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Constructing Distinct Kinds of Solutions for the Time-Dependent Coefficients Coupled Klein-Gordon-Schrödinger Equation

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Abstract—We seek exact solutions of the coupled Klein-Gordon-Schrödinger equation with variable coefficients with the aid of Lie classical approach. By using the Lie classical method, we are able to derive symmetries that are used for reducing the coupled system of partial differential equations into ordinary differential equations. From reduced differential equations we have derived some new exact solutions of coupled Klein-Gordon-Schrödinger equations involving some special functions such as Airy wave functions, Bessel functions, Mathieu functions etc.

Keywords—Klein-Gordon-Schrödinger Equation, Lie Classical Method, Exact Solutions.

I. Introduction

IN the study of nonlinear partial differential equations, the discovery of explicit solutions has great theoretical and practical importance. These explicit solutions for nonlinear systems are used as models for physical or numerical investigations and often reflect qualitatively on the behaviour of more complicated solutions. As the scientific literature grew richer, the task of determining these special solutions posed ever increasing challenge to the scientists and researchers. It was in this quest that over the years, a variety of methods for finding these special solutions by reducing the partial differential equations (PDEs) to one or more odinary differential equations (ODEs) have been devised. Included are the methods of group-invariant solutions, based on the theory of continuous group of transformations, better known as "Lie groups", acting on the space of independent and dependent variables for the system. The method is originally due to Sophus Lie [1], who eastablished that in the case of ODEs, invariance under a one-parameter symmetry group implies that the order of the equation can be reduced by one and in the case of PDEs, symmetry group implies the reduction of number of independent variables by one.

Because of the wide range of applications in several branches of physics, the variable coefficients Klein-Gordon-Schrödinger (VCKGS) equations are always the subject of studies both in physical and mathematical contexts. In this direction, we shall obtain the general set of determining equations for the infinitesimals of the Lie point symmetry groups [1-8] leaving VCKGS equation invariant. The symmetries will then be used to obtain physically interesting solutions of the equation. In this paper, we will consider the following form of coupled Klein-Gordon-Schrödinger equation [9, 10] with variable coefficients:

$$\begin{aligned} &u_{tt} + f(t)u_{xx} + g(t)u + h(t)|v|^2 = 0, \\ &iv_t + p(t)v_{xx} + q(t)uv = 0, \end{aligned} \tag{1}$$

where $v(x,t)$ is a complex function, $u(x,t)$ is a real one and $i^2 = -1$. Here, the coefficients $f(t), g(t), h(t), p(t), q(t)$ are the functions of independent variable t that corresponds to new or more realistic physical conditions. In the form of constant coefficients with $f(t) = -c^2, g(t) = 1, h(t) = 1, p(t) = 1, q(t) = 1$, this system is a classical model which describes the interaction between conservative complex neutron field and neutral meson Yukawa in quantum field theory.

In the literatures, the existence, uniqueness, and asymptotic behaviour of global solution of Klein-Gordon-Schrödinger equations are considered in [9, 10] and the exact solitary wave solution is given in [11]. Numerically, two conservative difference schemes are constructed in [12, 13] and the multisymplectic method is considered in [14]. In [15], the modified decomposition method was used for finding the solutions for the coupled KGS equations with initial conditions and the approximate solutions to the equations have been calculated without any need to a transformation techniques and linearization of the equations. Wang [16] established a rational physical model for controlling KGS dynamics system. In the case of perturbation caused by disturbances and uncertainties in control field, numerical quantum approach is presented for finding the quantum optimal control pairing of perturbative system. In [17], a class of discrete-time orthogonal spline collocation schemes for solving coupled KGS equations with initial and boundary conditions are considered. Some new generalized solitary solutions of the KGS equations are obtained by Wang et al. [18] using the Exp-function method, which include some known solutions obtained by the F-expansion method and the homogeneous balance method. Darwish et al. [19] devised an algebraic method to uniformly construct a series of explicit exact solutions for the coupled KGS equations. By applying the Jacobi elliptic function expansion method, the periodic solutions for a class of coupled nonlinear Klein-Gordon equations, which include coupled nonlinear Klein-Gordon equation, coupled nonlinear Klein-Gordon-Schrödinger equations and coupled nonlinear Klein-Gordon-Zakharov equations, are obtained in [20]. Anjan Biswas and Houria Triki [21] obtains the 1-soliton solution of the KGS equation with power law nonlinearity. The solitary wave ansatz is used to carry out the integration. With the aid of Lie classical

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approach and modified (G'/G)-expansion method, we have obtained the exact traveling wave solutions of the coupled KGS equation in [22].

The layout of this paper is as follows. In section 2, we have investigated the symmetries of VCKGS equation by using Lie classical approach and in section 3, we utilized the obtained symmetries to reduce the no of independent variables in studied equation and futher constructed their different kinds of solutions. Certain conclusions and discussions are made in last section.

II. Classical Lie Symmetries and Optimal System

The concept of similarity connected with the idea of invariance under a group transformation is quite fundamental in theoretical physics, not only for obtaining new solutions but also for elucidating some obscure branches of physical laws and for a detailed study of Lie group theory the interested reader is referred to the well-known books [2-4]. In this paper, in virtue of classical Lie group method, we will discuss the classical similarity reductions and exact solutions for the VCKGS equation (1). First of all, we take the complex function $v(x,t)$ as

$$v(x,t) = r(x,t) + is(x,t), \tag{2}$$

which decomposes the system (1) into the following system of equtaions:

$$\begin{aligned} &u_{tt} + f(t)u_{xx} + g(t)u + h(t)(r^2+s^2) = 0,\\ &r_t + p(t)s_{xx} + q(t)us = 0,\\ &-s_t + p(t)r_{xx} + q(t)ur = 0. \end{aligned} \tag{3}$$

To find the symmetries, let us consider the Lie group of point transformations as

$$\begin{aligned} u^* &= u + \epsilon\phi(x,t,u,r,s) + O(\epsilon^2),\\ r^* &= r + \epsilon\psi(x,t,u,r,s) + O(\epsilon^2),\\ s^* &= s + \epsilon\eta(x,t,u,r,s) + O(\epsilon^2),\\ x^* &= x + \epsilon\xi(x,t,u,r,s) + O(\epsilon^2),\\ t^* &= t + \epsilon\tau(x,t,u,r,s) + O(\epsilon^2), \end{aligned} \tag{4}$$

which leaves the system (3) invariant. The method for determining the symmetry group of (3) consists of finding the infinitesimals ϕ, ψ, η, ξ and τ, which are functions of x, t, u, r, s. Assuming that the system (3) is invariant under the transformations (4), we get the following relation from the coefficients of the first order of ϵ:

$$\begin{aligned} &\phi^{tt} + f(t)\phi^{xx} + \tau f'(t)u_{xx} + g(t)\phi + u\tau g'(t) + 2h(t)r\psi\\ &+r^2\tau h'(t) + 2h(t)s\eta + s^2\tau h'(t) = 0,\\ &\psi^t + p(t)\eta^{xx} + \tau p'(t)s_{xx} + q(t)u\eta + q(t)\phi s + \tau q'(t)us = 0,\\ &-\eta^t + p(t)\psi^{xx} + \tau p'(t)r_{xx} + q(t)u\psi + q(t)\phi r + \tau q'(t)ur = 0, \end{aligned} \tag{5}$$

where $\eta^t, \eta^{xx}, \psi^t, \psi^{xx}, \phi^{xx}$ and ϕ^{tt} are extended (prolonged) infinitesimals acting on an enlarged space that includes all derivatives of the dependent variables $s_t, s_{xx}, r_t, r_{xx}, u_{xx}$ and u_{tt} . The infinitesimals are determined from invariance condition (5), by setting the coefficients of different differentials equal to zero. We obtain a large number of PDEs in ϕ, ψ, η, ξ and τ that need to be satisfied. The general solution of this large system provides following forms for the infinitesimal elements ϕ, ψ, η, ξ and τ:

$$\xi = c_2x + c_5,\ \tau = c_3t + c_6,\ \phi = c_4u,\ \psi = c_1r,\ \eta = c_1s, \tag{6}$$

where c_1, c_2, c_3, c_4, c_5 and c_6 are arbitrary constants and time variables $f(t), g(t), h(t), p(t), q(t)$ satisfy the following conditions:

$$\begin{aligned} &\tau f'(t) - 2f(t)\xi_x + 2\tau_t f(t) = 0,\\ &\tau p'(t) + \tau_t p(t) - 2p(t)\xi_x = 0,\\ &\tau g'(t) + 2\tau_t g(t) = 0,\\ &\tau h'(t) + 2\tau_t h(t) - \phi_u h(t) + 2h(t)\eta_s = 0,\\ &\tau q'(t) + \tau_t q(t) + q(t)\phi_u = 0. \end{aligned} \tag{7}$$

The infinitesimal generators of the corresponding Lie algebra are given by

$$\begin{aligned} &Z_1 = s\tfrac{\partial}{\partial s} + r\tfrac{\partial}{\partial r},\ Z_2 = x\tfrac{\partial}{\partial x},\ Z_3 = t\tfrac{\partial}{\partial t},\ Z_4 = u\tfrac{\partial}{\partial u},\\ &Z_5 = \tfrac{\partial}{\partial x},\ Z_6 = \tfrac{\partial}{\partial t}. \end{aligned} \tag{8}$$

In general, there are infinite number of subalgebras of this Lie algebra formed from any linear combination of generators $Z_j; j = 1,2,3,4,5,6$ and to any sublagebra one can get the reduction using characterstic equations:

$$\tfrac{dx}{\xi} = \tfrac{dt}{\tau} = \tfrac{du}{\phi} = \tfrac{dr}{\psi} = \tfrac{ds}{\eta}. \tag{9}$$

To find the optimal system, we have to firstly compute the commutator and adjoint relations. For the commutator table, the Lie brackets are obtained using the expression $[V_i, V_j] = V_iV_j - V_jV_i$ and the adjoint action is given by the Lie series

$$Ad(exp(\epsilon V_i))V_j = V_j - \epsilon[V_i, V_j] + \frac{\epsilon^2}{2}[V_i,[V_i,V_j]] - ..., \tag{10}$$

where ϵ is a parameter. Using above relations, we can find commutator and adjoint relations for (8) which are interpreted in following tables:

TABLE I
Commutator Table

	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6
Z_1	0	0	0	0	0	0
Z_2	0	0	0	0	$-Z_5$	0
Z_3	0	0	0	0	0	$-Z_6$
Z_4	0	0	0	0	0	0
Z_5	0	Z_5	0	0	0	0
Z_6	0	0	Z_6	0	0	0

TABLE II
Adjoint Table

	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6
Z_1	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6
Z_2	Z_1	Z_2	Z_3	Z_4	Z_5e^{ϵ}	Z_6
Z_3	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6e^{ϵ}
Z_4	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6
Z_5	Z_1	$Z_2 - \epsilon Z_5$	Z_3	Z_4	Z_5	Z_6
Z_6	Z_1	Z_2	$Z_3 - \epsilon Z_6$	Z_4	Z_5	Z_6

Thus, the optimal system for (8) consists of following vector fields:

$$\begin{aligned} &(i)\ Z_1 + \alpha Z_2 + \beta Z_3 + \gamma Z_4,\ (ii)\ Z_2 + pZ_3 + qZ_4,\\ &(iii)\ Z_3 + cZ_4 + Z_5,\ (iv)\ Z_3 + dZ_4,\ (v)\ Z_4 + aZ_5 + Z_6,\\ &(vi)\ Z_4 + bZ_5,\ (vii)Z_5 + Z_6,\ (viii)Z_5,\ (ix)\ Z_6, \end{aligned} \tag{11}$$

III. Symmetry Reductions and Exact Solutions

In this section, we will make use of the optimal system of vector fields (11) and reduce the Eq. (1) to ODEs. The similarity variables and the similarity solutions of the Eq. (1) can be obtained by solving characterstic equation (9) and the coefficient functions are given by equation (7). The general solution of these equations involves two constants, one become independent variable ζ and other plays the role of new dependent variables $F(\zeta), G(\zeta), H(\zeta)$.

A. Vector Field $Z_1 + \alpha Z_2 + \beta Z_3 + \gamma Z_4$

For this vector field, on solving the equations (7) and (9) we obtain

$$\begin{aligned}&u(x,t) = t^{\frac{\gamma}{\beta}} F(\zeta),\ \ v(x,t) = t^{\frac{1}{\beta}} e^{iG(\zeta)} H(\zeta),\ \ \zeta = xt^{-\frac{\alpha}{\beta}},\\ &f(t) = K_1 t^{\frac{2\alpha-2\beta}{\beta}},\ \ g(t) = \frac{K_2}{t^2},\ \ h(t) = K_3 t^{\frac{\gamma-2-2\beta}{\beta}},\\ &p(t) = K_4 t^{\frac{2\alpha-\beta}{\beta}},\ \ q(t) = K_5 t^{\frac{-\beta-\gamma}{\beta}},\end{aligned} \tag{12}$$

where K_1, K_2, K_3, K_4, K_5 are arbitrary constants. Substituting (12) into Eq. (1), we have the functions $F(\zeta), G(\zeta), H(\zeta)$ which must satisfy the following system of ODEs:

$$\begin{aligned}&\tfrac{\gamma}{\beta}(\tfrac{\gamma}{\beta}-1)F(\zeta) - \tfrac{\alpha\gamma}{\beta^2}\zeta F'(\zeta) - \tfrac{\alpha}{\beta}(\tfrac{\gamma-\alpha-\beta}{\beta})\zeta F'(\zeta)\\ &+\tfrac{\alpha^2}{\beta^2}\zeta^2 F''(\zeta) + K_1 F''(\zeta) + K_2 F(\zeta) + K_3 H(\zeta)^2 = 0,\\ &\tfrac{1}{\beta}H(\zeta) - \tfrac{\alpha}{\beta}\zeta H'(\zeta) + 2K_4 G'(\zeta)H'(\zeta) + K_4 G''(\zeta)H(\zeta) = 0,\\ &\tfrac{\alpha}{\beta}\zeta G'(\zeta)H(\zeta) + K_4 H''(\zeta) - K_4 G'(\zeta)^2 H(\zeta) + K_5 F(\zeta)H(\zeta) = 0.\end{aligned} \tag{13}$$

To construct solutions for above system, let us suppose that system (13) assumes the solution in the following form:

$$\begin{aligned}F(\zeta) &= a_0 + a_1\zeta + a_2\zeta^2,\\ G(\zeta) &= b_0 + b_1\zeta + b_2\zeta^2,\\ H(\zeta) &= c_0 + c_1\zeta + c_2\zeta^2,\end{aligned} \tag{14}$$

where $a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2$ are constants to be determined. Using (14) in ODEs (13), we get a set of algebraic equations which can be solved to get the following solution of eq. (1):

$$\begin{aligned}&u(x,t) = -1/9\, t^{\frac{\gamma}{\beta}} \left(-\alpha + 2\alpha^2 - 1\right) x^2 \left(t^{-\frac{\alpha}{\beta}}\right)^2 {K_4}^{-1}\beta^{-2}{K_5}^{-1},\\ &v(x,t) = t^{\beta-1} \mathrm{e}^{i\left(b_0 + 1/6\,(\alpha-1)x^2\left(t^{-\frac{\alpha}{\beta}}\right)^2 {K_4}^{-1}\beta^{-1}\right)} c_1\, xt^{-\frac{\alpha}{\beta}},\end{aligned} \tag{15}$$

where $K_3 = 1/9\,\frac{(2\alpha+1)(\alpha-1)\left(4\alpha^2+2\alpha\beta-4\alpha\gamma-\gamma\beta+K_2\beta^2+\gamma^2\right)}{K_5\beta^4 {c_1}^2 K_4}$, $K_1 = 0$.

B. Vector Field $Z_2 + pZ_3 + qZ_4$

For this case, the similarity variable, similarity solution and coefficient functions are as follows:

$$\begin{aligned}&u(x,t) = t^{\frac{q}{p}} F(\zeta),\ \ v(x,t) = e^{iG(\zeta)} H(\zeta),\ \ \zeta = xt^{-\frac{1}{p}},\\ &f(t) = K_1 t^{\frac{2-2p}{p}},\ \ g(t) = \frac{K_2}{t^2},\ \ h(t) = K_3 t^{\frac{q-2p}{p}},\\ &p(t) = K_4 t^{\frac{2-p}{p}},\ \ q(t) = K_5 t^{\frac{-p-q}{p}},\end{aligned} \tag{16}$$

Substituting (16) into Eq. (1), it follows the corresponding reduced ODEs:

$$\begin{aligned}&\tfrac{q(q-p)}{p^2}F(\zeta) - \tfrac{q}{p^2}\zeta F'(\zeta) - \tfrac{q-p-1}{p^2}\zeta F'(\zeta) + \tfrac{1}{p^2}\zeta^2 F''(\zeta)\\ &+K_1 F''(\zeta) + K_2 F(\zeta) + K_3 H(\zeta)^2 = 0,\\ &-\tfrac{1}{p}\zeta H'(\zeta) + 2K_4 G'(\zeta)H'(\zeta) + K_4 G''(\zeta)H(\zeta) = 0,\\ &\tfrac{1}{p}\zeta G'(\zeta)H(\zeta) + K_4 H''(\zeta) - K_4 G'(\zeta)^2 H(\zeta) + K_5 F(\zeta)H(\zeta) = 0.\end{aligned} \tag{17}$$

Solving above system of equations using substitution (14), we get the following values of $u(x,t), v(x,t)$:

$$\begin{aligned}&u(x,t) =\\ &t^{\frac{q}{p}}\left(\tfrac{9}{8}\,\tfrac{K_3 p^2 {c_0}^2}{3-2q+p} - \tfrac{3p{c_0}^2 K_3\, xt^{-p-1}}{16(3-2q+p)\sqrt{-\frac{K_1}{9p+9-18q}}} + \tfrac{9\,K_3\,{c_0}^2(p+1-2q)x^2\left(t^{-p-1}\right)^2}{16(3-2q+p)K_1}\right),\\ &v(x,t) = \mathrm{e}^{i\left(b_0 + 2\sqrt{-\frac{K_1}{9p+9-18q}}xt^{-p-1}{K_4}^{-1} + 1/6\,x^2\left(t^{-p-1}\right)^2{K_4}^{-1}p^{-1}\right)}\\ &\left(c_0 - 1/12\,c_0\,xt^{-p-1}\frac{1}{\sqrt{-\frac{K_1}{9p+9-18q}}}p^{-1}\right),\end{aligned} \tag{18}$$

where $K_2 = \frac{-33+34q-17p-9q^2+9pq}{9p^2}$, $K_5 = \frac{-32K_1(3-2q+p)}{81K_4(p+1-2q)K_3p^2{c_0}^2}$.

C. Vector Field $Z_3 + cZ_4 + Z_5$

This case yields the following forms of invariants and coefficient functions:

$$\begin{aligned}&u(x,t) = t^c F(\zeta),\ \ v(x,t) = e^{iG(\zeta)} H(\zeta),\ \ \zeta = x - \log(t),\\ &f(t) = K_1 t^{-2},\ \ g(t) = \frac{K_2}{t^2},\ \ h(t) = K_3 t^{c-2},\ \ p(t) = K_4 t^{-1},\\ &q(t) = K_5 t^{-c-1},\end{aligned} \tag{19}$$

On using (19) into eq. (1), it corresponds to the following system of reuced ODEs:

$$\begin{aligned}&c(c-1)F(\zeta) - (2c-1)F'(\zeta) + F''(\zeta) + K_1 F''(\zeta)\\ &+K_2 F(\zeta) + K_3 H(\zeta)^2 = 0,\\ &-H'(\zeta) + 2K_4 G'(\zeta)H'(\zeta) + K_4 G''(\zeta)H(\zeta) = 0,\\ &G'(\zeta)H(\zeta) + K_4 H''(\zeta) - K_4 H(\zeta)(G'(\zeta))^2 + K_5 F(\zeta)H(\zeta) = 0.\end{aligned} \tag{20}$$

The variable coefficients Klein-Gordon-Schrödinger equation has the following solution corresponding to above system of ODEs:

$$\begin{aligned}&u(x,t) = -1/4\,\frac{t^c\left({C_2}^2 + 4{K_4}^2\right)}{{C_2}^2 K_5 K_4},\\ &v(x,t) = -1/2\,\mathrm{e}^{i\left(1/2\,\frac{x-\log(t)}{K_4} + C_2\right)}\sqrt{-C_4\left(\mathrm{e}^{\frac{x-\log(t)}{C_2}}\right)^{-2}\left(\mathrm{e}^{\frac{C_3}{C_2}}\right)^{-2}}\\ &\left(\left(\mathrm{e}^{\frac{x-\log(t)}{C_2}}\right)^2\left(\mathrm{e}^{\frac{C_3}{C_2}}\right)^2 + 1\right),\end{aligned} \tag{21}$$

where C_2, C_3, C_4 are arbitrary constants and $K_1 = -1, K_2 = -c^2 + c, K_3 = 0$.

D. Vector Field $Z_3 + dZ_4$

On solving the equations (7) and (9), we get

$$\begin{aligned}&u(x,t) = t^d F(\zeta),\ \ v(x,t) = e^{iG(\zeta)} H(\zeta),\ \ \zeta = x,\\ &f(t) = K_1 t^{-2},\ \ g(t) = \frac{K_2}{t^2},\ \ h(t) = K_3 t^{d-2},\ \ p(t) = K_4 t^{-1},\\ &q(t) = K_5 t^{-d-1},\end{aligned} \tag{22}$$

We get the following system of ODEs using above transformations:

$$\begin{aligned}&d(d-1)F(\zeta)+K_1F''(\zeta)+K_2F(\zeta)+K_3H(\zeta)^2=0,\\&2K_4G'(\zeta)H'(\zeta)+K_4G''(\zeta)H(\zeta)=0,\\&K_4H''(\zeta)-K_4(G'(\zeta))^2H(\zeta)+K_5F(\zeta)H(\zeta)=0.\end{aligned}\tag{23}$$

The above reduced system of ODEs corresponds to following solution of eq. (1):

$$\begin{aligned}u(x,t)&=-6\,\frac{t^d K_4\,{C_5}^2\wp(C_4+C_5\,x,0,C_1)}{K_5},\\v(x,t)&=-6\,\frac{\mathrm{e}^{iC_1}\sqrt{K_1\,K_4\,K_3\,K_5}{C_5}^2\wp(C_4+C_5\,x,0,C_1)}{K_3\,K_5},\end{aligned}\tag{24}$$

where C_1, C_4, C_5 are arbitrary constants, $K_2=-d^2+d$ and $\wp$ denotes WeirstrassP function.

E. Vector Field $Z_4+aZ_5+Z_6$

Following the same way as above, we get

$$\begin{aligned}&u(x,t)=e^tF(\zeta),\;\; v(x,t)=e^{iG(\zeta)}H(\zeta),\;\; \zeta=x-at,\\&f(t)=K_1,\;\; g(t)=K_2,\;\; h(t)=K_3e^t,\;\; p(t)=K_4,\\&q(t)=K_5e^{-t},\end{aligned}\tag{25}$$

Substituting (25) into eq. (1), it follows the following system of reduced ODEs:

$$\begin{aligned}&F(\zeta)-2aF'(\zeta)+a^2F''(\zeta)+K_1F''(\zeta)+K_2F(\zeta)\\&+K_3H(\zeta)^2=0,\\&-aH'(\zeta)+2K_4G'(\zeta)H'(\zeta)+K_4G''(\zeta)H(\zeta)=0,\\&aG'(\zeta)H(\zeta)+K_4H''(\zeta)-K_4(G'(\zeta))^2H(\zeta)\\&+K_5F(\zeta)H(\zeta)=0.\end{aligned}\tag{26}$$

On solving the above system of ODEs, we get the following solution:

$$\begin{aligned}u(x,t)&=-1/4\,\frac{\mathrm{e}^t\left(a^2{C_2}^2+4\,{K_4}^2\right)}{{C_2}^2K_5\,K_4},\\v(x,t)&=-1/2\,\mathrm{e}^{i\left(1/2\,\frac{a(x-at)}{K_4}+C_2\right)}\sqrt{-C_4\left(\mathrm{e}^{\frac{x-at}{C_2}}\right)^{-2}\left(\mathrm{e}^{\frac{C_3}{C_2}}\right)^{-2}}\\&\left(\left(\mathrm{e}^{\frac{x-at}{C_2}}\right)^2\left(\mathrm{e}^{\frac{C_3}{C_2}}\right)^2+1\right),\end{aligned}\tag{27}$$

where C_2, C_3, C_4 are arbitrary constants and $K_1=-a^2$, $K_2=-1$, $K_3=0$.

F. Vector Field Z_4+bZ_5

For this case, the similarity variable, similarity solution and coefficient functions are obtained as follows:

$$\begin{aligned}&u(x,t)=e^{\frac{x}{b}}F(\zeta),\;\; v(x,t)=e^{iG(\zeta)}H(\zeta),\;\; \zeta=t,\\&f(t)=f(t),\;\; g(t)=g(t),\;\; h(t)=0,\;\; p(t)=p(t),\;\; q(t)=0,\end{aligned}\tag{28}$$

On using (28), eq. (1) transforms to following system of ODEs:

$$\begin{aligned}&b^2F''(\zeta)+f(\zeta)F(\zeta)+b^2g(\zeta)F(\zeta)=0,\\&H'(\zeta)=0,\\&G'(\zeta)H(\zeta)=0.\end{aligned}\tag{29}$$

For $f(t)=t, g(t)=t^2$, we get the following solution of eq. (1):

$$\begin{aligned}&u(x,t)=\\&\mathrm{e}^{\frac{x}{b}}\left(C_1\,F_1(-1/16\,\tfrac{-4\,b^4+i}{b^4};\,1/2;\,\tfrac{1/4\,i\left(2\,b^2t+1\right)^2}{b^4})\mathrm{e}^{\frac{-1/2\,it\left(1+b^2t\right)}{b^2}}\right)\\&+\mathrm{e}^{\frac{x}{b}}\left(C_2\,F_1(\tfrac{-1}{16}\,\tfrac{-12b^4+i}{b^4};\,\tfrac{3}{2};\,\tfrac{\frac{1}{4}i\left(2b^2t+1\right)^2}{b^4})\left(2b^2t+1\right)\mathrm{e}^{\frac{\frac{-1}{2}it\left(1+b^2t\right)}{b^2}}\right),\\&v(x,t)=\mathrm{e}^{iC_2}C_1,\end{aligned}\tag{30}$$

where C_1, C_2 are arbitrary constants and F_1 represents the hypergeometric function. For $f(t)=\sin(t), g(t)=1$, eq. (1) possesses the following solution:

$$\begin{aligned}&u(x,t)=\mathrm{e}^{\frac{x}{b}}(C_1\,C(4,-2\,b^{-2},-1/4\,\pi+1/2\,t)\\&+C_2\,S(4,-2\,b^{-2},-1/4\,\pi+1/2\,t)),\\&v(x,t)=\mathrm{e}^{iC_2}C_1,\end{aligned}\tag{31}$$

where C_1, C_2 are arbitrary constants and C, S represents Mathieu Cosine and Mathieu Sine functions respectively.

G. Vector Field Z_5+Z_6

For this vector field, solving equations (7) and (9), we get

$$\begin{aligned}&u(x,t)=F(\zeta),\;\; v(x,t)=e^{iG(\zeta)}H(\zeta),\;\; \zeta=x-t,\\&f(t)=K_1,\;\; g(t)=K_2,\;\; h(t)=K_3,\;\; p(t)=K_4,\;\; q(t)=K_5,\end{aligned}\tag{32}$$

By using above transformations in VCKGS equation, we get the following system of ODEs:

$$\begin{aligned}&F''(\zeta)+K_1F''(\zeta)+K_2F(\zeta)+K_3H(\zeta)^2=0,\\&-H'(\zeta)+2K_4G'(\zeta)H'(\zeta)+K_4G''(\zeta)H(\zeta)=0,\\&G'(\zeta)H(\zeta)+K_4H''(\zeta)-K_4H(\zeta)(G'(\zeta))^2+K_5F(\zeta)H(\zeta)=0.\end{aligned}\tag{33}$$

Solving above system of ODEs, we get the following solutions of eq. (1):

$$\begin{aligned}&(i)\;\; u(x,t)=-1/4\,\tfrac{1+16\,{K_4}^2{C_3}^2}{K_5\,K_4},\\&\quad v(x,t)=\mathrm{e}^{i\left(1/2\,\frac{x-t}{K_4}+C_2\right)}\left(1/2\,C_4+C_4\,(\sinh(C_2+C_3\,(x-t)))^2\right),\\&(ii)\;\; u(x,t)=-1/4\,\tfrac{1+36\,{K_4}^2{C_3}^2}{K_5\,K_4},\\&\quad v(x,t)=\mathrm{e}^{i\left(1/2\,\frac{x-t}{K_4}+C_2\right)}(-3/4\,C_4\,\cosh(C_2+C_3\,(x-t))\\&+C_4\,(\cosh(C_2+C_3\,(x-t)))^3),\end{aligned}\tag{34}$$

where C_2, C_3, C_4 are arbitrary constants and $K_2=0, K_3=0$.

H. Vector Field Z_5

For this vector field, solving the equations (7) and (9), we get

$$\begin{aligned}&u(x,t)=F(\zeta),\;\; v(x,t)=e^{iG(\zeta)}H(\zeta),\;\; \zeta=t,\\&f(t)=f(t),\;\; g(t)=g(t),\;\; h(t)=h(t),\;\; p(t)=p(t),\;\; q(t)=q(t),\end{aligned}\tag{35}$$

The above forms of invariants and coefficient functions transforms VCKGS equation to following system of ODEs:

$$\begin{aligned}&F''(\zeta)+g(\zeta)F(\zeta)+h(\zeta)H(\zeta)^2=0,\\&-G'(\zeta)H(\zeta)+q(\zeta)F(\zeta)H(\zeta)=0,\\&H'(\zeta)=0.\end{aligned}\tag{36}$$

For $h(\zeta)=0, g(\zeta)=\zeta$, eq (1) yields the following solution:

$$\begin{aligned} u(x,t) &= C_3\,Ai(-t)+C_4\,Bi(-t),\\ v(x,t) &= \mathrm{e}^{i\left(\int (C_3\,Ai(-t)+C_4\,Bi(-t))q(t)dt+C_1\right)}C_2, \end{aligned} \tag{37}$$

where C_1, C_2, C_3, C_4 are arbitrary constants and $Ai(t), Bi(t)$ are Airy wave functions. For $h(\zeta 0=0, g(\zeta)=\zeta^2$, we get the following solution:

$$\begin{aligned} u(x,t) &= C_1\sqrt{t}\,\mathrm{J}_{\frac{1}{4}}\left(1/2\,t^2\right)+C_2\sqrt{t}\mathrm{Y}_{\frac{1}{4}}\left(1/2\,t^2\right),\\ v(x,t) &= \mathrm{e}^{i\left(\int q(t)\left(C_1\sqrt{t}\mathrm{J}_{\frac{1}{4}}\left(1/2\,t^2\right)+C_2\sqrt{t}\mathrm{Y}_{\frac{1}{4}}\left(1/2\,t^2\right)\right)dt+C_1\right)}C_2, \end{aligned} \tag{38}$$

where C_1, C_2 are arbitrary constants and J and Y are Bessel functions of first and second kind respectively.

I. Vector Field Z_6

Following the same way as above, we get

$$\begin{aligned} &u(x,t)=F(\zeta),\ \ v(x,t)=e^{iG(\zeta)}H(\zeta),\ \ \zeta=x,\\ &f(t)=K_1,\ \ g(t)=K_2,\ \ h(t)=K_3,\ \ p(t)=K_4,\ \ q(t)=K_5, \end{aligned} \tag{39}$$

The above tranformations corresponds to the following reductions of eq (1):

$$\begin{aligned} &K_1F''(\zeta)+K_2F(\zeta)+K_3H(\zeta)^2=0,\\ &K_4H''(\zeta)-K_4H(\zeta)(G'(\zeta))^2+K_5F(\zeta)H(\zeta)=0,\\ &2G'(\zeta)H'(\zeta)+G''(\zeta)H(\zeta)=0. \end{aligned} \tag{40}$$

The above system of ODEs can be solved to get the following solutions:

$$\begin{aligned} (i)\ \ u(x,t) &= -\frac{K_4\left(2\,C_4\left(\sin(C_1+C_3\,x)\right)^2{C_3}^2-2\,C_4\left(\cos(C_1+C_3\,x)\right)^2{C_3}^2\right)}{\left(-1/2\,C_4+C_4\left(\cos(C_1+C_3\,x)\right)^2\right)K_5},\\ v(x,t) &= \mathrm{e}^{iC_1}\left(-1/2\,C_4+C_4\,\left(\cos\left(C_1+C_3\,x\right)\right)^2\right),\\ (ii)\ \ u(x,t) &= -\frac{9\,K_4\,{C_4}^4\left(\mathrm{e}^{C_1+C_3\,x}\right)^{12}{C_3}^2-K_4\,{C_2}^2}{K_5\,{C_4}^4\left(\mathrm{e}^{C_1+C_3\,x}\right)^{12}},\\ v(x,t) &= \mathrm{e}^{iC_1}C_4\,\left(\mathrm{e}^{C_1+C_3\,x}\right)^3,\\ (iii)\ \ u(x,t) &= \frac{K_4\,{C_2}^2}{K_5\left(C_4+C_5\,\left(C_1+C_3\,x\right)\right)^4},\\ v(x,t) &= \mathrm{e}^{iC_1}\left(C_4+C_5\,\left(C_1+C_3\,x\right)\right), \end{aligned} \tag{41}$$

where C_1, C_2, C_3, C_4, C_5 are arbitrary constants and $K_2=0, K_3=0$.

IV. Discussions and Concluding Remarks

Keeping in view the efficacy and physical importance of the Klein-Gordon-Schrödinger equations, we have studied here the coupled KGS equation in variable form. By the Lie classical approach, we investigated the symmetries of VCKGS equations and utilized these symmetries for obtaining group infinitesimals that are helpful in the reduction of a system of PDEs to a system of ODEs. After that by solving the reduced ODEs, new exact solutions were obtained.

Remark 1: By applying the Lie classical approach, we are able to find vector fields that are used to derive exact solutions of the nonlinear system (1) in variable form. Corresponding to certain vector fields, we obtained solutions involving special functions such as Airy wave functions, Bessel functions, Mathieu functions etc.

Remark 2: Here, the variable form of KGS equation has been studied for new symmetry reductions and exact solutions that is not found in the literature.

It is worth mentioning here that the authenticity of all the solutions has been checked with the aid of Maple software.

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