Computing center conditions for non-analytic vector fields with constant angular speed

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Abstract—We investigate the planar quasi-septic non-analytic systems which have a center-focus equilibrium at the origin and whose angular speed is constant. The system could be changed into an analytic system by two transformations, with the help of computer algebra system MATHEMATICA, the conditions of uniform isochronous center are obtained.

Keywords—Non-analytic; Center–focus problem; Lyapunov constant; Uniform isochronous center

I. INTRODUCTION

ONE of open problems for planar polynomial differential systems is how to characterize their centers and isochronous centers. Article [1] points out that "A center of an analytic system is isochronous if and only if there exists an analytic change of coordinates such that the original system is reduced to a linear system", so an isochronous center is also called a linearizable center. If all the solution of system rotate around the origin, and their angular speed is constant along rays, finding isochronous centers is equivalent to proving the existence of centers, and the center is called to be an uniform isochronous center. Polynomial systems with an uniform isochronous center can always be written in the following form:

$$
\begin{cases}
\frac{dx}{dt} = -y + xH(x, y), \\
\frac{dy}{dt} = x + yH(x, y).
\end{cases}
\tag{1}
$$

where $H(x, y)$ is a polynomial.

The system has a unique fixed point $O(0,0)$ which is a center for the linear approximation and enjoys the property of uniform isochronicity: the motion along the trajectories occurs with the constant angular velocity equal to 1. The fixed point can then be a focus or center. If it is a center, it will be automatically isochronous. The problem of characterizing the centers of polynomial uniformly isochronous systems, which attracted the attention of many authors. When $H(x, y)$ is a homogeneous polynomial, the problem of characterizing the centers has been solved, see [7]. But when $H(x, y)$ is a nonhomogeneous polynomial, the problem remains open, see [2], [3], [4], [5], and the references therein.

In this work, we study an uniformly isochronous system-not

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necessarily polynomial ones:

$$
\begin{aligned}
\frac{dx}{dt} = {} & -y + x((x^2 + y^2)^{r-1}(A_{20}x^2 + A_{11}xy + A_{02}y^2) \\
& + (x^2 + y^2)^{3(r-1)}(A_{60}x^6 + A_{51}x^5y + A_{42}x^4y^2 \\
& + A_{33}x^3y^3 + A_{24}x^2y^4 + A_{15}xy^5 + A_{06}y^6)), \\
\frac{dy}{dt} = {} & x + y((x^2 + y^2)^{r-1}(A_{20}x^2 + A_{11}xy + A_{02}y^2) \\
& + (x^2 + y^2)^{3(r-1)}(A_{60}x^6 + A_{51}x^5y + A_{42}x^4y^2 \\
& + A_{33}x^3y^3 + A_{24}x^2y^4 + A_{15}xy^5 + A_{06}y^6)).
\end{aligned}
\tag{2}
$$

The main goal of this paper is to use the integral factor method theory to distinguish center–focus and give the conditions of uniform isochronous center.

As far as we know that there were few results about nonanalytic *Poincarè* system, this paper is divided into four section. In Section 2, we restate some known results necessary to demonstrate the main results. In Section 3, we use the recursive algorithm to obtain the conditions of uniform isochronous center for system.

II. PRELIMINARY KNOWLEDGE

The ideas of this section come from [6], where the center–focus problem of critical points in the planar dynamical systems are studied. We first recall the related notions and results. For more details, please refer to [6].

In [6], the authors defined complex center and complex isochronous center for the following complex system

$$
\begin{aligned}
\frac{dz}{dT} &= z + \sum_{k=2}^{\infty} \sum_{\alpha+\beta=k} a_{\alpha\beta}z^{\alpha}w^{\beta} = Z(z, w), \\
\frac{dw}{dT} &= -w - \sum_{k=2}^{\infty} \sum_{\alpha+\beta=k} b_{\alpha\beta}w^{\alpha}z^{\beta} = -W(z, w),
\end{aligned}
\tag{3}
$$

and gave two recursive algorithms to determine necessary conditions for a center and an isochronous center. We now restate the definitions and algorithms.

Lemma 2.1: ([6]) For system (3), we can derive uniquely the following formal series:

$$
\xi = z + \sum_{k+j=2}^{\infty} c_{kj}z^k w^j, \quad \eta = w + \sum_{k+j=2}^{\infty} d_{kj}w^k z^j, \tag{4}
$$

where $c_{k+1,k} = d_{k+1,k} = 0, k = 1, 2, \cdots$, such that

$$
\begin{aligned}
\frac{d\xi}{dT} &= \xi + \sum_{j=1}^{\infty} p_j \xi^{j+1}\eta^j, \\
\frac{d\eta}{dT} &= -\eta - \sum_{j=1}^{\infty} q_j \eta^{j+1}\xi^j.
\end{aligned}
\tag{5}
$$

Definition 2.1: ([6]) Let $\mu_0 = 0, \mu_k = p_k - q_k, \tau_k = p_k + q_k, k = 1, 2, \cdots$. μ_k is called the k_{th} singular point quantity

of the origin of system (3) and τ_k is called the k_{th} period constant of the origin of system (3).

Theorem 2.1: ([6]) For system (3), the origin is a complex center if and only if $\mu_k = 0, k = 1, 2, \cdots$. The origin is a complex isochronous center if and only if $\mu_k = \tau_k = 0, k = 1, 2, \cdots$.

Theorem 2.2: ([6]) For system (3), we can derive successively the terms of the following formal series:

$$M(z,w) = \sum_{\alpha+\beta=0}^{\infty} c_{\alpha\beta} z^{\alpha} w^{\beta}, \qquad (6)$$

such that

$$\frac{\partial(MZ)}{\partial z} - \frac{\partial(MW)}{\partial w} = \sum_{m=1}^{\infty} (m+1)\mu_m (zw)^m, \qquad (7)$$

where $c_{00} = 1, \forall c_{kk} \in R, k = 1, 2, \cdots$, and for any integer m, μ_m is determined by the following recursive formulae:

$$c_{\alpha\beta} = \frac{1}{\beta-\alpha} \sum_{k+j=3}^{\alpha+\beta+2} [(\alpha+1)a_{k,j-1} - (\beta+1)b_{j,k-1}]$$
$$\times c_{\alpha-k+1,\beta-j+1}, \qquad (8)$$
$$\mu_m = \sum_{k+j=3}^{2m+2} (a_{k,j-1} - b_{j,k-1}) c_{m-k+1,m-j+1}.$$

III. CONDITIONS OF UNIFORM ISOCHRONOUS CENTER

Now the conditions of uniform isochronous center of two classes of systems will be investigated with the help of Mathematics.

First, we consider a special case of system (2),

$$\frac{dx}{dt} = -y + x((x^2+y^2)^{r-1}(A_{11}xy) + (x^2+y^2)^{3(r-1)}$$
$$\times (A_{60}x^6 + A_{51}x^5y + A_{42}x^4y^2 + A_{33}x^3y^3$$
$$+ A_{24}x^2y^4 + A_{15}xy^5 + A_{06}y^6)),$$
$$\frac{dy}{dt} = x + y((x^2+y^2)^{r-1}(A_{11}xy) + (x^2+y^2)^{3(r-1)}$$
$$\times (A_{60}x^6 + A_{51}x^5y + A_{42}x^4y^2 + A_{33}x^3y^3$$
$$+ A_{24}x^2y^4 + A_{15}xy^5 + A_{06}y^6)).$$
$$(9)$$

Through the transformations

$$u = x(x^2+y^2)^2, \ v = y(x^2+y^2)^2,$$
$$z = u + iv, \ w = u - iv, \ T = it, \ i = \sqrt{-1},$$

system (9) can be transformed into the following system:

$$\frac{dz}{dT} = z - \frac{1}{4}A_{11}rz^3 + \frac{1}{4}A_{11}zw^2$$
$$+ \frac{i}{64}(A_{06}r + iA_{15}r - A_{24}r - iA_{33}r + A_{42}r + iA_{51}r$$
$$+ -A_{60}r)z^7 \frac{i}{7680}(-720A_{06}r - 480iA_{15}r + 240A_{24}r$$
$$+ 240A_{42}r + 480iA_{51}r - 720A_{60}r)z^6w$$
$$+ \frac{i}{3072}(720A_{06}r + 240iA_{15}r + 48A_{24}r + 144iA_{33}r$$
$$- 48A_{42}r + 240iA_{51}r - 720A_{60}r)z^5w^2 + \frac{i}{2304}$$
$$\times (-720A_{06}r - 144A_{24}r - 144A_{42}r - 720A_{60}r)z^4w^3$$
$$+ \frac{i}{3072}(720A_{06}r - 240iA_{15}r + 48A_{24}r - 144iA_{33}r$$
$$- 48A_{42}r - 240iA_{51}r - 720A_{60}r)z^3w^4$$
$$+ \frac{i}{7680}(-720A_{06}r + 480iA_{15}r + 240A_{24}r + 240A_{42}r$$
$$- 480iA_{51}r - 720A_{60}r)z^2w^5 + \frac{i}{46080}(720A_{06}r$$
$$- 720iA_{15}r - 720A_{24}r + 720iA_{33}r + 720A_{42}r$$
$$- 720iA_{51}r - 720A_{60}r)zw^6,$$
$$(10)$$

$$\frac{dw}{dT} = -w - \frac{1}{4}A_{11}rz^2w + \frac{1}{4}A_{11}rw^3$$
$$+ \frac{i}{46080}(720A_{06}r + 720iA_{15}r - 720A_{24}r - 720iA_{33}r$$
$$+ 720A_{42}r + 720iA_{51}r - 720A_{60}r)z^6w$$
$$+ \frac{i}{7680}(-720A_{06}r - 480iA_{15}r + 240A_{24}r + 240A_{42}r$$
$$+ 480iA_{51}r - 720A_{60}r)z^5w^2$$
$$+ \frac{i}{3072}(720A_{06}r + 240iA_{15}r + 48A_{24}r + 144iA_{33}r$$
$$- 48A_{42}r + 240iA_{51}r - 720A_{60}r)z^4w^3$$
$$+ \frac{i}{2304}(-720A_{06}r - 144A_{24}r - 144A_{42}r - 720A_{60}r)z^3w^4$$
$$+ \frac{i}{3072}(720A_{06}r - 240iA_{15}r + 48A_{24}r - 144iA_{33}r$$
$$- 48A_{42}r - 240iA_{51}r - 720A_{60}r)z^2w^5$$
$$+ \frac{i}{7680}(-720A_{06}r + 480iA_{15}r + 240A_{24}r + 240A_{42}r$$
$$- 480iA_{51}r - 720A_{60}r)zw^6 + \frac{i}{64}(A_{06}r - iA_{15}r - A_{24}r$$
$$+ iA_{33}r + A_{42}r - iA_{51}r - A_{60}r)w^7.$$
$$(11)$$

Next we discuss the calculation of singular point quantities and center conditions at the origin of system (10). Applying the recursive formulae in Theorem 2.2, we compute singular point quantities and simplify them, then we have

Theorem 3.1: The first five singular point quantities at the origin of system (10) are as follows:

$$u_1 = 0,$$
$$u_2 = 0,$$
$$u_3 = -\frac{i}{8}(5A_{06} + A_{24} + A_{42} + 5A_{60})r, \qquad (12)$$
$$u_4 = -\frac{i}{16}A_{11}(5A_{06} - A_{42} - 10A_{60})r^2,$$
$$u_5 = -\frac{3i}{160}A_{11}^2(15A_{06} - A_{42})r^3.$$

From Theorem 3.1, we get

Theorem 3.2: For system (10), the first five singular point quantities are zero if and only if one of the following conditions holds:

$$A_{11} = 0, \ 5A_{06} + A_{24} + A_{42} + 5A_{60} = 0;$$
$$A_{24} = -A_{42} = -15A_{06}, \ A_{60} = -A_{06}; \qquad (13)$$

Readily verify the following.

Theorem 3.3: For system (10), all the singular point quantities at the origin are zero if and only if the first two singular point quantities are zero, i.e., one of the conditions in Theorem 3.2 holds. Correspondingly, the conditions in Theorem 3.2 are the center conditions of the origin.

Proof. If conditions (3.3) holds, the sufficiency part of the Theorem follows from the fact that (3.2) is a quasi homogeneous system of degree 7 whose coefficients satisfy the equation $5A_{06} + A_{24} + A_{42} + 5A_{60} = 0$ representing a necessary and sufficient centering condition, see [7].

When conditions (3.4) holds, system (3.2) is reversible and its trajectories are symmetric with respect to both coordinate axes.

Theorem 3.4: The origin of system (10) is a uniform isochronous center if and only if one of conditions in theorem 3.3 holds.

In the previous discussion we considered a particular case. Now, we consider the general system (2). Through the same transformations as above. System (2) can be transformed into

the following system:

$$\frac{dz}{dT} = z + \frac{i}{64}(16A_{02}r + 16iA_{11}r - 16A_{20}r)z^3$$
$$+ \frac{i}{64}(-32A_{02}r - 32A_{20}r)z^2w$$
$$+ \frac{i}{128}(32A_{02}r - 32iA - 11r - 32A_{20}r)zw^2$$
$$+ \frac{i}{64}(A_{06}r + iA_{15}r - A_{24}r - iA_{33}r + A_{42}r + iA_{51}r$$
$$- A_{60}r)z^7 + \frac{i}{7680}(-720A_{06}r - 480iA_{15}r + 240A_{24}r$$
$$+ 240A_{42}r + 480iA_{51}r - 720A_{60}r)z^6w$$
$$+ \frac{i}{3072}(720A_{06}r + 240iA_{15}r + 48A_{24}r$$
$$+ 144iA_{33}r - 48A_{42}r + 240iA_{51}r - 720A_{60}r)z^5w^2$$
$$+ \frac{i}{2304}(-720A_{06}r - 144A_{24}r - 144A_{42}r$$
$$- 720A_{60}r)z^4w^3 + \frac{i}{3072}(720A_{06}r - 240iA_{15}r$$
$$+ 48A_{24}r - 144iA_{33}r - 48A_{42}r - 240iA_{51}r$$
$$- 720A_{60}r)z^3w^4 + \frac{i}{7680}(-720A_{06}r + 480iA_{15}r$$
$$+ 240A_{24}r + 240A_{42}r - 480iA_{51}r - 720A_{60}r)z^2w^5$$
$$+ \frac{i}{46080}(720A_{06}r - 720iA_{15}r - 720A_{24}r + 720iA_{33}r$$
$$+ 720A_{42}r - 720iA_{51}r - 720A_{60}r)zw^6,$$
$$\frac{dw}{dT} = -w + \frac{i}{64}(16A_{02}r - 16iA_{11}r - 16A_{20}r)w^3$$
$$+ \frac{i}{64}(-32A_{02}r - 32A_{20}r)z^2w$$
$$+ \frac{i}{128}(32A_{02}r + 32iA - 11r - 32A_{20}r)zw^2$$
$$+ \frac{i}{46080}(720A_{06}r + 720iA_{15}r - 720A_{24}r - 720iA_{33}r$$
$$+ 720A_{42}r + 720iA_{51}r - 720A_{60}r)z^6w$$
$$+ \frac{i}{7680}(-720A_{06}r - 480iA_{15}r + 240A_{24}r + 240A_{42}r$$
$$+ 480iA_{51}r - 720A_{60}r)z^5w^2$$
$$+ \frac{i}{3072}(720A_{06}r + 240iA_{15}r + 48A_{24}r + 144iA_{33}r$$
$$- 48A_{42}r + 240iA_{51}r - 720A_{60}r)z^4w^3$$
$$+ \frac{i}{2304}(-720A_{06}r - 144A_{24}r - 144A_{42}r$$
$$- 720A_{60}r)z^3w^4 + \frac{i}{3072}(720A_{06}r - 240iA_{15}r + 48A_{24}r$$
$$- 144iA_{33}r - 48A_{42}r - 240iA_{51}r - 720A_{60}r)z^2w^5$$
$$+ \frac{i}{7680}(-720A_{06}r + 480iA_{15}r + 240A_{24}r + 240A_{42}r$$
$$- 480iA_{51}r - 720A_{60}r)zw^6$$
$$+ \frac{i}{64}(A_{06}r - iA_{15}r - A_{24}r + iA_{33}r + A_{42}r$$
$$- iA_{51}r - A_{60}r)w^7.$$

$$(14)$$

Applying the recursive formulae in Theorem 2.2, we compute singular point quantities and simplify them, the first singular point quantities at the origin of system (14) is

$$u_1 = -i(A_{20} + A_{02})r,$$

If $A_{20} = -A_{02}$, then the change of variables $x = x\cos\theta + y\sin\theta, y = -x\sin\theta + y\cos\theta$, with θ defined from the condition

$$A_{20}\cos^2\theta + A_{11}\sin\theta\cos\theta + A_{02}\sin^2\theta = 0$$

reduces (2) to a system of the form (9):

$$\frac{dx}{dt} = -y + x((x^2 + y^2)^{r-1}(\tilde{A}_{11}xy) + (x^2 + y^2)^{3(r-1)}$$
$$\times (\tilde{A}_{60}x^6 + \tilde{A}_{51}x^5y + \tilde{A}_{42}x^4y^2 + \tilde{A}_{33}x^3y^3$$
$$+ \tilde{A}_{24}x^2y^4 + \tilde{A}_{15}xy^5 + \tilde{A}_{06}y^6)),$$
$$\frac{dy}{dt} = x + y((x^2 + y^2)^{r-1}(\tilde{A}_{11}xy) + (x^2 + y^2)^{3(r-1)}$$
$$\times (\tilde{A}_{60}x^6 + \tilde{A}_{51}x^5y + \tilde{A}_{42}x^4y^2 + \tilde{A}_{33}x^3y^3$$
$$+ \tilde{A}_{24}x^2y^4 + \tilde{A}_{15}xy^5 + \tilde{A}_{06}y^6)).$$

$$(15)$$

whose coefficients are expressible in terms of the coefficients of (2). In particular, we have

$$\tilde{A}_{11} = A_{11}\cos^2\theta - 4A_{20}\cos\theta\sin\theta - A_{11}\sin^2\theta,$$

$$\tilde{A}_{60} = \frac{2a_0 - a_2 + 2a_4 - a_6}{32}, \ \tilde{A}_{40} = \frac{6a_0 - a_2 - 10a_4 + 15a_6}{32},$$

$$\tilde{A}_{24} = \frac{6a_0 + a_2 - 10a_4 - 15a_6}{32}, \ \tilde{A}_{06} = \frac{2a_0 + a_2 + 2a_4 + a_6}{32},$$

where

$$a_0 = 5A_{60} + A_{24} + A_{42} + 5A_{06},$$

$$a_2 = (15A_{60} - A_{24} + A_{42} - 15A_{06})\cos2\theta + (5A_{51} + 3A_{33} + 5c_{15})\sin2\theta,$$

$$a_4 = (3A_{60} - A_{42} - A_{24} + 3A_{06})\cos4\theta + 2(A_{51} - A_{15})\sin4\theta,$$

$$a_6 = (A_{60} - A_{42} + A_{24} - A_{06})\cos6\theta + (A_{51} - A_{33} + c_{15})\sin6\theta.$$

Using Theorem 3.2, we see that the origin is a center of system (15) if and only if one of the following two conditions is satisfied:

Theorem 3.5: The origin of system (15) is a uniform isochronous center if and only if one of conditions holds:

$$A_{20} + A_{02} = 0, \widetilde{A}_{11} = 0, \ 5\widetilde{A}_{06} + \widetilde{A}_{24} + \widetilde{A}_{42} + 5\widetilde{A}_{60} = 0; \quad (16)$$

$$A_{20} + A_{02} = 0, \widetilde{A}_{24} = -\widetilde{A}_{42} = -15\widetilde{A}_{06}, \ \widetilde{A}_{60} = -A_{06}; \quad (17)$$

So we could get the following theorem easily

Theorem 3.6: The origin of system (2) is a uniform isochronous center if and only if one of conditions holds:

$$A_{20} = A_{11} = A_{02} = 0, \ 5A_{60} + A_{42} + A_{24} + 5c_{06} = 0, \quad (18)$$

$$A_{20} + A_{02} = 0, \ 5A_{60} + A_{42} + A_{24} + 5c_{06} = 0,$$

$$A_{11}(15A_{60} + A_{42} - A_{24} - 15A_{06})$$
$$+ 2A_{20}(5A_{51} + 3A_{33} + 5A_{15}) = 0,$$

$$(A_{11}^2 - 4A_{20}^2)(3A_{60} - A_{42} - A_{24} + 3A_{06}) \quad (19)$$
$$+ 8A_{20}A_{11}(A_{51} - A_{15}) = 0,$$

$$A_{11}(A_{11}^2 - 12A_{20}^2)(A_{60} - A_{42} + A_{24} - A_{06})$$
$$+ 2A_{20}(A_{11}^2 - 4A_{20}^2)(A_{51} - A_{33} + A_{15}) = 0.$$

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