International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:7, No:9, 2013

Confidence Interval for the Inverse of a Normal Mean with a Known Coefficient of Variation

Arunee Wongkhao, Suparat Niwitpong, Sa-aat Niwitpong

***Abstract*—**In this paper, we propose two new confidence intervals for the inverse of a normal mean with a known coefficient of variation. One of new confidence intervals for the inverse of a normal mean with a known coefficient of variation is constructed based on the pivotal statistic Z where Z is a standard normal distribution and another confidence interval is constructed based on the generalized confidence interval, presented by Weerahandi. We examine the performance of these confidence intervals in terms of coverage probabilities and average lengths via Monte Carlo simulation.

***Keywords*—**The inverse of a normal mean, confidence interval, generalized confidence intervals, known coefficient of variation.

I. Introduction

SUPPOSE $X_i \sim N(\mu,\sigma^2), i=1,2,...,n$ and assume that the coefficient of variation is known, $\tau=\sigma/\mu$, where μ,σ^2 are population mean and population variance of X, respectively. We are interesting in constructing the confidence interval for the inverse of a normal mean, $\theta=\mu^{-1}$.

The problem of estimating μ^{-1} is studied in many areas i.e. in experimental nuclear Physics, agricultural and economic researches. In experimental nuclear Physics, Lamanna et al. [1] studied a charged particle momentum $p=\mu^{-1}$ where μ is the track curvature of a particle. Zaman [2], [3] discussed this problem in the one-dimensional special case of the single period control problem and the estimation of structural parameters of a simultaneous equation as recognized. Withers and Nadarajah [4] proposed the theorem to construct the point estimators for the inverse powers of a normal mean. Although the point estimator for the inverse of a normal mean can be a useful measure, the greatest use of it to construct a confidence interval for the inverse of a normal mean for the population quantity. We propose two new the confidence intervals for the inverse of a normal mean with a known coefficient of variation. One of our new confidence interval is constructed based on the asymptotic normality of the test statistic Z where Z is a standard normal distribution, another the confidence interval is constructed based on the generalized confidence interval, presented by Weerahandi [5].

The rest of the paper is organized as follows. The two confidence intervals for the inverse of a normal mean based on the exact method and the generalized confidence interval are discussed in Section II. Simulation studies show the coverage probabilities and average length widths for each interval and their results are presented in Section III, and some conclusions are given in Section IV.

Arunee Wongkhao, Suparat Niwitpong, and Sa-aat Niwitpong are with Department of Applied Statistic, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand (e-mail: rpoocmu@gmail.com, suparat8@yahool.com, snw@kmutnb.ac.th).

II. Confidence Interval for the Inverse of a Normal Mean with a Known Coefficient of Variation

A. Exact Method

In this section, we consider the expected length and variance of the estimator for θ, $\hat{\theta}=\bar{X}^{-1}$. This result will show that our estimator is an asymptotic biased estimator.

Theorem1. Suppose $X_i \sim N(\mu,\sigma^2), i=1,2,...,n$ where μ,σ^2 are respectively population mean and population variance of X. Then the estimator of θ is $\hat{\theta}=\bar{X}^{-1}$, the expectation of $E(\hat{\theta})$ and $E(\hat{\theta}^2)$ when the coefficient of variation is known, $\tau=\sigma/\mu$ are

$$\theta\left[1+\sum_{k=1}^{\infty}\frac{(2k)!}{2^k k!}\left(\frac{\tau^2}{n}\right)^k\right] \text{ and } \theta^2\left[\sum_{k=0}^{\infty}\frac{(2k+1)!}{2^k k!}\left(\frac{\tau^2}{n}\right)^k\right].$$

Proof of Theorem1. Applying the Taylor series expansion of $(\bar{X})^{-1}$ at $\bar{X}=\mu$ as shown in Mahmoudvand and Hassani [6], the estimator $\hat{\theta}$ can be written as

$$\begin{aligned}\hat{\theta}=\frac{1}{\bar{X}}&=\frac{1}{\mu}+f'\left(\frac{1}{\mu}\right)\left(\bar{X}-\mu\right)+f''\left(\frac{1}{\mu}\right)\left(\bar{X}-\mu\right)^2+... \qquad (1)\\ &=\frac{1}{\mu}+\left(-\frac{1}{\mu^2}\right)(\bar{X}-\mu)+\left(\frac{2}{2\mu^3}\right)(\bar{X}-\mu)^2+...\\ &=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(\bar{X}-\mu)^{k-1}}{\mu^k}\end{aligned}$$

Hence,

$$\begin{aligned}E\left(\frac{1}{\bar{X}}\right)&=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{E\left(\bar{X}-\mu\right)^{k-1}}{\mu^k}\\ &=\sum_{k=0}^{\infty}2^k\frac{\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}\mu^{2k+1}}\left(\frac{\sigma^2}{n}\right)^k\end{aligned}$$

$$=\frac{1}{\mu}\left[1+\sum_{k=1}^{\infty}2^{k}\frac{\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}\mu^{2k}}\left(\frac{\sigma^{2}}{n}\right)^{k}\right]$$

$$=\frac{1}{\mu}\left[1+\sum_{k=1}^{\infty}\frac{2^{k}\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}}\left(\frac{\tau^{2}}{n}\right)^{k}\right],\tau=\frac{\sigma}{\mu}$$

$$=\frac{1}{\mu}\left[1+\sum_{k=1}^{\infty}\frac{2^{k}\prod_{j=1}^{k}\left(k-j+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\sqrt{\pi}}\left(\frac{\tau^{2}}{n}\right)^{k}\right]$$

$$=\frac{1}{\mu}\left[1+\sum_{k=1}^{\infty}2^{k}\prod_{j=1}^{k}\left(k-j+\frac{1}{2}\right)\left(\frac{\tau^{2}}{n}\right)^{k}\right],\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

$$=\frac{1}{\mu}\left[1+\sum_{k=1}^{\infty}\prod_{j=1}^{k}(2k-2j+1)\left(\frac{\tau^{2}}{n}\right)^{k}\right]$$

$$=\frac{1}{\mu}\left[1+\sum_{k=1}^{\infty}\frac{(2k)!}{2^{k}k!}\left(\frac{\tau^{2}}{n}\right)^{k}\right] \qquad (2)$$

From (1), $E(\bar{X}^{-1})\to\theta, n\to\infty$ (3)

and $E(\hat{\theta}_1)=\theta$ where $\hat{\theta}_1=\frac{1}{w}\hat{\theta}$ and $w=\left[1+\sum_{k=1}^{\infty}\frac{(2k)!}{2^{k}k!}\left(\frac{\tau^{2}}{n}\right)^{k}\right]$.

Therefore, the estimation $\hat{\theta}_1$ is unbiased estimator of $\theta=\mu^{-1}$ and

$$E(\hat{\theta}^{2})=E\left(\frac{1}{\bar{X}^{2}}\right)$$

$$=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}kE(\bar{X}-\mu)^{k-1}}{\mu^{k+1}}$$

$$=\sum_{k=0}^{\infty}\frac{(2k+1)2^{k}\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}\mu^{2k+2}}\left(\frac{\sigma^{2}}{n}\right)^{k}$$

$$=\frac{1}{\mu^{2}}\left[\sum_{k=0}^{\infty}\frac{(2k+1)2^{k}\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}\mu^{2k+2}}\left(\frac{\tau^{2}}{n}\right)^{k}\right],\tau=\frac{\sigma}{\mu}$$

$$=\frac{1}{\mu^{2}}\left[\sum_{k=0}^{\infty}\frac{(2k+1)!\Gamma\left(\frac{1}{2}\right)}{2^{k}k!\sqrt{\pi}}\left(\frac{\tau^{2}}{n}\right)^{k}\right],\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

$$=\frac{1}{\mu^{2}}\left[\sum_{k=0}^{\infty}\frac{(2k+1)!}{2^{k}k!}\left(\frac{\tau^{2}}{n}\right)^{k}\right] \qquad (4)$$

This ends the proof.

Corollary1. From Theorem1, $Var(\hat{\theta})\approx\tau^{2}/n\mu^{2}$

Proof of Corollary1, consider only the first two terms of the right hand side of (1),

$$\hat{\theta}=\frac{1}{\bar{X}}\approx\frac{1}{\mu}-\frac{(\bar{X}-\mu)}{\mu^{2}}$$

$$Var(\hat{\theta})=Var\left(\frac{1}{\mu}\right)-Var\left[\frac{(\bar{X}-\mu)}{\mu^{2}}\right]$$

$$=\frac{1}{\mu^{4}}Var(\bar{X}-\mu)$$

$$=\frac{\tau^{2}}{n\mu^{2}} \qquad (5)$$

This ends the proof.

Now we will the fact that, from the Central Limit Theorem,

$$Z=\frac{\hat{\theta}-\theta}{\sqrt{Var(\hat{\theta})}}\sim N(0,1) \qquad (6)$$

Plugging (5) to (6), we have

$$Z=\frac{\frac{\hat{\theta}}{w}-\theta}{\sqrt{\frac{\tau^{2}}{n\mu^{2}}}}\sim N(0,1)$$

It is therefore easily seen that, the $100(1-\alpha)\%$ confidence interval for θ is

$$CI_{exact}=\left[\frac{\hat{\theta}}{w}\pm Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\tau^{2}}{n\bar{X}^{2}}}\right] \qquad (7)$$

where $Z_{1-\alpha/2}$ is an upper $1-\alpha/2$ quintile of the standard normal distribution.

B. Generalized Confidence Interval (GCI)

Weerahandi [5] defined the generalized pivotal as a statistic that has a distribution free of unknown parameters. The generalized theory of this method has been shown that a generalized pivotal quantity is not only a function of random variable, but also involves the observe values, parameters and nuisance parameters.

In this section, we propose the generalized confidence interval for the inverse of a normal mean with a known coefficient of variation. Let X be a random variable from normal distribution, $f\left(X,\theta,\sigma^{2}\right)$ where θ is a parameter of interest and σ^{2} is a nuisance parameter. Let x be the observed value of X.

The procedure to construct the confidence interval for the inverse of a normal mean $\theta = \mu^{-1}$ with a known coefficient variation $\tau = \sigma / \mu$ is as follow: the first construct a generalized pivotal quantity $R_\theta = R(X, x, \theta, \sigma^2)$ which is a function of the random variable X, its observed value x, and the parameters θ, σ^2. The general pivotal quantity based on the sufficient statistic $\bar{X} = n^{-1}\sum_{i=1}^{n} X_i, S^2 = (n-1)^{-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ and it can be defined as

$$R_\theta = R(X, x, \theta, \sigma^2) = \frac{\tau}{\sqrt{\frac{(n-1)s^2}{U}}} \tag{8}$$

where s^2 is the observed value of S^2 and $U = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$.

Thus R_θ is said to be the generalized pivotal quantity that satisfies the following conditions:

1) For a fixed x, the probability distribution of R_θ free of unknown parameters.
2) The observed value of R_θ as $r_{obs} = r(x, x, \theta, \sigma^2)$ does not depend on the nuisance parameter and the value of R_θ equal to μ^{-1} as $s^2 = S^2$.

Suppose $R_\theta(1-\alpha)$ is the $100(1-\alpha)th$ percentile of the distribution of R_θ, so $R_\theta(1-\alpha)$ is the $100(1-\alpha)\%$ generalized upper confidence interval for θ.

III. Simulation Studies

In this section, we write the function in R program to generate the data with means and variances. In simulation, we set mean $(\mu) = 1$, the coefficient of variation $(\tau = \sigma / \mu)$ = 0.1, 0.3, 0.5, 0.7, 0.8, 1 and the sample sizes (n) = 10, 20, 40, 100, 1000. The results of simulation studies are the coverage probabilities and average lengths of each confidence interval. The nominal level of 0.95 is calculated based on 10,000 replications. From Table I, we find that the coverage probability of the CI_{exact} confidence interval performs as well as the confidence interval CI_{gci} and they approach to 0.95 for all values of the coefficient of variation (τ). The confidence interval CI_{exact} performs better than the confidence interval CI_{gci} for the large sample size because of the confidence interval CI_{exact} based on the asymptotic normality of the test statistic Z. Table I also shows that ratio of the expected length of the two confidence intervals $E\left(CI_{gci} / CI_{exact}\right)$ is greater than 1, when the values of the coefficient of variation are less than or equal 0.7. This means the length of confidence interval CI_{exact} which is shorter than the length of the confidence interval CI_{gci}.

IV. Conclusion

In this paper, the two new confidence intervals for the inverse of a normal mean with a known coefficient of variation are studied at the first time. The simulation studies show that the coverage probabilities of both confidence interval CI_{exact} and confidence interval CI_{gci} are not significantly different and they approach to $(1-\alpha)$. When the values of coefficient of variation are small, the confidence interval CI_{exact} is shorter than CI_{gci}. The confidence interval CI_{exact} is also easy to use more than the confidence interval CI_{gci} which is based on a computational approach. So we suggest the confidence interval CI_{exact} when a coefficient of variation is known for practitioner.

TABLE I
Coverage Probability and a Ratio of Expected Length of the Intervals CI_{exact} and CI_{gci} when $\mu = 1$

n	τ	Coverage probability		Ratio of lengths
		CI_{gci}	CI_{exact}	$E(CI_{gci} / CI_{exact})$
10	0.1	0.9433	0.9500	7.9283
	0.3	0.9454	0.9465	2.6281
	0.5	0.9464	0.9457	1.5472
	0.7	0.9429	0.9427	1.0656
	0.8	0.9452	0.9382	0.9147
	1	0.9457	0.8830	0.6198
20	0.1	0.9473	0.9509	7.4176
	0.3	0.9499	0.9508	2.4691
	0.5	0.9548	0.9457	1.4676
	0.7	0.9421	0.9510	1.5797
	0.8	0.9461	0.9468	0.8968
	1	0.9458	0.9431	0.6987
40	0.1	0.9461	0.9519	7.1977
	0.3	0.9459	0.9452	2.3973
	0.5	0.9474	0.9488	1.2721
	0.7	0.9476	0.9450	0.7081
	0.8	0.9425	0.9490	0.8847
	1	0.9467	0.9515	0.6233
100	0.1	0.9481	0.9468	7.0841
	0.3	0.9424	0.9553	2.3610
	0.5	0.9449	0.9500	1.4141
	0.7	0.9484	0.9523	1.0083
	0.8	0.9453	0.9454	0.8791
	1	0.9463	0.9527	0.7018
1000	0.1	0.9470	0.9513	7.0156
	0.3	0.9468	0.9529	2.3375
	0.5	0.9472	0.9546	1.4029
	0.7	0.9436	0.9509	1.0022
	0.8	0.9458	0.9508	0.7862
	1	0.9471	0.9523	0.7708

ACKNOWLEDGMENT

The authors are grateful to Rajamangala University of Technology Lanna for their partial finance support.

REFERENCES

[1] E. Lamanna, G. Romano and C. Sgrbi, “Curvature measurements in nuclear emulsions,” Nuclear Instruments and Methods, vol. 187, pp. 387-391, 1981.

[2] A. Zaman, “Estimates without moments: the case of the reciprocal of a normal mean,” Journal of Econometrics, vol. 15, pp. 289-298, 1981a.

[3] A. Zaman, “A complete class theorem for the control problem and the future results on admissibility and inadmissibility,” Annals of Statistics, vol. 9, pp. 812-821, 1981b.

[4] C. S. withers and S. Nadarajah, “Estimators for the inverse powers of a normal mean,” Journal of Statistical Planning and Inference, vol. 143, pp. 441-455, 2013.

[5] S. Weerahandi, “Generalized confidence intervals,” Journal of the American Statistical Association, vol. 88, no. 423, pp. 899-905, Sep. 1993.

[6] R. Mahmoudvand and H. Hassani, “Two new confidence intervals for the coefficient of variation in a normal distribution,” Journal of Applied Statistics, vol. 36, no. 4, pp. 429-442, 2009.