABLE IV
THE FIRST FOUR NATURAL FREQUENCIES OF THE BEAM UNDER VARIOUS BOUNDARY CONDITIONS

Boundary Conditions	ω_1(rad/s)	ω_2(rad/s)	ω_3(rad/s)	ω_4(rad/s)
C-C	1203.4998	3223.9495	6100.0963	8125.0104
F-F	1220.3711	3300.8862	6262.8823	8183.6200
S-S	540.7359	2128.7098	4669.6608	8030.4573
at $\xi=0, 1$: $w=0, \theta=0, T=0$	1203.4877	3224.0146	6100.0627	8129.6383
at $\xi=0, 1$: $w=0, M=0, \psi=0$	602.5455	2116.4747	4715.4566	7304.1033
at $\xi=0, 1$: $S=0, \theta=0, T=0$	541.0426	2128.5001	4671.3989	7903.5821
at $\xi=0, 1$: $S=0, M=0, \psi=0$	1339.1478	3262.8299	6406.3464	7497.8193
at $\xi=0, 1$: $S=0, \theta=0, \psi=0$	540.7359	2128.7098	4669.6608	8030.4573



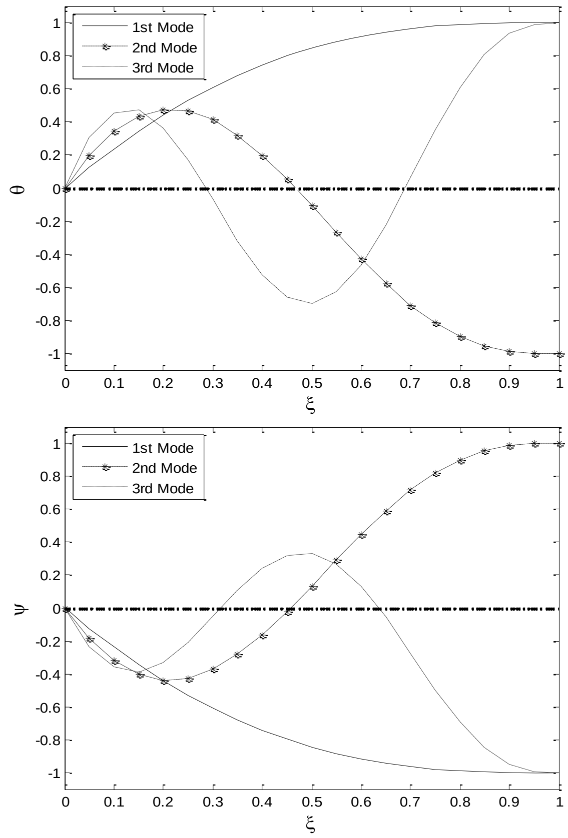


Fig. 1 Mode shapes of cantilever beam

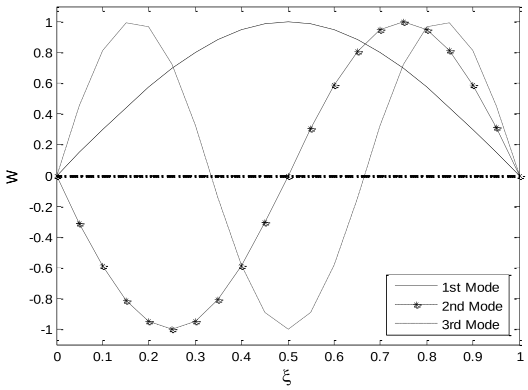


Fig. 2 Mode shapes of beam with BCs: at $\xi=0, 1$: $w=0, M=0, \psi=0$

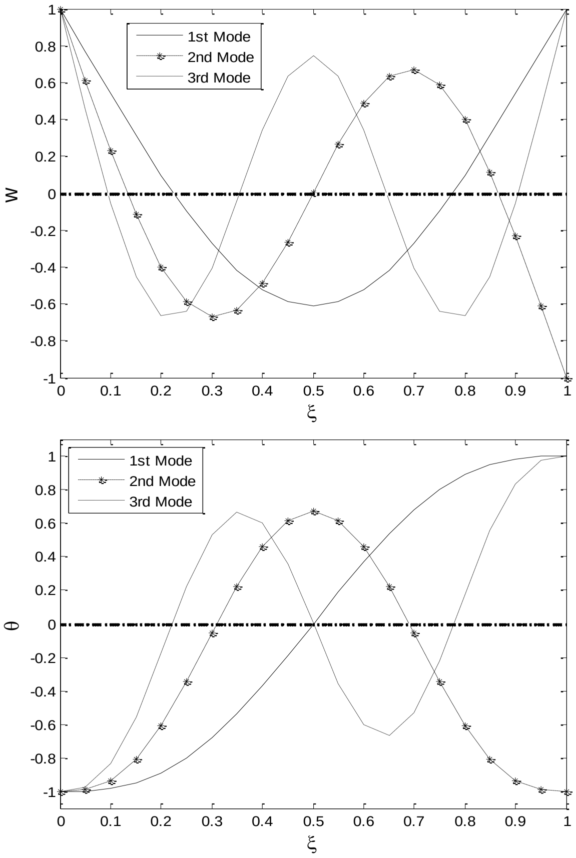


Fig. 3 Mode shapes of Free-Free beam

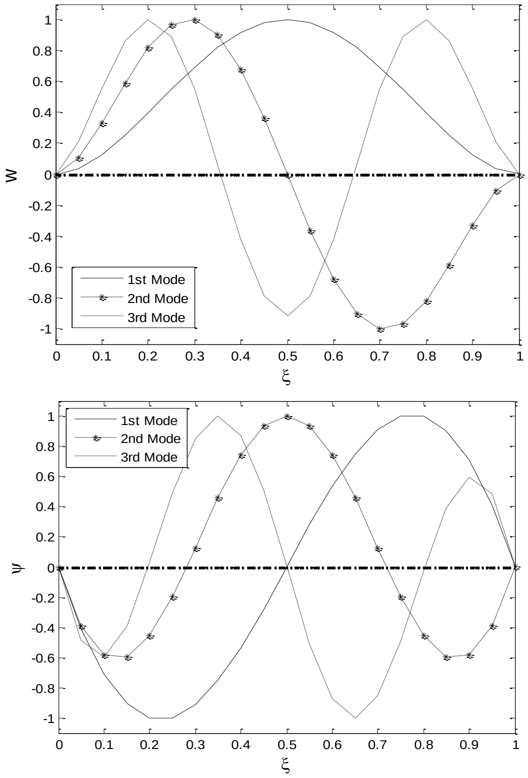


Fig. 4 Mode shapes of clamped-clamped beam

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