International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:6, No:8, 2012

Correspondence Theorem for Anti L-fuzzy Normal Subgroups

Jian Tang, Yunfei Yao

Abstract—In this paper the concept of the cosets of an anti L-fuzzy normal subgroup of a group is given. Furthermore, the group G/A of cosets of an anti L-fuzzy normal subgroup A of a group G is shown to be isomorphic to a factor group of G in a natural way. Finally, we prove that if $f : G_1 \longrightarrow G_2$ is an epimorphism of groups, then there is a one-to-one order-preserving correspondence between the anti L-fuzzy normal subgroups of G_2 and those of G_1 which are constant on the kernel of f.

Keywords—Group; anti L-fuzzy subgroups; anti L-fuzzy normal subgroups; cosets of an anti L-fuzzy normal subgroup.

I. Introduction

Let X be a nonempty set. A fuzzy subset of X is, by definition, an arbitrary mapping $A : X \longrightarrow [0, 1]$, where $[0, 1]$ is the usual interval of real numbers. The important concept of a fuzzy set put forth by Zadeh in 1965 [1] has opened up keen insights and applications in a wide range of scientific fields. Since its inception, the theory of fuzzy subsets has developed in many directions and found applications in a wide variety of fields. The study of fuzzy subsets and its application to various mathematical contexts has given rise to what is now commonly called fuzzy mathematics. Fuzzy algebra is an important branch of fuzzy mathematics.

In 1971, Rosenfeld [2] introduced the concept of fuzzy subgroup of a group and the study of fuzzy algebraic structures was started. In 1975, Negoita and Ralescu [3] considered a generalization of Rosenfeld's definition in which the unit interval $I = [0, 1]$ was replaced by an appropriate lattice structure. In 1979, Anthony and Sherwood [4] redefined a fuzzy subgroup of a group using the concept of triangular norm. Such fuzzy group theory has evoked tremendous interest among several other mathematicians, as [5-7].

In 1990, R.Biswas [8] introduced the concept of anti fuzzy subgroup of a group. Recently, Wang Sheng-hai [9] further obtained some basic properties of anti fuzzy subgroups and anti fuzzy normal subgroups of a group. In continuation of the study undertaken by Wang Sheng-hai [9], the cosets of an anti L-fuzzy normal subgroup of a group is developed in the present paper. And we prove that the group G/A of cosets of an anti L-fuzzy normal subgroup A of a group G is isomorphic to a factor group of G in a natural way. Finally, we show that if $f : G_1 \longrightarrow G_2$ is an epimorphism of groups, then there is a one-to-one order-preserving correspondence between the anti L-fuzzy normal subgroups of G_2 and those of G_1 which are constant on the kernel of f. In this paper, we illustrate that one can pass from the theory of groups to the theory of "fuzzy" groups. As an application of the results of this paper, the corresponding results of group are also obtained.

Jian Tang and Yunfei Yao are with the School of Mathematics and Computational Science, Fuyang Normal College, Fuyang, Anhui, 236037, P.R.China (e-mail: tangjian0901@126.com, yaoyunfeiyry@126.com).

Manuscript received: May 16, 2012.

II. Preliminaries and some notations

Throughout this paper G stands for a group and L stands for an arbitrary completely distributive lattice in which contains least and greatest elements, in the sense that it is complete and satisfies the law

$$(\wedge\{a_i; i \in I\}) \vee (\wedge\{b_j; j \in J\}) = \wedge\{a_i \vee b_j; i \in I, j \in J\}$$

for any $a_i, b_j \in L$ $(i \in I, j \in J)$.

Definition 2.1 : An anti L-fuzzy subgroup is a function $A : G \longrightarrow L$ satisfying the following conditions:

(1) $A(xy) \leq A(x) \vee A(y)$;
(2) $A(x^{-1}) \leq A(x)$

for all x, y in G.

Where the product of x and y is denoted by xy and the inverse of x by x^{-1}.

It is easy to show that an anti L-fuzzy subgroup of a group G satisfied $A(x) \geq A(e)$ and $A(x^{-1}) = A(x)$ for all x in G, where e is the identity element of G.

Proposition 2.2: A function $A : G \longrightarrow L$ is an anti L-fuzzy subgroup if and only if $A(xy^{-1}) \leq A(x) \vee A(y)$ for all $x, y \in G$.

Proof. The proof is straightforward, and we omit it.

Proposition 2.3: If $A : G \longrightarrow L$ is an anti L-fuzzy subgroup, then $A(xy^{-1}) = A(e)$ implies $A(x) = A(y), \forall x, y \in G$.

Proof. Let $A(xy^{-1}) = A(e)$. Since A is an anti L-fuzzy subgroup of G, we have

$$\begin{aligned} A(y) &= A(xx^{-1}y) \leq A(x) \vee A(x^{-1}y) \\ &= A(x) \vee A(e) = A(x). \end{aligned}$$

Similarly, we may show that

$$A(x) \leq A(y),$$

since $A(y^{-1}x) = A(x^{-1}y) = A(e)$.

Definition 2.4 : An anti L-fuzzy subgroup A of a group G is called an anti L-fuzzy normal subgroup if for all $x, y \in G$ it satisfies the following condition:

$$A(y^{-1}xy) = A(x).$$

Clearly, $A(y^{-1}xy) = A(x)$ is equivalent to $A(xy) = A(yx)$ for all $x, y \in G$.

Proposition 2.5: If $A : G \longrightarrow L$ is an anti L-fuzzy normal subgroup, then $G_A = \{x \in G; A(x) = A(e)\}$ is a normal subgroup of G.

Proof. First, G_A be a nonempty subset of G, since $e \in G_A$. Let $x, y \in G_A$. Then we have

$$A(xy^{-1}) \leq A(x) \vee A(y) = A(e) \vee A(e) = A(e),$$

and it is clear that $A(xy^{-1}) \geq A(e)$. Thus $xy^{-1} \in G_A$ and G_A is a subgroup of G.

Furthermore, for any $x, y \in G_A$, we have

$$A(y^{-1}xy) = A(x) = A(e),$$

which implies that $y^{-1}xy \in G_A$. We have thus shown that G_A is a normal subgroup of G.

The reader is referred to [8-10] for notation and terminology not defined in this paper.

III. Cosets of an anti L-fuzzy normal subgroup

Definition 3.1 : Let $A : G \longrightarrow L$ is an anti L-fuzzy subgroup of G. For any x in G, the L-fuzzy subset $xA : G \longrightarrow L$ defined by

$$(xA)(y) = A(x^{-1}y), \forall y \in G,$$

is called a left coset of A.

Similarly, we may define a right coset of the anti L-fuzzy subgroup A as follows:

$$(Ax)(y) = A(yx^{-1}), \forall x, y \in G.$$

Proposition 3.2: If $A : G \longrightarrow L$ is an anti L-fuzzy subgroup of G, then A is an anti L-fuzzy normal subgroup of G if and only if $xA = Ax$ for any x in G.

Proof. For any x, z in G, we have

$$\begin{aligned} xA = Ax \quad &\Leftrightarrow \quad (xA)(z) = (Ax)(z) \\ &\Leftrightarrow \quad A(x^{-1}z) = A(zx^{-1}) \\ &\Leftrightarrow \quad A(z^{-1}x) = A(xz^{-1}) \\ &\Leftrightarrow \quad A(yx) = A(xy), \forall x, y \in G. \end{aligned}$$

This completes the proof.

Therefore, if $A : G \longrightarrow L$ is an anti L-fuzzy normal subgroup of G, we can use the expression "xA(or Ax) is a coset of the anti L-fuzzy normal subgroup A".

Proposition 3.3: If $A : G \longrightarrow L$ is an anti L-fuzzy normal subgroup of G, then

$$xA = yA \ \text{ if and only if } \ x^{-1}y \in G_A.$$

Proof. Suppose first that $xA = yA$. Then $A(x^{-1}y) = (xA)(y) = (yA)(y) = A(y^{-1}y) = A(e)$, which implies that $x^{-1}y \in G_A$ by Proposition 2.5.

Conversely, suppose that $x^{-1}y \in G_A$. It follows that $A(x^{-1}y) = A(e)$, then $(xA)(z) = A(x^{-1}z) = A(x^{-1}yy^{-1}z) \leq A(x^{-1}y) \vee A(y^{-1}z) = A(y^{-1}z) = (yA)(z)$ for all z in G. Similarly, we have $(yA)(z) \leq (xA)(z)$ for all z in G. Hence $(xA)(z) = (yA)(z)$ for all z in G and we have $xA = yA$.

Proposition 3.4: Let $A : G \longrightarrow L$ be an anti L-fuzzy normal subgroup of G and x, y, u, v any elements in G. If $xA = uA$ and $yA = vA$, then $(xy)A = (uv)A$.

Proof. If $xA = uA$ and $yA = vA$, then we have $x^{-1}u, y^{-1}v \in G_A$ by Proposition 3.3, and so $(xy)^{-1}uv = y^{-1}(x^{-1}u)y(y^{-1}v) \in G_A$ since G_A is a normal subgroup of G. Therefore, $(xy)A = (uv)A$.

Remark: Proposition 3.4 allows us to define one binary operation "$*$" on the set G/A of cosets of the anti L-fuzzy normal subgroup A as follows:

$$(xA) * (yA) = (xy)A$$

for any $x, y \in G$. It is easy to see that G/A is a group under this binary operation with identity $A(= eA)$ and $(xA)^{-1} = x^{-1}A$ for all $x \in G$.

Definition 3.5: The group G/A of cosets of the anti L-fuzzy normal subgroup A is called the factor group or the quotient group of G by A.

Now we consider the natural epimorphism $\psi : G \longrightarrow G/A \quad x \mapsto xA$. Let $x \in G$. Then $x \in Ker\psi$ if and only if $xA = A$. Also, $xA = A$ if and only if $x \in G_A$ by Proposition 3.3. Which implies that $Ker\psi = G_A$. Using the above statements we can easily verify the following theorem:

Theorem 3.6: The group G/A of cosets of an anti L-fuzzy normal subgroup A of a group G is ismorphic to a factor group G/G_A of G. The isomorphic correspondence is given by

$$xA \longleftrightarrow xG_A.$$

IV. Correspondence theorem

In this section unless stated otherwise G_1 and G_2 are groups with the identity elements e_1 and e_2, respectively, $f : G_1 \longrightarrow G_2$ is a homomorphism of groups and $A_1 : G_1 \longrightarrow L$ and $A_2 : G_2 \longrightarrow L$ are anti L-fuzzy normal subgroups. We define the image of A_1 under f to be the L-fuzzy subset $f(A_1) : G_2 \longrightarrow L$ such that

$$f(A_1)(y) = \begin{cases} \bigwedge\{A_1(x); x \in f^{-1}(y)\}, & \text{if} \quad f^{-1}(y) \neq \emptyset \\ \qquad 1 & , \ \text{if} \quad f^{-1}(y) = \emptyset \end{cases}$$

for any $y \in G_2$.

The inverse image of A_2 is an L-fuzzy subset $f^{-1}(A_2) : G_1 \longrightarrow L$, defined by $f^{-1}(A_2)(x) = A_2(f(x))$ for any $x \in G_1$.

Lemma 4.1: If A is constant on Ker$f$, then $f(A)(f(x)) = A(x)$ for all $x \in G$.

Proof. Let $f(x) = y$. Then we have $f(A)(y) = \bigwedge\{A(x); x \in f^{-1}(y)\}$. Since $f(x^{-1}z) = (f(x))^{-1}f(z) = e_2$ for all $z \in f^{-1}(y)$, which implies $x^{-1}z \in$ Ker$f$. Also since A is constant on Ker$f$, and so $A(x^{-1}z) = A(e_1)$. Therefore $A(x) = A(z)$ for all $z \in f^{-1}(y)$ by Proposition 2.3. Hence we complete the proof of this Lemma.

Proposition 4.2: If $f : G_1 \longrightarrow G_2$ is a homomorphism and $A_1 : G_1 \longrightarrow L$ and $A_2 : G_2 \longrightarrow L$ are anti L-fuzzy normal subgroups, then

(1) *If f is surjective, then $f(A_1)$ is an anti L-fuzzy normal subgroup;*

(2) *$f^{-1}(A_2)$ is an anti L-fuzzy normal subgroup which is constant on Ker$f$;*

(3) *If f is surjective, then $f(f^{-1}(A_2)) = A_2$.*

Proof. (1) Let $u, v \in G_2$. Since f is surjective, and so there exist $x, y \in G_1$ such that $f(x) = u$, $f(y) = v$. It follows that $xy^{-1} \in f^{-1}(uv^{-1})$ and $y^{-1}xy \in f^{-1}(v^{-1}uv)$. Therefore, by completely distributivity of L, we have

$$\begin{aligned} & f(A_1)(uv^{-1}) \\ = \ & \wedge\{A_1(z); z \in f^{-1}(uv^{-1})\} \\ \leq \ & \wedge\{A_1(xy^{-1}); x \in f^{-1}(u), y \in f^{-1}(v)\} \\ \leq \ & \wedge\{A_1(x) \vee A_1(y); x \in f^{-1}(u), y \in f^{-1}(v)\} \\ = \ & (\wedge\{A_1(x); x \in f^{-1}(u)\}) \vee \\ & (\wedge\{A_1(y); y \in f^{-1}(v)\}) \\ = \ & f(A_1)(u) \vee f(A_1(v) \end{aligned}$$

and

$$\begin{aligned} & f(A_1)(v^{-1}uv) \\ = \ & \wedge\{A_1(z); z \in f^{-1}(v^{-1}uv)\} \\ \leq \ & \wedge\{A_1(y^{-1}xy); x \in f^{-1}(u), y \in f^{-1}(v)\} \\ \leq \ & \wedge\{A_1(x); x \in f^{-1}(u)\} \\ = \ & f(A_1)(u). \end{aligned}$$

This proves (1).

(2) For any $x, y \in G_1$, we have

$$\begin{aligned} f^{-1}(A_2)(xy^{-1}) &= A_2(f(xy^{-1})) \\ &= A_2(f(x)(f(y))^{-1}) \\ &\leq A_2(f(x)) \vee A_2(f(y)) \\ &= f^{-1}(A_2)(x) \vee f^{-1}(A_2)(y)) \end{aligned}$$

and

$$\begin{aligned} f^{-1}(A_2)(y^{-1}xy) &= A_2(f(y^{-1}xy)) \\ &= A_2((f(y))^{-1}f(x)f(y)) \\ &= A_2(f(x)) \\ &= f^{-1}(A_2)(x). \end{aligned}$$

Which implies that $f^{-1}(A_2)$ is an anti L-fuzzy normal subgroup.

Moreover, $f^{-1}(A_2)(x) = A_2(f(x)) = A_2(e_2)$ for all $x \in$ Ker$f$. This proves (2).

(3) For any $y \in G_2$, since f is surjective, there exists $x \in G_1$ such that $y = f(x)$. Thus we, by Lemma 4.1 and (2), have

$$f(f^{-1}(A_2))(y) = f^{-1}(A_2)(x) = A_2(f(x)) = A_2(y).$$

This proves (3).

(4) If A_1 is constant on Ker$f$, then

$$f^{-1}(f(A_1))(x) = f(A_1)(f(x)) = A_1(x)$$

for all $x \in G$ by Lemma 4.1. Which means that $f^{-1}(f(A_1)) = A_1$.

We have thus proved Proposition 4.2 which leads to the following correspondence theorem:

Theorem 4.3 (Theorem of Correspondence): If $f : G_1 \longrightarrow G_2$ is an epimorphism of groups and L is a completely distributive lattice, then there is a one-to-one order-preserving correspondence between the anti L-fuzzy normal subgroups of G_2 and those of G_1 which are constant on Ker$f$.

Proof. Let $F(G_2)$ be the set of all anti L-fuzzy normal subgroups of G_2 and $F(G_1)$ be the set of all anti L-fuzzy normal subgroups of G_1 which are constant on Ker$f$. Let $\varphi : F(G_1) \longrightarrow F(G_2)$ and $\psi : F(G_2) \longrightarrow F(G_1)$ by defined by $\varphi(A_1) = f(A_1)$ and $\psi(A_2) = f^{-1}(A_2)$.

By Proposition 4.2, we show easily that φ and ψ are well-defined functions and inverses of each other, setting the required one-to-one correspondence. Moreover, it is easy to see that this correspondence preserves the order.

V. Conclusion

Using the concept of the cosets of an anti L-fuzzy normal subgroup on a group, we have studied the anti L-fuzzy normal subgroups on a group and proved that if $f : G_1 \longrightarrow G_2$ is an epimorphism of groups, then there is a one-to-one order-preserving correspondence between the anti L-fuzzy normal subgroups of G_2 and those of G_1 which are constant on the kernel of f. The aim that we establish those results is threefold: Firstly, to generalize fundamental results from (ordinary) group theory; Secondly, to find out some new results; Thirdly, to clarify the links between fuzzy group theory and the classical group theory. Those will be the object of a forthcoming paper.

Acknowledgment

The authors would like to thank the anonymous referees for their interest in our work. The work is supported by the National Specific Subject of the Ministry of Education and the Ministry of Finance of the PRC (TS11496), the Anhui Provincial Excellent Youth Talent Foundation (2012SQRL115ZD), the University Natural Science Project of Anhui Province (KJ2012B133, KJ2012Z311) and the Fuyang Normal College Natural Science Foundation (2007LZ01).

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