Constant Order Predictor Corrector Method for the Solution of Modeled Problems of First Order IVPs of ODEs

A. A. James, A. O. Adesanya, M. R. Odekunle, D. G. Yakubu

***Abstract*—**This paper examines the development of one step, five hybrid point method for the solution of first order initial value problems. We adopted the method of collocation and interpolation of power series approximate solution to generate a continuous linear multistep method. The continuous linear multistep method was evaluated at selected grid points to give the discrete linear multistep method. The method was implemented using a constant order predictor of order seven over an overlapping interval. The basic properties of the derived corrector was investigated and found to be zero stable, consistent and convergent. The region of absolute stability was also investigated. The method was tested on some numerical experiments and found to compete favorably with the existing methods.

***Keyword*—**Interpolation, Approximate Solution, Collocation, Differential system, Half step, Converges, Block method, Efficiency.

***AMS Subject Classification*—**65L05, 65L06, 65D30.

I. Introduction

It is remarkable to note that many physical phenomena in sciences, engineering, and medicine, to mention few, are modeled by equations involving derivatives, which are generally referred to as differential equations. A differential equation in which the unknown parameter is a function of one independent variable is called ordinary differential equations, while that involving two or more independent variables is called a partial differential equation.

The general form of the first order initial value problems of ordinary differential equations is in the form;

$$y' = f(x, y),\ y(x_0) = y_0 \tag{1}$$

where f(x,y) is a given real valued function in the strip S=[a,b]⊂[-∞,∞] which is continuous within the region. We assumed that f(x,y) satisfies Lipchitz conditions that guaranteed the existence and uniqueness of the solution to (2).

Scholars have developed linear multistep method for the solution of (1). They developed methods varying from the discrete linear multistep method to the continuous linear multistep method. According to [1], the continuous linear multistep method has greater advantages over the discrete method, in that it gives better error estimation, provides a simplified form of coefficient for further analytical work at different points, and guarantees easy appropriation of solutions at all interior points within the interval of integration. Among the authors that proposed the continuous linear multistep method are; [2]-[4] to mention a few. They individually proposed methods which are implemented in predictor corrector mode, and adopted Taylor series expansion to supply the starting value.

Generally, the major setback of the predictor-corrector method is the high cost of implementation, as subroutines are very complicated to write because of the special techniques required to supply starting values. Therefore we seek to address this setback by proposing a method that shares the properties of both the block method and the predictor corrector method. It should be recalled that [5]. First proposed block method as a predictor to a predictor corrector algorithm [6]-[10] adopted the Milne approach and concluded that though the method is more expensive to implement but it gives better results than the block method; hence the method follows the Milne approach.

II. Method and Materials

A. Derivation of Our New Corrector

We consider power series approximate solution in the form

$$y(x) = \sum_{j=0}^{s+r-1} a_j x^j \tag{2}$$

where S and r are the number of interpolation and collocation points respectively.

The first derivative of (2) gives

$$y'(x) = \sum_{j=0}^{s+r-1} j a_j x^{j-1} \tag{3}$$

Adewale James is with the Mathematics Faculty, American University of Nigeria Yola, Nigeria, (Phone: +234-08077092831 e-mail: adewale.james@aun.edu.ng).

Adesanya Olaide is currently a Senior Lecturer at the Department of Mathematics, ModibboAdama University of Technology, Yola, Nigeria. (e-mail: torlar10@yahoo.com).

Remilekun Odekunle is a professor of Computation Mathematics at the Department of Mathematics, ModibboAdama University of Technology, Yola, Adamawa State.

Dauda Yakubu is a Professor of Mathematics with interest in both Ordinary and partial differential equations, Currectly Lecturing at the Depatment of Mathematics, Federal University of Bauchi, Nigeria. (e-mail: daudagyakubu@yahoo.com).

Substituting (3) into (2) gives

$$f(x,y)=\sum_{j=0}^{s+r-1} j a_j x^{j-1} \tag{4}$$

interpolating (3) at$x_n, x_{n+\frac{1}{6}}, x_{n+\frac{1}{3}},$ and collocating (4) at$x_{n+s}, s=0(\frac{1}{6})1$gives and a system of non-linear equation in the form

$$AX=U \tag{5}$$

where

$$X=\begin{bmatrix}1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9\\ 1 & x_{n+\frac{1}{6}} & x^2_{n+\frac{1}{6}} & x^3_{n+\frac{1}{6}} & x^4_{n+\frac{1}{6}} & x^5_{n+\frac{1}{6}} & x^6_{n+\frac{1}{6}} & x^7_{n+\frac{1}{6}} & x^8_{n+\frac{1}{6}} & x^9_{n+\frac{1}{6}}\\ 1 & x_{n+3} & x^2_{n+\frac{1}{3}} & x^3_{n+\frac{1}{3}} & x^4_{n+\frac{1}{3}} & x^5_{n+\frac{1}{3}} & x^6_{n+\frac{1}{3}} & x^7_{n+\frac{1}{3}} & x^8_{n+\frac{1}{3}} & x^9_{n+\frac{1}{3}}\\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 & 8x_n^7 & 8x_n^8\\ 0 & 1 & 2x_{n+\frac{1}{6}} & 3x^2_{n+\frac{1}{6}} & 4x^3_{n+\frac{1}{6}} & 5x^4_{n+\frac{1}{6}} & 6x^5_{n+\frac{1}{6}} & 7x^6_{n+\frac{1}{6}} & 8x^7_{n+\frac{1}{6}} & 8x^8_{n+\frac{1}{6}}\\ 0 & 1 & 2x_{n+\frac{1}{3}} & 3x^2_{n+\frac{1}{3}} & 4x^3_{n+\frac{1}{3}} & 5x^4_{n+\frac{1}{3}} & 6x^5_{n+\frac{1}{3}} & 7x^6_{n+\frac{1}{3}} & 8x^7_{n+\frac{1}{3}} & 8x^8_{n+\frac{1}{3}}\\ 0 & 1 & 2x_{n+\frac{1}{2}} & 3x^2_{n+\frac{1}{2}} & 4x^3_{n+\frac{1}{2}} & 5x^4_{n+\frac{1}{2}} & 6x^5_{n+\frac{1}{2}} & 7x^6_{n+\frac{1}{2}} & 8x^7_{n+\frac{1}{2}} & 8x^8_{n+\frac{1}{2}}\\ 0 & 1 & 2x_{n+\frac{2}{3}} & 3x^2_{n+\frac{2}{3}} & 4x^3_{n+\frac{2}{3}} & 5x^4_{n+\frac{2}{3}} & 6x^5_{n+\frac{2}{3}} & 7x^6_{n+\frac{2}{3}} & 8x^7_{n+\frac{2}{3}} & 8x^8_{n+\frac{2}{3}}\\ 0 & 1 & 2x_{n+\frac{5}{6}} & 3x^2_{n+\frac{5}{6}} & 4x^3_{n+\frac{5}{6}} & 5x^4_{n+\frac{5}{6}} & 6x^5_{n+\frac{5}{6}} & 7x^6_{n+\frac{5}{6}} & 8x^7_{n+\frac{5}{6}} & 8x^8_{n+\frac{5}{6}}\\ 0 & 1 & 2x_{n+1} & 3x^2_{n+1} & 4x^3_{n+1} & 5x^4_{n+1} & 6x^5_{n+1} & 7x^6_{n+1} & 8x^7_{n+1} & 8x^8_{n+1}\end{bmatrix},$$

$$A=\begin{bmatrix}a_0\\a_1\\a_2\\a_3\\a_4\\a_5\\a_6\\a_7\\a_8\\a_9\end{bmatrix},\ U=\begin{bmatrix}y_n\\ y_{n+\frac{1}{6}}\\ y_{n+\frac{1}{3}}\\ f_n\\ f_{n+\frac{1}{6}}\\ f_{n+\frac{1}{3}}\\ f_{n+\frac{1}{2}}\\ f_{n+\frac{2}{3}}\\ f_{n+\frac{5}{6}}\\ f_{n+1}\end{bmatrix}$$

Solving (5) for the $a_j s$ and substituting back into (4) gives a continuous multistep method in the form

$$y(t)=\alpha_\mu(t)y_{n+\mu}+h\left(\textstyle\sum_{j=0}^{1}\beta(t)f_{n+j}+\beta_k(t)\,f_{n+j}\right) \tag{6}$$

where

$$\mu=0,\frac{1}{6},\frac{1}{3},k=\frac{1}{6}\left(\frac{1}{6}\right)1,f_{n+k}=f(x_n+kh),t=\frac{x-x_n}{h}$$

where

$$y_{n+j},y_{n+k},f_{n+j},and\ f_{n+k}$$

Evaluating (6) at $t=1$ gives

$$y_{n+1}-\frac{197000}{58879}y_n-\frac{746496}{855791}y_{n+\frac{1}{6}}+\frac{1002375}{588791}y_{n+\frac{1}{3}} \\ =h\left(\begin{array}{c}\frac{172225}{1236459}f_n-\frac{701280}{412153}f_{n+\frac{1}{6}}+\frac{666000}{412153}f_{n+\frac{1}{3}}+\frac{788000}{1236459}f_{n+\frac{1}{2}}- \\ \frac{28125}{412153}f_{n+\frac{2}{3}}+\frac{115200}{412153}f_{n+\frac{5}{6}}+\frac{57410}{1236459}f_{n+1}\end{array}\right) \tag{7}$$

Equation (7) is our corrector

B. Derivation of the Constant Order Predictor

Reference [8] had developed a block method which we adopted as our constant order predictor. They considered interpolating (3) x_n and collocating (4) at $x_{n+r}, r=0(\frac{1}{6})1$to obtained a discrete block given as

$$A^{(0)}Y_m=ey_n+h^k dF(Y_n)+h^k bF(Y_m) \tag{8}$$

where

$$d=\begin{bmatrix}\frac{19087}{362880} & \frac{1139}{22680} & \frac{137}{2688} & \frac{143}{2835} & \frac{3715}{72576} & \frac{41}{840}\end{bmatrix}$$

$$b=\begin{bmatrix}\frac{2713}{13120} & \frac{-15487}{120960} & \frac{293}{2835} & \frac{-6737}{120960} & \frac{263}{15720} & \frac{-863}{362880}\\ \frac{47}{189} & \frac{11}{7560} & \frac{166}{2835} & \frac{-269}{7560} & \frac{11}{945} & \frac{-37}{22680}\\ \frac{27}{112} & \frac{387}{4480} & \frac{17}{105} & \frac{-243}{4480} & \frac{9}{560} & \frac{-27}{13440}\\ \frac{232}{945} & \frac{65}{945} & \frac{752}{2835} & \frac{29}{945} & \frac{8}{945} & \frac{-4}{2835}\\ \frac{725}{3024} & \frac{2125}{24192} & \frac{125}{567} & \frac{2875}{24192} & \frac{235}{3024} & \frac{-275}{72576}\end{bmatrix}$$

$$e=\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix}$$

where

$$Y_m=\left[y_{n+\frac{1}{6}},y_{n+\frac{1}{3}},y_{n+\frac{1}{2}},y_{n+\frac{2}{3}},y_{n+\frac{5}{6}},y_{n+1}\right]^T$$

$A^{(0)}=6\times 6$ identity matrix

$$F(Y_m)=\left[f_{n+\frac{1}{6}},f_{n+\frac{1}{3}},f_{n+\frac{1}{2}},f_{n+\frac{2}{3}},f_{n+\frac{5}{6}},f_{n+1}\right]$$

$f(y_n)=f_n,\ e,d,are\ r\times r\ matrix,$ where r is the number of collocation point.

III. Analysis of the Basic Properties of Corrector Method

A. Order of the Method and Constant Order

We defined a linear operator on (7) as

$$L[y(x):h]=y(t)-\alpha_\mu(t)y_{n+\mu}+h\left(\sum_{j=0}^{1}B_j(t)f_{n+j}+B_k(t)f_{n+k}\right)$$

Expanding $y_{n+\mu}, f_{n+j}$ and f_{n+k}in Taylor series and expansion and company coefficient of h gives

$$L\{y(x);h\}=c_0y(x)+c_1hy'(x)+\cdots c_ph^py^{(p)}(x)+\cdots c_{p+1}h^{p+1}y^{(p+1)}(x)+\cdots$$

B. Definition

The difference operator L and the associated continuous multistep method (6.3.5) are said to be of order p if $c_0=c_1=$

$\cdots = c_p, c_{p+1} \neq 0$ is called the error constant and implies that the local truncation error is given by

$$t_{n+k} = C_{p+2}\mathrm{h}^{(\mathrm{p}+1)}\mathrm{y}^{(\mathrm{p}+1)}(\mathrm{x}) + 0(\mathrm{h}^{\mathrm{p}+2})$$

The order of our discrete scheme is 9 with error constant

$$c_{p+1} = -4.8753 \times 10^{-11}$$

C. Zero Stability

A continuous hybrid linear multistep method is said to be zero stable if the zeros of the first characteristic polynomial $e(r)$ satisfies |r| = 1 is simple. The roots of the first characteristic polynomial

$$\rho(z) = z - \frac{197000}{58879} - \frac{746496}{855791} z^{\frac{1}{6}} + \frac{1002375}{588791} z^{\frac{1}{3}} \quad (9)$$

equating (9) to zero and solving for z gives roots as 0 and 1, hence our corrector is zero stable.

D. Consistency of the Corrector

The second characteristics polynomials of (7) are given

$$\sigma(z) = -\frac{172225}{1236459} - \frac{701280}{412153} z^{\frac{1}{6}} + \frac{666000}{412153} z^{\frac{1}{3}} + \frac{788000}{1236459} z^{\frac{1}{2}} - \frac{28125}{412153} z^{\frac{2}{3}} + \frac{115200}{412153} z^{\frac{5}{6}} + \frac{57410}{1236459} z$$

$\rho(1) = 0, \rho'(1) = \sigma(1)$ hence our method is consistent.

E. Convergence

The necessary and sufficient condition for a linear multistep method to be convergent is that, it must be consistent and zero stable, hence our method is convergent.

IV. Numerical Examples

A. Numeric Example 1

We consider the growth model described by the differential equation of the form

$$\frac{dN}{dt} = \alpha N, N(0) = 1000, t \in [0,1]$$

The above growth equation represents the rate of growth of bacteria in a colony. We shall assume that the model grows continuously without restriction. One may ask; how many bacteria are in a colony after some minutes if an individual produces an offspring at an average growth rate of 0.2? We also assume that N(t) is the population size at time t.

The theoretical solution is given by

$N(t) = 1000e^{0.2t}$ we note that the growth rate $\alpha = 0.2$ in the growth equation.

B. Numerical Example 2

The SIR model is an epidemiological model that computes the theoretical numbers of people infected with a contagious illness in a closed population over time. The name of this class of models derives from the fact that they involves coupled equations relating the number of susceptible people S(t), number of people infected I(t) and the number of people who have recovered R(t). This is a good and simple model for many infectious diseases including measles. The SIR model is described by the three coupled equations.

$$\frac{dS}{dt} = \mu(1 - S) - \beta IS$$

$$\frac{dI}{dt} = -\mu I - \gamma I + \beta IS$$

$$\frac{dR}{dt} = -\mu R + \gamma$$

where μ, γ, β are positive parameters. Define y to be

$$y = S + I + R$$

Adding the three coupled equations above, we obtain the following evolution equations for

$$y' = \mu(1 - y)$$

Taking $\mu = 0.5$ and attaching an initial condition y(0)=0.5 (for a particular closed population), we obtain,

$$y'(t) = 0.5(1 - y), y(0) = 0.5$$

Whose exact solution is

$$y(t) = 1 - 0.5^{-0.5t}.$$

TABLE I
FOR NUMERIC EXAMPLE 1

X	Exact result	Computed Result	ERN)	ERS
0.1	1020.2013400267558	1020.2013400267565	6.82121 (-13)	1.8303(-11)
0.2	1040.8107741923882	1040.8107741923861	2.04636 (-12)	1.2505(-11)
0.3	1061.8365465453596	1061.8365465453599	2.27373 (-13)	1.2278(-11)
0.4	1083.2870676749587	1083.2870676749576	1.13686 (-12)	3.1377(-11)
0.5	1105.1709180756477	1105.1709180756473	4.54747 (-13)	2.2168(-10)
0.6	1127.4968515793757	1127.4968515793755	2.27373 (-13)	2.0600(-10)
0.7	1150.2737988572273	1150.2737988572242	3.18323 (-12)	2.1714(-10)
0.8	1173.5108709918102	1173.5108709918097	4.54747 (-13)	2.2168(-10)
0.9	1197.2173631218104	1197.2173631218095	9.09494 (-13)	2.7444(-10)
1.0	1221.4027581601702	1221.4027581601722	2.04636 (-12)	4.8999(-10)

ERN= Error in New Method
ERS=Error in [11]

TABLE II
FOR NUMERIC EXAMPLE 2

X	Exact Result	ERB	ERN	ERS
0.1	0.5243852877496430	7.704948e-014	3.33066(-16)	5.574430e-012
0.2	0.5475812909820202	1.465494e-013	6.66133(-16)	3.946177e-012
0.3	0.5696460117874711	2.090550e-013	0.0000000	8.183232e-012
0.4	0.5906346234610092	2.652323e-013	3.33066(-16)	3.436118e-011
0.5	0.6105996084642975	3.151923e-013	1.11022(-15)	1.929743e-010
0.6	0.6295908896591411	3.599343e-013	4.44089(-16)	1.879040e-010
0.7	0.6476559551406433	3.994582e-013	8.8817(-16)	1.776835e-010
0.8	0.6648399769821803	4.342082e-013	4.44089(-16)	1.724676e-010
0.9	0.6811859241891134	4.647394e-013	4.44089(-16)	1.847545e-010
1.0	0.6967346701436833	4.911627e-013	7.77156(-16)	3.005770e-010

ERB-Error in Block method
ERN-Error in New method
ERS-Error in [11]

V. DISCUSSION OF THE RESULT

We have considered two numerical examples in this paper. Problem I and I1 were solved by [11] where they proposed a block method of order six, combining power series and exponential function as their approximate solution. The results are shown in Tables I and II. It has been shown clearly that our method gave better approximation than the existing methods.

VI. CONCLUSION

We have proposed a new method that harnesses the properties of the Predictor Corrector method and the Block method. The results affirm the claims of [7] and [8] as discussed in section one. We have equally established that increasing the interpolation points with the same block predictor improves the method. It has been established in literature that the higher the order of a numerical scheme, the higher the accuracy. In our future correspondence, we shall consider a case when the corrector gives results at a non-overlapping interval.

REFERENCES

[1] Awoyemi D.O., A Class of Continuous Method for General Second Order Initial Value Problems in Ordinary Differential Equation. J. of Computer Math, 1999; (72):29-37

[2] Onumanyi P, Awoyemi D.O, Jator S.N, and Serisena U.W, New Linear Multistep Method With Constant Coefficient for First Order Initial Value Problems J.N.M.S., 1994;13(7): 37 --51

[3] Adesanya A.O, Anake T.A, Oghoyon G.J, Continuous Implicit Method for the Solution of General Second Order Ordinary Differential Equation. J. Nigerian Association of Mathematical Physics, 2009; (40):71-78.

[4] Kayode S.J and Awoyemi D.O, A Multi derivative Collocation Method for fifth Order Ordinary Differential Equation.J. of Mathematics and Statistics. 2010; 6(1): 60-63.

[5] Milne W.E, Numerical Solution of Differential equation, Wiley Publisher, New York;1953

[6] Anake T. A, Awoyemi D. O and Adesanya A. O. One-Step Implicit Hybrid Block Method for the Direct Solution of General Second Order Ordinary Differential Equations. IAENG International Journal of Applied Mathematics, 2012;42(4):224 -228

[7] Anake T. A, Awoyemi, D. O and Adesanya A. O. One-Step Implicit Hybrid Block Method for the Direct Solution of General Second Order Ordinary Differential Equations. IAENG International Journal of Applied Mathematics, 2012: 42(4):224 -228

[8] Adesanya A Olaide, Odekunle M. Remilekun and James A. Adewale, Order seven continuous hybrid method for the solution of first order ordinary differential equations. Canadian Journal on Science and Engineering Mathematics Vol. 3 No. 4: 2012;

[9] Areo E.A, Ademiluyi R.A, and Babatola P.O, Three-step hybrid linear multistep method for the solution of first order initial value problem in ordinary differential equation ,J.N.A.M.P, 2011; (19):155-158.

[10] Yahaya Y.A. and Umar Mohammed. A family of implicit 4-step block Hybrid Collocation method for accurate and efficient solution of ODEs , Nigerian Journal of Mathematics and Applications. 2010; Vol.20: pp43-52.

International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:7, No:11, 2013

[11] Sunday J. Odekunle M.R. and Adeyanya A.O. Order six block integrator for the solution of first-order ordinary differential equations, IJMSC , 2013;Vol.3 (1).1: 87-96.