

# Compression and Filtering of Random Signals under Constraint of Variable Memory

Anatoli Torokhti and Stan Miklavcic

**Abstract**—We study a new technique for optimal data compression subject to conditions of causality and different types of memory. The technique is based on the assumption that some information about compressed data can be obtained from a solution of the associated problem without constraints of causality and memory. This allows us to consider two separate problem related to compression and decompression subject to those constraints. Their solutions are given and the analysis of the associated errors is provided.

**Keywords**—stochastic signals, optimization problems in signal processing.

## I. INTRODUCTION

A Study of data compression methods is motivated by the necessity to reduce expenditures incurred with the transmission, processing and storage of large data arrays. Such methods have also been applied successfully to the solution of problems related to clustering, feature selection and forecasting. While the topics have been intensively studied (see e.g. [1]–[10]), a number of related fundamental questions are still open. One of them concerns specific restrictions associated with different types of causality and memory.

### A. First motivation: causality and memory

Data compression techniques mainly consist of two operations, compression itself and decompression (or reconstruction). In reality, both operations are subject to the conditions of causality and memory.

By the heuristic definition of causality, the present value of the output of a filter is not affected by future values of the input [11]. To determine the output signal at time  $t_0$ , the causal filter should “remember” the input signal up to time  $t_0$ .

A filter with *finite memory*  $\Delta$  is “able” to determine the output signal at time  $t_0$  from a fragment of the input signal on the segment  $[t_0 - \Delta, t_0]$  only. In other words, the filter with finite memory  $\Delta$  should “remember” the input signal on the segment of length  $\Delta$  [5].

A broader notion of a filter with *variable finite memory* [5] means that such a filter should remember segments of the input signal on time segments of different length [5].

A formalization of these concepts are given in Section III-A below.

Our first motivation comes from the observation that the filters we propose should be causal with variable finite memory.

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### B. Second motivation: reformulation of the problem

Let  $(\Omega, \Sigma, \mu)$  be a probability space, where  $\Omega = \{\omega\}$  is the set of outcomes,  $\Sigma$  a  $\sigma$ -field of measurable subsets in  $\Omega$  and  $\mu : \Sigma \rightarrow [0, 1]$  an associated probability measure on  $\Sigma$  with  $\mu(\Omega) = 1$ .

Below, in Sections I-B1–I-B3, we consider some possible statements of the problem and discuss associated assumptions. An alternative new statement of the problem developed in this paper is formulated in Section II.

1) *Informal statement of the problem:* In an informal way, the data compression problem we consider can be expressed as follows.

Let  $\mathbf{y} \in L^2(\Omega, \mathbb{R}^n)$  be observable data and  $\mathbf{x} \in L^2(\Omega, \mathbb{R}^m)$  be a reference signal that is to be estimated from  $\mathbf{y}$  in such a way that, (a) the data  $\mathbf{y}$  should be compressed to a shorter vector  $\mathbf{z} \in L^2(\Omega, \mathbb{R}^r)^1$  with  $r < \min\{m, n\}$  and (b)  $\mathbf{z}$  should be decompressed (reconstructed) to a signal  $\tilde{\mathbf{x}} \in L^2(\Omega, \mathbb{R}^m)$  that is ‘close’ to  $\mathbf{x}$  in some appropriate sense. Both operations should be *causal* and *have variable finite memory*. In this paper, the term ‘close’ is used with respect to the minimum of the norm (2) of the difference between  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$ .

2) *Some possible formalizations of the problem:* The problem can be formulated in several alternate ways. Here, we consider two of them.

*The first way* is as follows. Let  $\mathcal{B} : L^2(\Omega, \mathbb{R}^n) \rightarrow L^2(\Omega, \mathbb{R}^r)$  signify compression so that  $\mathbf{z} = \mathcal{B}(\mathbf{y})$  and let  $\mathcal{A} : L^2(\Omega, \mathbb{R}^r) \rightarrow L^2(\Omega, \mathbb{R}^m)$  designate data decompression, i.e.,  $\tilde{\mathbf{x}} = \mathcal{A}(\mathbf{z})$ . We suppose that  $\mathcal{B}$  and  $\mathcal{A}$  are linear operators defined by the relationships

$$[\mathcal{B}(\mathbf{y})](\omega) = B[\mathbf{y}(\omega)] \quad \text{and} \quad [\mathcal{A}(\mathbf{z})](\omega) = A[\mathbf{z}(\omega)] \quad (1)$$

where  $B \in \mathbb{R}^{n \times r}$  and  $A \in \mathbb{R}^{r \times m}$ . In the remainder of this paper we shall use the same symbol to represent both the linear operator acting on a random vector and its associated matrix.

We define the norm to be

$$\|\mathbf{x}\|_{\Omega}^2 = \int_{\Omega} \|\mathbf{x}(\omega)\|_2^2 d\mu(\omega) \quad (2)$$

where  $\|\mathbf{x}(\omega)\|_2$  is the Euclidean norm of  $\mathbf{x}(\omega)$ . Let us denote by  $J(A, B)$ , the norm of the difference between  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$ , constructed by  $A$  and  $B$ :

$$J(A, B) = \|\mathbf{x} - (A \circ B)(\mathbf{y})\|_{\Omega}^2. \quad (3)$$

The problem is to find  $B^0 : L^2(\Omega, \mathbb{R}^n) \rightarrow L^2(\Omega, \mathbb{R}^r)$  and  $A^0 : L^2(\Omega, \mathbb{R}^r) \rightarrow L^2(\Omega, \mathbb{R}^m)$  such that

$$J(A^0, B^0) = \min_{A, B} J(A, B) \quad (4)$$

<sup>1</sup>Components of  $\mathbf{z}$  are often called *principal components* [1].

subject to conditions of causality and variable finite memory for  $A$  and  $B$ .

A *second way* to formulate the problem is as follows. Let  $\mathcal{F} : L^2(\Omega, \mathbb{R}^n) \rightarrow L^2(\Omega, \mathbb{R}^m)$  be a linear operator defined by

$$[\mathcal{F}(\mathbf{y})](\omega) = F[\mathbf{y}(\omega)] \quad (5)$$

where  $F \in \mathbb{R}^{n \times m}$ . Let  $\text{rank } F = r$  and

$$J(F) = \|\mathbf{x} - \mathcal{F}(\mathbf{y})\|_{\Omega}^2.$$

Find  $\mathcal{F}^0 : L^2(\Omega, \mathbb{R}^n) \rightarrow L^2(\Omega, \mathbb{R}^m)$  such that

$$J(F^0) = \min_F J(F) \quad (6)$$

subject to

$$\text{rank } F \leq \min\{m, n\} \quad (7)$$

and conditions of causality and variable finite memory for  $F$ .

### 3) Inherent assumptions and possible solution approaches:

If there are no constraints associated with causality and variable finite memory, then solutions of problems (4) and (6)-(7) are based on the assumptions that certain covariance matrices are known. In this regard, see, for example, [2]–[5]. In particular, the solution of problem (4) provided in [2] (with no constraints of causality and variable finite memory) follows from an iterative scheme that requires a knowledge of two covariance matrices at each step of the scheme. Thus, if the method proposed in [2] requires  $p$  iterative steps, then it requires a knowledge of  $2p$  covariance matrices.

In the case of no constraints of causality and variable finite memory, the known solution of problem (6)-(7) requires knowledge of two covariance matrices only [3], [4], [5].

4) *Difficulties associated with problems (4) and (6)-(7)* : A special difficulty with solving problem (4) is that it involves two unknowns,  $A$  and  $B$ , but only one functional to minimize. An iterative approach to its *approximate* solution based on the method presented in [2] requires knowledge of a number of covariance matrices. Another difficulty is that  $A$  and  $B$  must keep their special form associated with causality and variable finite memory. Related definitions are given in Section I-B2.

Unlike (4), the problem (6)–(7) has only one unknown. Nevertheless, the main difficulty associated with problem (6)–(7) is similar: an implementation of the conditions of causality and variable finite memory into a structure of  $F$  implies a representation of  $F$  as a product of two factors, each with a specific shape related to causality and memory. An approach to its *exact* solution based on the methodology presented in [3], [4], [5] would require knowledge of two covariance matrices only, but implies constraints related to a special shape of each factor in a decomposition of  $F$  as a product of two factors. Therefore, as with problem (4), the difficulty again relates to dealing with two unknowns with only one functional to minimize.

## II. STATEMENT OF THE PROBLEM

### A. Basic idea

To avoid the difficulties discussed above, we propose a new approach to the solution of the data compression problem. The basic idea is as follows.

Let  $\mathbf{x} \in L^2(\Omega, \mathbb{R}^m)$ ,  $\mathbf{y} \in L^2(\Omega, \mathbb{R}^n)$  and  $\mathbf{z} \in L^2(\Omega, \mathbb{R}^r)$ , and let  $A$  and  $B$  be defined as in Section I-B2. Let  $\mathcal{M}(r, n, \eta_B)$  and  $\mathcal{M}(m, r, \eta_A)$  be sets of causal matrices,  $B \in \mathbb{R}^{r \times n}$  and  $A \in \mathbb{R}^{m \times r}$ , with variable finite memories,  $\eta_B$  and  $\eta_A$ , respectively, defined as in Section III-A below. We assume that information about vector  $\mathbf{z}$  in the form of associated covariance matrices can be obtained, in particular, from the known solution [5] of problem (6)-(7) *with no constraints associated with causality and memory*.

Therefore, in this paper, the data compression problem subject to  $B \in \mathcal{M}(r, n, \eta_B)$  and  $A \in \mathcal{M}(m, r, \eta_A)$  is proposed to state in the form of two separate problems (8) and (9) formulated below.

It is shown in Section III-A that operators  $B$  and  $A$  satisfying conditions of causality and variable finite memory must have special forms. In Section III-A this issue is discussed in detail.

### B. The problem

Consider

$$J_1(B) = \|\mathbf{z} - B(\mathbf{y})\|_{\Omega}^2.$$

Let  $B^0$  be such that

$$J_1(B^0) = \min_B J_1(B) \quad \text{subject to } B \in \mathcal{M}(r, n, \eta_B). \quad (8)$$

Definitions of the set  $\mathcal{M}(r, n, \eta_B)$  and the set  $\mathcal{M}(m, r, \eta_A)$  used in (9) are given in Section III-A below. We write  $\mathbf{z}^0 = B^0(\mathbf{y})$ .

Next, let

$$J_2(A) = \|\mathbf{x} - A(\mathbf{z}^0)\|_{\Omega}^2$$

and let  $A^0$  be such that

$$J_2(A^0) = \min_A J_2(A) \quad \text{subject to } A \in \mathcal{M}(m, r, \eta_A). \quad (9)$$

We denote  $\mathbf{x}^0 = A^0(\mathbf{z}^0)$ .

The problem considered in this paper is to find operators  $B^0$  and  $A^0$  that satisfy minimization criteria (8) and (9), respectively.

Operator  $B^0$  provides a compression of the signal  $\mathbf{y}$  to the form  $\mathbf{z}^0$ . It is shown in Section III-B that the solution requires knowledge of four covariance matrices formed from the vectors  $\mathbf{z}$  and  $\mathbf{y}$ . Further,  $A^0$  reconstructs the compressed signal  $\mathbf{z}^0$  to the form  $\mathbf{x}^0$  so that  $\mathbf{x}^0$  is an optimal representation of  $\mathbf{x}$  in the sense of the constrained minimization problem (9). This stage implies knowledge of four covariance matrices formed from  $\mathbf{x}$  and  $\mathbf{z}^0$ .

### C. Differences of our statement of the problem

The major differences between the above statement of the problem and the earlier statements found in [2]–[5] are as follows. Firstly,  $A$  and  $B$  should be causal with variable finite memory. Secondly, we represent the initial problem considered in Section I-B1 in the form of a concatenation of two new separate problems (8) and (9).

The crucial assumption we make is that information about vector  $\mathbf{z}$  can be obtained from the known solution [5] of problem (6)-(7) with no constraints associated with causality and memory.

## III. MAIN RESULTS

## A. Formalization of concept of variable memory

Let  $\tau_1 < \tau_2 < \dots < \tau_n$  be time instants and  $\alpha, \beta, \vartheta : \mathbb{R} \rightarrow L^2(\Omega, \mathbb{R})$  be continuous functions. Suppose  $\alpha_k = \alpha(\tau_k)$ ,  $\beta_k = \beta(\tau_k)$  and  $\vartheta_k = \vartheta(\tau_k)$  are real valued random variables having finite second moments. We write

$$\mathbf{x} = [\alpha_1, \alpha_2, \dots, \alpha_m]^T \quad \mathbf{y} = [\beta_1, \beta_2, \dots, \beta_n]^T$$

and

$$\mathbf{z} = [\vartheta_1, \dots, \vartheta_r]^T.$$

Let  $\tilde{\mathbf{z}}$  be a compressed form of data  $\mathbf{y}$  defined by  $\tilde{\mathbf{z}} = B(\mathbf{y})$  with  $\tilde{\mathbf{z}} = [\tilde{\vartheta}_1, \dots, \tilde{\vartheta}_r]^T$ , and  $\tilde{\mathbf{x}}$  be a de-compression of  $\tilde{\mathbf{z}}$  defined by  $\tilde{\mathbf{x}} = A(\tilde{\mathbf{z}})$  with  $\tilde{\mathbf{x}} = [\tilde{\alpha}_1, \dots, \tilde{\alpha}_m]^T$ .

In many applications<sup>2</sup>, to obtain  $\tilde{\vartheta}_k$  for  $k = 1, \dots, r$ , it is necessary for the compressor  $B$  to use only a limited number of input components,  $\eta_{B_k} = 1, \dots, r$ . A number of such input components  $\eta_{B_k}$  is here called a *kth local memory* for  $B$ .

To define a notation of memory for the compressor  $B$ , we use parameters  $p$  and  $g$  which are positive integers such that

$$1 \leq p \leq n \quad \text{and} \quad n - r + 2 \leq g \leq n.$$

**Definition 1:** The vector  $\eta_B = [\eta_{B_1}, \dots, \eta_{B_r}]$  is called a *variable memory* of the compressor  $B$ . In particular,  $\eta_B$  is called a *complete* variable memory if  $\eta_{B_1} = g$  and  $\eta_{B_k} = n$  when  $k = n - g + 1, \dots, n$ . Vector  $\eta_B$  is called a *truncated* variable memory of  $B$  if, for  $r \leq p \leq n$ ,  $\eta_{B_1} = p - r + 1$  and  $\eta_{B_r} = p$ . Here,  $p$  relates to the last possible nonzero entry in the bottom row of  $B$  and  $g$  relates to the last possible nonzero entry in the first row.

The notation  $\eta_A = [\eta_{A_1}, \dots, \eta_{A_m}]$  has a similar meaning for the de-compressor  $A$ . Here,  $\eta_{A_j}$  is the *jth local memory* of  $A$ . In other words,  $\eta_{A_j}$  is the number of input components used by the de-compressor  $A$  to obtain the estimate  $\tilde{\alpha}_j$  with  $j = 1, \dots, m$ .

The parameters  $q$  and  $s$  which are positive integers such that

$$1 \leq q \leq r \quad \text{and} \quad 2 \leq s \leq m$$

are used below to define two types of memory for  $A$ .

**Definition 2:** Vector  $\eta_A$  is called a *complete* variable memory of the de-compressor  $A$  if  $\eta_{A_1} = q$  and  $\eta_{A_j} = r$  when  $j = s + r - 1, \dots, m$ . Vector  $\eta_A$  is called a *truncated* variable memory of  $A$  if  $\eta_{A_j} = 0$  for  $j = 1, \dots, s - 1$ ,  $\eta_{A_s} = s$  and  $\eta_{A_j} = r$  when  $j = s + r - 1, \dots, m$ . Here,  $q$  relates to the first possible nonzero entry in the last column of  $A$  and  $s$  relates to the first possible nonzero entry in the first column.

The memory constraints described above imply that certain elements of the matrices  $B = \{b_{ij}\}_{i,j=1}^{r,n}$  and  $A = \{a_{ij}\}_{i,j=1}^{m,r}$  must be set equal to zero. In this regard, for matrix  $B$  with  $r \leq p \leq n$ , we require that

$$b_{i,j} = 0 \quad (10)$$

$$\text{if } j = p - r + i + 1, \dots, n, \text{ for } \begin{cases} p = r, \dots, n - 1, \\ i = 1, \dots, r \end{cases} \quad (11)$$

<sup>2</sup>Examples include computer medical diagnostics [14] and problems of bio-informatics [15].

$$\text{and } \begin{cases} p = n, \\ i = 1, \dots, r - 1, \end{cases} \quad (12)$$

and, for  $1 \leq p \leq r - 1$ , it is required that

$$b_{i,j} = 0 \quad (13)$$

$$\text{if } \begin{cases} i = 1, \dots, r - p, \\ j = 1, \dots, n, \end{cases} \quad (14)$$

$$\text{and if } \begin{cases} i = r - p + 1, \dots, r, \\ j = i - r + p + 1, \dots, n. \end{cases} \quad (15)$$

See Examples 1 and 2 below.

For matrix  $A$  with  $r \leq p \leq n$ , we require

$$a_{i,j} = 0 \quad (16)$$

$$\text{if } j = q + i, \dots, r \text{ for } q = 1, \dots, r - 1, i = 1, \dots, r - q,$$

and, for  $2 \leq s \leq m$ , it is required that

$$a_{i,j} = 0 \quad (17)$$

$$\text{if } j = s + i, \dots, r \text{ for } s = 1, \dots, m, i = 1, \dots, s + r - 1,$$

See Examples 3 and 4 below.

The above conditions imply the following definitions.

**Definition 3:** A matrix  $B$  satisfying the constraint (10)–(11) is said to be a causal operator with the *truncated* variable memory  $\eta_B = [p - r + 1, \dots, p]$ . The set of such matrices is denoted by  $\mathcal{M}_T(r, n, \eta_B)$ .

**Example 1:** Let  $n = 8$ ,  $r = 3$  and  $p = 7$ . If the symbol  $\bullet$  denotes an entry that may be non-zero, then  $B$  of the form

$$B = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & 0 \end{bmatrix}$$

is a causal operator with the *truncated* variable memory  $\eta_B = [5, 6, 7]$ .

**Definition 4:** A matrix  $B$  satisfying the constraint (15)–(14) is said to be a causal operator with the *complete* variable memory  $\eta_B = [g, g + 1, \dots, n]$ . Here,  $\eta_{B_k} = n$  when  $k = n - g + 1, \dots, n$ . The set of such matrices is denoted by  $\mathcal{M}_C(r, n, \eta_B)$ .

**Example 2:** Let  $n = 6$ ,  $r = 4$  and  $g = 4$ . Then  $B$  such that

$$B = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{bmatrix}$$

is the causal operator with the *complete* variable memory  $\eta_B = [4, 5, 6, 6]$ .

**Definition 5:** A matrix  $A$  satisfying the constraint (16)–(17) is said to be a causal operator with the *complete* variable memory  $\eta_A = [r - q + 1, \dots, r]$ . Here,  $\eta_{A_j} = r$  when  $j = q, \dots, m$ . The set of such matrices is denoted by  $\mathcal{M}_C(m, r, \eta_A)$ .

**Example 3:** Let  $m = 5$ ,  $r = 4$  and  $q = 3$ . Then  $A$  of the form

$$A = \begin{bmatrix} \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix}$$

is a causal operator with the *complete* variable memory  $\eta_A = [2, 3, 4, 4, 4]$ .

**Definition 6:** A matrix  $A$  satisfying the constraint (17)–(18) is said to be a causal operator with the *truncated* variable memory  $\eta_A = [0, \dots, 0, 1, \dots, r]$ . Here,  $\eta_{A_j} = 0$  when  $j = 1, \dots, s-1$ , and  $\eta_{A_j} = r$  when  $j = s+r-1, \dots, m$ . The set of such matrices is denoted by  $\mathcal{M}_T(m, r, \eta_A)$ .

**Example 4:** Let  $m = 6$ ,  $r = 4$  and  $s = 3$ . Then matrix  $A$  such that

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix}$$

is a causal operator with the *truncated* variable memory  $\eta_A = [0, 0, 1, 2, 3, 4]$ .

#### B. Solution of problems (8) and (9)

To proceed any further we shall require some more notation. Let

$$\langle \alpha_i, \beta_j \rangle = \int_{\Omega} \alpha_i(\omega) \beta_j(\omega) d\mu(\omega) < \infty,$$

$$E_{xy} = \{\langle \alpha_i, \beta_j \rangle\}_{i,j=1}^{m,n} \in \mathbb{R}^{m \times n}, \quad (18)$$

$$\mathbf{y}_1 = [\beta_1, \dots, \beta_{g-1}]^T, \quad \mathbf{y}_2 = [\beta_g, \dots, \beta_n]^T, \quad (19)$$

$$\mathbf{z}_1 = [\vartheta_1, \dots, \vartheta_{g-1}]^T \quad \text{and} \quad \mathbf{z}_2 = [\vartheta_g, \dots, \vartheta_n]^T. \quad (20)$$

The pseudo-inverse matrix [12] for any matrix  $M$  is denoted by  $M^\dagger$ . The symbol  $\mathbb{O}$  designates the zero matrix.

**Definition 7:** [5] Two random vectors  $\mathbf{u}$  and  $\mathbf{w}$  are said to be mutually orthogonal if  $E_{uw} = \mathbb{O}$ .

**Lemma 1:** [5] If we define

$$\mathbf{w}_1 = \mathbf{y}_1 \quad \text{and} \quad \mathbf{w}_2 = \mathbf{y}_2 - P_y \mathbf{y}_1$$

where  $P_y = E_{y_1 y_2} E_{y_1 y_1}^\dagger + D_y (I - E_{y_1 y_1} E_{y_1 y_1}^\dagger)$  with  $D_y$  an arbitrary matrix, then  $\mathbf{w}_1$  and  $\mathbf{w}_2$  are mutually orthogonal random vectors.

**1) Solution of problem (8). The case of complete variable memory:** Let us first consider problem (8) when  $B$  has the complete variable memory  $\eta_B = [g, g+1, \dots, n]$  (see Definition 4).

Let us partition  $B$  in four matrices  $K_B, L_B, S_{B1}$  and  $S_{B2}$  so that

$K_B \in \mathbb{R}^{n_b \times (g-1)}$  is a rectangular matrix with  $n_b = n - g + 1$ ,  $L_B \in \mathbb{R}^{n_b \times n_b}$  is a lower triangular matrix, and  $S_{B1} \in \mathbb{R}^{(r-n_b) \times (g-1)}$ ,  $S_{B2} \in \mathbb{R}^{(r-n_b) \times n_b}$  are rectangular matrices.

We have

$$\begin{aligned} B(\mathbf{y}) &= \begin{bmatrix} K_B & L_B \\ S_{B1} & S_{B2} \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} K_B(\mathbf{y}_1) + L_B(\mathbf{y}_2) \\ S_{B1}(\mathbf{y}_1) + S_{B2}(\mathbf{y}_2) \end{bmatrix} \\ &= \begin{bmatrix} K_B(\mathbf{w}_1) + L_B(\mathbf{w}_2 + P_y(\mathbf{w}_1)) \\ S_{B1}(\mathbf{w}_1) + S_{B2}(\mathbf{w}_2 + P_y(\mathbf{w}_1)) \end{bmatrix} \\ &= \begin{bmatrix} T_B(\mathbf{w}_1) + L_B(\mathbf{w}_2) \\ S_B(\mathbf{w}_1) + S_{B2}(\mathbf{w}_2) \end{bmatrix}, \end{aligned}$$

where

$$T_B = K_B + L_B P_y \quad \text{and} \quad S_B = S_{B1} + S_{B2} P_y. \quad (21)$$

Then

$$\begin{aligned} J_1(B) &= \|\mathbf{z} - B(\mathbf{y})\|_{\Omega}^2 \\ &= \left\| \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} - \begin{bmatrix} T_B & L_B \\ S_B & S_{B2} \end{bmatrix} \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix} \right\|_{\Omega}^2 \\ &= J^{(1)}(T_B, L_B) + J^{(2)}(S_B, S_{B2}), \end{aligned} \quad (22)$$

where

$$J^{(1)}(T, L) = \|\mathbf{z}_1 - [T_B(\mathbf{w}_1) + L_B(\mathbf{w}_2)]\|_{\Omega}^2$$

and

$$J^{(2)}(S_B, S_{B2}) = \|\mathbf{z}_2 - [S_B(\mathbf{w}_1) + S_{B2}(\mathbf{w}_2)]\|_{\Omega}^2.$$

By analogy with Lemma 37 in [5],

$$\begin{aligned} \min_{B \in \mathcal{M}(r, n, \eta_B)} J_1(B) &= \min_{T_B, L_B} J^{(1)}(T_B, L_B) \\ &\quad + \min_{S_B, S_{B2}} J^{(2)}(S_B, S_{B2}). \end{aligned} \quad (23)$$

Therefore, problem (8) is reduced to finding matrices  $T_B^0, L_B^0, S_B^0$  and  $S_{B2}^0$  such that

$$J^{(1)}(T_B^0, L_B^0) = \min_{T_B, L_B} J^{(1)}(T_B, L_B) \quad (24)$$

and

$$J^{(2)}(S_B^0, S_{B2}^0) = \min_{S_B, S_{B2}} J^{(2)}(S_B, S_{B2}). \quad (25)$$

Taking into account the orthogonality of vectors  $\mathbf{w}_1$  and  $\mathbf{w}_2$ , and working in analogy with the argument on pp. 348–352 in [5], it follows that matrices  $S_B^0$  and  $S_{B2}^0$  are given by

$$S_B^0 = E_{z_2 w_1} E_{w_1 w_1}^\dagger + H_B (I - E_{w_1 w_1} E_{w_1 w_1}^\dagger) \quad (26)$$

and

$$S_{B2}^0 = E_{z_2 w_2} E_{w_2 w_2}^\dagger + H_{B2} (I - E_{w_2 w_2} E_{w_2 w_2}^\dagger), \quad (27)$$

where  $H_B$  and  $H_{B2}$  are arbitrary matrices.

Next, to find  $T_B^0$  and  $L_B^0$  we use the following notation.

For  $r = 1, 2, \dots, \ell$ , let  $\rho$  be the rank of the matrix  $E_{w_2 w_2} \in \mathbb{R}^{n_2 \times n_2}$  with  $n_b = n - g + 1$ , and let

$$E_{w_2 w_2}^{1/2} = Q_{w, \rho} R_{w, \rho}$$

be the QR-decomposition for  $E_{w_2 w_2}^{1/2}$  where  $Q_{w, \rho} \in \mathbb{R}^{n_2 \times \rho}$  and  $Q_{w, \rho}^T Q_{w, \rho} = I$  and  $R_{w, \rho} \in \mathbb{R}^{\rho \times n_2}$  is upper trapezoidal with rank  $\rho$ . We write  $G_{w, \rho} = R_{w, \rho}^T$  and use the notation

$$G_{w, \rho} = [g_1, \dots, g_\rho] \in \mathbb{R}^{n_2 \times \rho}$$

where  $g_j \in \mathbb{R}^{n_2}$  denotes the  $j$ -th column of  $G_{w,\rho}$ . We also write

$$G_{w,s} = [g_1, \dots, g_s] \in \mathbb{R}^{n_2 \times s} \quad (28)$$

for  $s \leq \rho$  to denote the matrix consisting of the first  $s$  columns of  $G_{w,\rho}$ . For the sake of simplicity, let us set

$$G_s := G_{w,s}. \quad (29)$$

Next,

$$\mathbf{e}_1^T = [1, 0, 0, 0, \dots], \quad \mathbf{e}_2^T = [0, 1, 0, 0, \dots], \quad \text{etc.}$$

denote the unit row vectors whatever the dimension of the space.

Finally, any square matrix  $M$  can be written as  $M = M_\Delta + M_\nabla$  where  $M_\Delta$  is lower triangular and  $M_\nabla$  is strictly upper triangular. We write  $\|\cdot\|_F$  for the Frobenius norm.

**Lemma 2:** If  $A = B + C$  and  $b_{ij}c_{ij} = 0$  for all  $i, j$  then

$$\|A\|_F^2 = \|B\|_F^2 + \|C\|_F^2.$$

**Theorem 1:** Let  $B$  have the complete variable memory  $\eta_B = [g, g+1, \dots, n]$ . Then the solution to problem (8) is provided by the matrix  $B^0$ , which has the form

$$B^0 = \begin{bmatrix} K_B^0 & L_B^0 \\ S_{B1}^0 & S_{B2}^0 \end{bmatrix},$$

where the blocks  $K_B^0 \in \mathbb{R}^{n_b \times (g-1)}$ ,  $S_{B1}^0 \in \mathbb{R}^{(r-n_b) \times (g-1)}$  and  $S_{B2}^0 \in \mathbb{R}^{(r-n_b) \times n_b}$  are rectangular, and the block  $L_B^0 \in \mathbb{R}^{n_b \times n_b}$  is lower triangular. These blocks are given as follows. The block  $K_B^0$  is given by

$$K_B^0 = T_B^0 - L_B^0 P_y \quad (30)$$

with

$$T_B^0 = E_{z_1 w_1} E_{w_1 w_1}^\dagger + N_{B1} (I - E_{w_1 w_1} E_{w_1 w_1}^\dagger) \quad (31)$$

where  $N_{B1}$  is an arbitrary matrix. The block  $L_B^0 = \begin{bmatrix} \lambda_1^0 \\ \vdots \\ \lambda_{n_b}^0 \end{bmatrix}$ , for each  $s = 1, 2, \dots, n_2$ , is defined by its rows

$$\lambda_s^0 = \mathbf{e}_s^T E_{z_1 w_2} E_{w_2 w_2}^\dagger G_s G_s^\dagger + f_s^T (I - G_s G_s^\dagger) \quad (32)$$

with  $f_s^T \in \mathbb{R}^{1 \times n_2}$  arbitrary. The blocks  $S_{B1}^0$  and  $S_{B2}^0$  are given by

$$S_{B1}^0 = S_B^0 - S_{B2}^0 P_y \quad (33)$$

and (27), respectively. In (33),  $S_B^0$  is presented by (26).

The minimum error associated with the matrix  $B^0$  is given by

$$\begin{aligned} \|\mathbf{z} - B^0 \mathbf{y}\|_\Omega^2 &= \sum_{s=1}^{\rho} \sum_{j=s+1}^{n_2} |e_s^T E_{z_1 w_2} E_{w_2 w_2}^\dagger g_j|^2 \\ &+ \sum_{j=1}^2 \|E_{z_j z_j}^{-1/2}\|_F^2 - \sum_{i=1}^2 \sum_{j=1}^2 \|E_{z_i w_i} E_{w_j w_j}^\dagger\|_F^2. \end{aligned}$$

**2) Solution of problem (9). The case of complete variable memory:** Let us now consider problem (9) when  $A$  has the complete variable memory  $\eta_A = [r-q+1, \dots, r]$  (see Definition 5).

In analogy with our partitioning of matrix  $B$ , we partition matrix  $A$  in four matrices  $K_A, L_A, S_{A1}$  and  $S_{A2}$  where

$K_A \in \mathbb{R}^{q \times (r-q)}$  is a rectangular matrix,

$L_A \in \mathbb{R}^{q \times q}$  is a lower triangular matrix, and

$S_{A1} \in \mathbb{R}^{(m-q) \times (r-q)}$ ,  $S_{A2} \in \mathbb{R}^{(m-q) \times q}$  are rectangular matrices.

Let us partition  $\mathbf{z}^0$  so that

$$\mathbf{z}^0 = \begin{bmatrix} \mathbf{z}_1^0 \\ \mathbf{z}_2^0 \end{bmatrix}$$

with  $\mathbf{z}_1^0 \in L^2(\Omega, \mathbb{R}^{r-q})$  and  $\mathbf{z}_2^0 \in L^2(\Omega, \mathbb{R}^q)$ . We also write

$$\mathbf{x}_1 = [\alpha_1 \dots, \alpha_{r-q}]^T \quad \text{and} \quad \mathbf{x}_2 = [\alpha_{r-q+1}, \dots, \alpha_m]^T,$$

and denote by  $\mathbf{v}_1 \in L^2(\Omega, \mathbb{R}^{r-q})$  and  $\mathbf{v}_2 \in L^2(\Omega, \mathbb{R}^q)$ , orthogonal vectors according to Lemma 1 as

$$\mathbf{v}_1 = \mathbf{z}_1^0 \quad \text{and} \quad \mathbf{v}_2 = \mathbf{z}_2^0 - P_z \mathbf{z}_1^0,$$

where  $P_z = E_{z_1 z_2} E_{z_1 z_1}^\dagger + D_z (I - E_{z_1 z_1} E_{z_1 z_1}^\dagger)$  with  $D_z$  an arbitrary matrix.

By analogy with (28)–(29), we write

$$G_{v,s} = [g_1, \dots, g_s] \in \mathbb{R}^{q \times s}$$

where  $G_{v,s}$  is constructed from a QR-decomposition of  $E_{v_2 v_2}^{1/2}$ , in a manner similar to the construction of matrix  $G_{w,s}$ .

Furthermore, we shall define  $G_s := G_{v,s}$ .

**Theorem 2:** Let  $A$  have the complete variable memory  $\eta_A = [r-q+1, \dots, r]$ . Then the solution to problem (9) is provided by the matrix  $A^0$ , which has the form

$$A^0 = \begin{bmatrix} K_A^0 & L_A^0 \\ S_{A1}^0 & S_{A2}^0 \end{bmatrix},$$

where the blocks  $K_A^0 \in \mathbb{R}^{q \times (r-q)}$ ,  $S_{A1}^0 \in \mathbb{R}^{(m-q) \times (r-q)}$  and  $S_{A2}^0 \in \mathbb{R}^{(m-q) \times q}$  are rectangular, and the block  $L_A^0 \in \mathbb{R}^{q \times q}$  is lower triangular. These blocks are given as follows. The block  $K_A^0$  is given by

$$K_A^0 = T_A^0 - L_A^0 P \quad (34)$$

with

$$T_A^0 = E_{x_1 v_1} E_{v_1 v_1}^\dagger + N_{A1} (I - E_{v_1 v_1} E_{v_1 v_1}^\dagger) \quad (35)$$

where  $N_{A1}$  is an arbitrary matrix. The block  $L_A^0 = \begin{bmatrix} \lambda_1^0 \\ \vdots \\ \lambda_q^0 \end{bmatrix}$ , for each  $s = 1, 2, \dots, q$ , is defined by its rows

$$\lambda_s^0 = \mathbf{e}_s^T E_{x_1 v_2} E_{v_2 v_2}^\dagger G_s G_s^\dagger + f_s^T (I - G_s G_s^\dagger) \quad (36)$$

with  $f_s^T \in \mathbb{R}^{1 \times q}$  arbitrary. The blocks  $S_{A1}^0$  and  $S_{A2}^0$  are given by

$$S_{A1}^0 = S_A^0 - S_{A2}^0 P$$

and

$$S_{A2}^0 = E_{x_2 v_2} E_{v_2 v_2}^\dagger + H_{A2} (I - E_{v_2 v_2} E_{v_2 v_2}^\dagger),$$

where

$$S_A^0 = E_{x_2 v_1} E_{v_1 v_1}^\dagger + H_A (I - E_{v_1 v_1} E_{v_1 v_1}^\dagger)$$

and  $H_{A2}$  and  $H_A$  are arbitrary matrices.

The minimum error associated with the matrix  $A^0$  is given by

$$\begin{aligned} \|\mathbf{x} - A^0 \mathbf{z}^0\|_\Omega^2 &= \sum_{s=1}^{\rho} \sum_{j=s+1}^q |e_s^T E_{x_1 v_2} E_{v_2 v_2}^\dagger g_j|^2 \\ &+ \sum_{j=1}^2 \|E_{x_j x_j}^{1/2}\|_F^2 - \sum_{i=1}^2 \sum_{j=1}^2 \|E_{x_i v_i} E_{v_j v_j}^\dagger\|_F^{1/2} \end{aligned}$$

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