International Journal of Mechanical, Industrial and Aerospace Sciences
ISSN: 2517-9950
Vol:2, No:11, 2008

Combining Minimum Energy and Minimum Direct Jerk of Linear Dynamic Systems

V. Tawiwat, and P. Jumnong

***Abstract*—**Both the minimum energy consumption and smoothness, which is quantified as a function of jerk, are generally needed in many dynamic systems such as the automobile and the pick-and-place robot manipulator that handles fragile equipments. Nevertheless, many researchers come up with either solely concerning on the minimum energy consumption or minimum jerk trajectory. This research paper proposes a simple yet very interesting when combining the minimum energy and jerk of indirect jerks approaches in designing the time-dependent system yielding an alternative optimal solution. Extremal solutions for the cost functions of the minimum energy, the minimum jerk and combining them together are found using the dynamic optimization methods together with the numerical approximation. This is to allow us to simulate and compare visually and statistically the time history of state inputs employed by combining minimum energy and jerk designs. The numerical solution of minimum direct jerk and energy problem are exactly the same solution; however, the solutions from problem of minimum energy yield the similar solution especially in term of tendency.

***Keywords*—**Optimization, Dynamic, Linear Systems, Jerks.

I. Introduction

MOST of the robots and advanced mobile machines nowadays are designed so that they are either optimized on their energy consumption or on their greatest smoothness of motion, [3]. Consequently, the trajectory planning and designs of these robots are done exclusively through many approaches such as the minimum energy and minimum jerk, [4]. Nevertheless, in some applications, the robot is needed to work very smoothly in order to avoid damaging the specimen that the robot is handling while consuming least amount of energy at the same time. In other words, we may want to minimize the jerk of the movement of the robot as to give it the smoothest motion as well as optimize that robot in the energy consumption issue.

The general format of the dynamic problems is consisting of the equation of motion, the initial conditions, and the boundary conditions. The area of interest in this paper will involve the problems with two-point-boundary-value conditions. Each of the problems may contain many possible solutions depending on the objective of application. Obviously, the robot that aims to run at lowest cost of energy will be designed to have the lowest actuator inputs during the motion. This is basically the optimization problem of the dynamic systems. Research shows that many of the researchers pay a lot of their attention on the minimization of energy while many tend to seek for the smoothness of the system. According to the second law of Newton's laws, there is a relationship between acceleration and summation of all forces including the control inputs of any linear dynamic system. By taking derivative with respect to time, there is a relationship between derivative of the acceleration called Jerk and derivative of all forces including the derivative of the control inputs of the dynamic system. In this paper, the derivative of the control inputs with respect to time are called indirect jerks. This has been proof that the solutions from both yield the same answers while considering indirect jerk has some specific advantage such as the optimality conditions and the CPU runtime [6]. However, in order to compare here the direct jerk has been used instead since both have the same numerical solutions.

Therefore, this research paper aims to search for the solutions while combining the minimum energy and minimum jerk by using the optimization method so that this new alternative can be put into applications.

P. Jumnong is with Mechanical Engineering Department, Suranaree University of Technology, Nakhonratchasima 30000, Thailand (e – mail: pai_sra@yahoo.co.th, tawiwat@hotmail.com).

II. Problem Statement

Dynamic systems can be described as the first order derivative function of state as

$$\dot{x}_i = f_i(x_1,...,x_n,u_1,...u_m,t); \quad i = 1,...,n, \tag{1}$$

where $x \in R^n$, $u \in R^m$ and t are state, control input, and time respectively, [5]. The problem of interest is to find the states $x(t)$ and control inputs $u(t)$ that make our system operates according to the desired objective of minimum both energy and jerk. Note that this paper is focusing on the system with fixed end time and fixed end points. Therefore, states and control inputs that serve the necessary condition must also be able to bring the system from initial conditions $x(t_0)$ at initial time t_0 to the end point $x(t_f)$ at time t_f.

The optimization problem of minimum energy will take the form of

$$J = \int_{t_0}^{t_f} \sum_{i=1}^{m} u_i^2 dt, \tag{2}$$

where u_i is the control input, which can be force or torque applied to the system, and $i = 1,...,m$. J is the cost function of the energy consumed by the system from initial time t_0 to end time t_f.

The same kind of concept is used to the minimum jerk problem. It is well known that jerk is the change of input force with respect to time. It is, thus, the third derivative with respect to time of *x*, or first order derivative of control input *u*. Therefore,

$$Jerk = \dddot{x} \propto \dot{u} \ . \tag{3}$$

Defining

$$\dot{u} = \tilde{u} \ , \tag{4}$$

so that (1) becomes

$$\dot{x}_i = f_i(x_1,...,x_{n+m},\tilde{u}_1,...\tilde{u}_m,t); \quad i = 1,...,n+m \ . \tag{5}$$

From now on, $\tilde{u}$ is treated as a variable and as the control input of our dynamic system. Consequently, (2) can be rewritten for the objective function of the minimum indirect jerk problem as

$$J = \int_{t_0}^{t_f} \sum_{i=1}^{m} \tilde{u}_i^2 dt \ . \tag{6}$$

Similarly, (2) also can be rewritten for the objective function of the minimum direct jerk problem as

$$J = \int_{t_0}^{t_f} \sum_{i=1}^{n} \dddot{x}_i^2 dt \ . \tag{7}$$

This time, *J* is the cost function of the jerks. Once the jerk variable has been added to the dynamic system, the objective function can have both energy and jerk combined as

$$J = \int_{t_0}^{t_f} \sum_{i=n+1}^{n+m} \dddot{x}_i^2 + \sum_{i=1}^{m} \tilde{u}_i^2 dt \tag{8}$$

Which allow one to investigate the solution and compare with the problem that considers only minimum energy or minimum jerk.

III. Necessary Conditions

In this paper, we use the calculus of variations in solving for the extremal solutions of the dynamic system, [1]. Representing the control input with *u*, the principle of calculus of variations helps us solve the optimization problem by finding the time history of the control input that would minimize the cost function of the form

$$J = \phi(t, x_1,...,x_n)_{t_f} + \int_{t_0}^{t_f} L(t, x_1,...,x_n,u_1,...,u_m)dt \ , \tag{9}$$

where

$$\phi(t, x_1,...,x_n)_{t_f} \ , \tag{10}$$

is the cost based on the final time and the final states of the system, and

$$\int_{t_i}^{t_f} L(t, x_1,...,x_n,u_1,...,u_m)dt \ , \tag{11}$$

is an integral cost dependent on the time history of the state and control variables. Since the cost of the final states would be equal in all feasible time histories of the control input; therefore, the first term of (9) is omitted.

To find the extremum of the function, the dynamic equations are augmented via Lagrange Multipliers to the cost functional as follow:

$$J'(x_1,...,x_n,u_1,...,u_m) = \int_{t_i}^{t_f} L'(t, x_1,...,x_n,u_1,...,u_m)dt \ . \tag{12}$$

Where

$$L'(t, x_1,...,x_n,u_1,...,u_m) = L + \sum_{i=1}^{n} \lambda_i (f_i) \ , \tag{13}$$

and $\lambda_i(t)$ are Lagrange multipliers. Consequently, (12) becomes:

$$J'(x_1,...,x_n,u_1,...,u_m) = \int_{t_i}^{t_f} [L(t, x_1,...,x_n,u_1,...,u_m) + \sum_{i=1}^{n} \lambda_i(t)[\dot{x}_i - f_i(t, x_1,...,x_n,u_1,...,u_m)]]dt \tag{14}$$

Since the problem with fixed end time and end points are considered, initial time t_0, end time t_f, initial state $x(t_0)$, and final state $x(t_f)$ must be set prior to solving the problem. The differentiable functions are dependent on the boundary condition of $x(t_0)= x_0$, $x(t_f)= x_f$, $u(t_0)= u_0$ and $u(t_f)= u_f$ where time used falls in the interval $t_i \le t \le t_f$.

Let function $L(t, x_1,...,x_n,u_1,...,u_m,\dot{x}_1,...,\dot{x}_n)$ be represented as a functional

$$J[x_1,...,x_n,u_1,...,u_m] = \int_{t_0}^{t_f} L(t, x_1,...,x_n,u_1,...,u_m,\dot{x}_1,...,\dot{x}_n)dt \tag{15}$$

Let $x(t_0)$ be incremented by $h_{xj}(t_0)$, $u(t_0)$ be incremented by $h_{uk}(t_0)$, and still satisfy the boundary conditions, then $h_{xj}(t_0) = h_{xj}(t_f) = h_{uk}(t_0) = h_{uk}(t_f) = 0$. So, the change in functional ΔJ will be

$$\Delta J = J[x_1 + h_{x1},...,x_n + h_{xj},u_1 + h_{u1},...,u_m + h_{uk}] - J[x_1,...,x_n,u_1,...,u_m]$$

$$= \int_{t_i}^{t_f} \left[\begin{matrix} L\begin{pmatrix} t, x_1 + h_{x1},\ldots, x_n + h_{xj}, \dot{x}_1 + \dot{h}_{x1},\ldots,\dot{x}_n + \dot{h}_{xj}, \\ u_1 + h_{u1},\ldots,u_m + h_{uk}, \dot{u}_1 + \dot{h}_{u1},\ldots,\dot{u}_m + \dot{h}_{uk} \end{pmatrix} \\ - L(t, x_1,\ldots,x_n,\dot{x}_1,\ldots,\dot{x}_n,u_1,\ldots,u_m,\dot{u}_1,\ldots,\dot{u}_m) \end{matrix} \right] dt \quad (16)$$

Applying Taylor's Series to (16), disregard the higher order terms, and apply it to the problem results in

$$\begin{aligned} \delta J' = & \int_{t_i}^{t_f} \sum_{j=1}^{n} \left(\frac{\partial L'}{\partial x_j} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{x}_j}\right) h_{x_j}\, dt \\ & + \int_{t_i}^{t_f} \sum_{k=1}^{m} \left(\frac{\partial L'}{\partial u_k} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{u}_k}\right) h_{u_k}\, dt \\ & + \sum_{k=1}^{m} \left(\frac{\partial L'}{\partial \dot{u}_k} h_{u_k}|_{t_f} - \frac{\partial L'}{\partial \dot{u}_k} h_{u_k}|_{t_i}\right) \\ & + \sum_{j=1}^{m} \left(\frac{\partial L'}{\partial \dot{x}_j} h_{x_j}|_{t_f} - \frac{\partial L'}{\partial \dot{x}_j} h_{x_j}|_{t_i}\right) \end{aligned} \quad (17)$$

Since $h_{x_j}|_{t_f} = h_{x_j}|_{t_i} = 0$ and $\frac{\partial L'}{\partial \dot{u}_k} = 0$, the last two terms of (17) become zero. In order that the cost functional of jerk in (14) can be solved for minimal solution, the condition that make $\delta J' = 0$ at arbitrary variation of h_{x_j} and h_{u_k} are needed. From (17), obviously the mentioned conditions are as follow:

$$\frac{\partial L'}{\partial x_j} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{x}_j} = 0, \quad (18)$$

and

$$\frac{\partial L'}{\partial u_k} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{u}_k} = 0, \quad (19)$$

for $j = 1, \ldots, n$ and $k = 1, \ldots, m$.

Equations (17) and (18) are the necessary conditions that will lead to solve for Lagrange multipliers $\lambda_j(t)$, and control inputs $u_k(t)$. Alternatively, we can use the derived relationship below to solve for the unknowns necessary conditions:
For

$$\dot{x}_i = f_i(x_1,\ldots,x_n,u_1,\ldots u_m,t), \quad i = 1,\ldots,n \quad (20)$$

Necessary conditions are (20) and

$$\dot{\lambda}_j = -\frac{\partial L}{\partial x_j} - \sum_{i=1}^{n} \lambda_i \frac{\partial f_i}{\partial x_j}, \quad j = 1,\ldots,n, \quad (21)$$

$$\frac{\partial L}{\partial u_k} + \sum_{i=1}^{n} \lambda_i \frac{\partial f_i}{\partial u_k} = 0, \quad k = 1,\ldots,m. \quad (22)$$

As of above the necessary conditions are in the form of differential and algebraic equations which are known as two-point boundary valued problem, [2].

IV. Example Problems

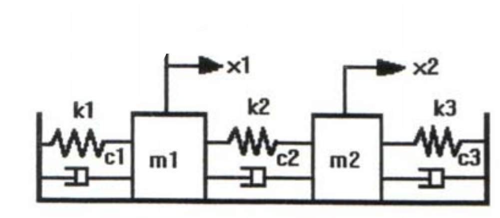

Fig. 1 Two degree of-freedom of spring mass and damper system

The procedure outlined in this paper for dynamic optimization is illustrated with the following example of a two degree-of-freedom spring-mass-damper system sketched in equation as

$$A\dot{x} = Bu \quad (23)$$

The matrices A and B for this system is as follows:

$$A = \begin{bmatrix} -M^{-1}C & -M^{-1}K \\ I_2 & 0 \end{bmatrix} \quad (24)$$

$$B = \begin{bmatrix} \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (25)$$

where the matrices M , C and K are:

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, C = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \quad (26)$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \quad (27)$$

The equation (23) can also be rewritten in the second order differential equation according to the second law of Newton. The parameters used in the model in MKS units are: $m_1 = m_2 = 1.0$, $c_1 = c_3 = 1.0$, $c_2 = 2.0$, $k_1 = k_2 = k_3 = 3.0$. The boundary conditions are $x(t_0) = (0\ 0\ 0\ 0)^T$ and $x(t_f) = (1\ 1\ 0\ 0)^T$, where

$t_0 = 0$ and $t_f = 1.0$. In addition, the constraint of the control input $-4 \le u_i \le 4,\ i = 1,2.$

A. Minimum Energy Problem

The cost function of minimum direct jerk is defined as

$$J = \int_0^1 u_1^2 + u_2^2 \, dt\,. \tag{28}$$

In order for the cost function in (28) to be minimized, the Calculus of Variations as stated in previous section has been used.

B. Minimum Direct Jerk Problem

The cost function of minimum indirect jerk is also defined as

$$J = \int_0^1 \dddot{x}_1^2 + \dddot{x}_2^2 \, dt\,. \tag{29}$$

Similarly for (29) to be minimized, the Calculus of Variations must be applied here.

C. Minimum Energy and Minimum Jerk Problem

The cost function of minimum indirect jerk is also defined as

$$J = \int_0^1 \dddot{x}_1^2 + \dddot{x}_2^2 + \tilde{u}_1^2 + \tilde{u}_2^2 \, dt\,. \tag{30}$$

This problem is also in the form that needs the Calculus of Variations.

D. Numerical Results

The minimum jerk problem has the exact same format as the minimum energy problem in (2). However, since the time derivative of control inputs are considered, the (23) must be rewritten as to include the consideration of jerk into the system:

$$\dddot{x}_1 + 3\ddot{x}_1 - 2\ddot{x}_2 + 6\dot{x}_1 - 3\dot{x}_2 = \frac{du_1}{dt} = \tilde{u}_1$$

$$\dddot{x}_2 - 2\ddot{x}_1 + 3\ddot{x}_2 - 3\dot{x}_1 + 6\dot{x}_2 = \frac{du_2}{dt} = \tilde{u}_2.$$

These new equations are suitable to add jerks in to the objective function as shown in (30). Therefore, the extra boundary conditions can be applied at both ends that are $u(t_0) = (0\,0)^T$ and $u(t_f) = (0\,0)^T$. These conditions can be applied in the numerical scheme through the original dynamic equations as follow:

$$\ddot{x}_1 + 3\dot{x}_1 - 2\dot{x}_2 + 6x_1 - 3x_2 = u_1$$

$$\ddot{x}_2 - 2\dot{x}_1 + 3\dot{x}_2 - 3x_1 + 6x_2 = u_2.$$

By using software developed by Tawiwat Veeraklaew, [6], these problems can be solved to obtain the optimal solutions. The idea behind this software is to transform the necessary conditions of the dynamic optimization to static optimization. Then one kind of the well known methods called nonlinear programming or linear programming has been used to solve for all parameters that are parameterized through collocation technique. The comparison for each variable such as state and control variables of the dynamic systems in this example are shown in figure below as Fig. 2 to Fig. 9.

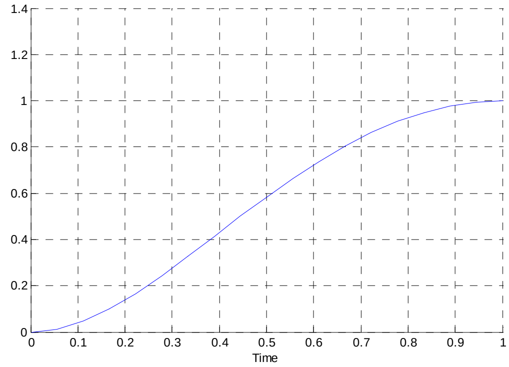

Fig. 2 Solutions of $x_1(t)$ from minimum direct jerk and combining minimum jerk and energy

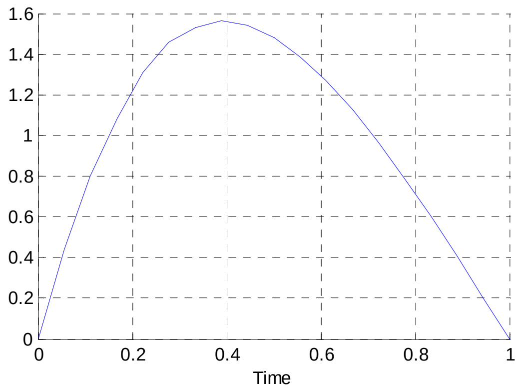

Fig. 3 Solutions of $\dot{x}_1(t)$ from minimum direct jerk and combining minimum jerk and energy

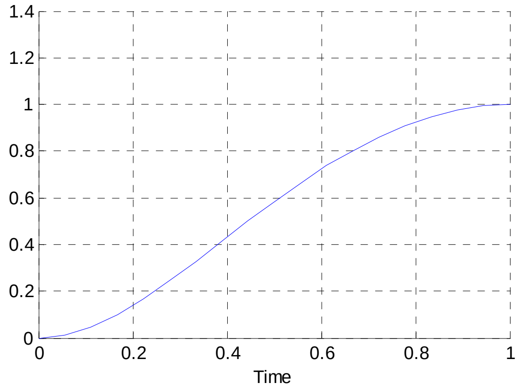

Fig. 4 Solutions of $x_2(t)$ from minimum direct jerk and combining minimum jerk and energy

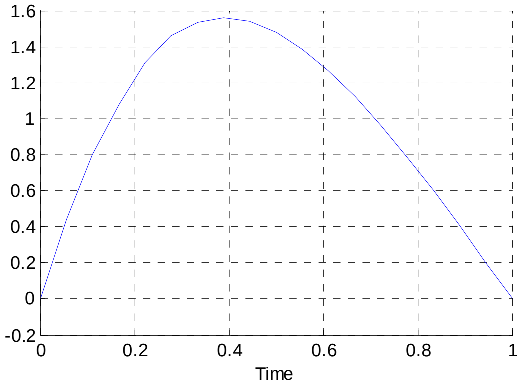

Fig. 5 Solutions of $\dot{x}_2(t)$ from minimum direct jerk and combining minimum jerk and energy

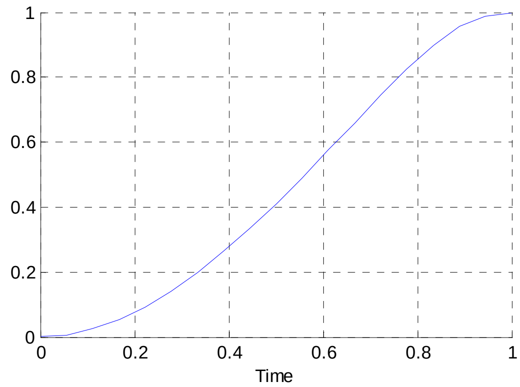

Fig. 6 Solutions of $x_1(t)$ from minimum energy

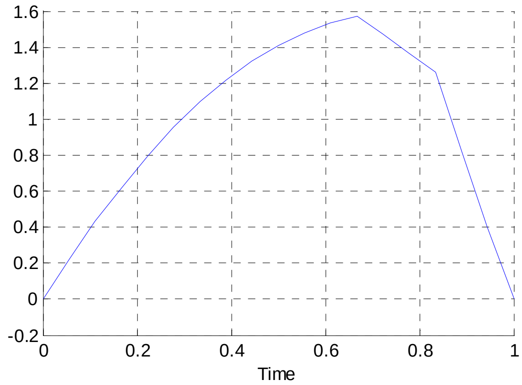

Fig. 7 Solutions of $\dot{x}_1(t)$ from minimum energy

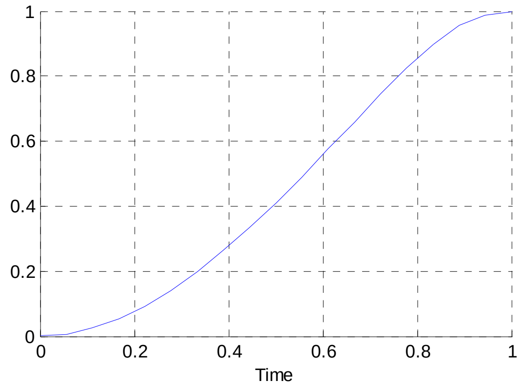

Fig. 8 Solutions of $x_2(t)$ from minimum energy

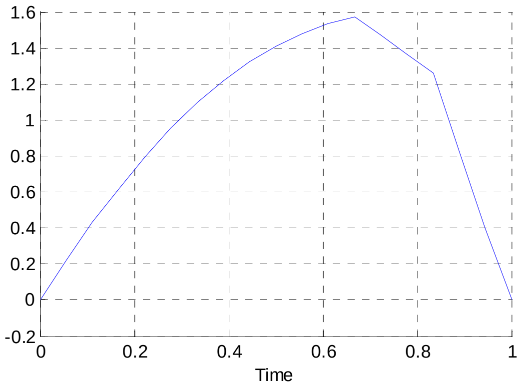

Fig. 9 Solutions of $\dot{x}_2(t)$ from minimum energy

From the solutions above, $x_1(t)$, $\dot{x}_1(t)$, $x_2(t)$ and $\dot{x}_2(t)$ from both minimum direct jerk and combining minimum jerk and energy have exactly the same solutions which can be seen obviously as shown in Fig. 2 to Fig. 5. The solution from minimum energy are different comparing to both problem in Fig. 2 to Fig.5; however, the tendency of the solutions of $x_1(t)$, $\dot{x}_1(t)$, $x_2(t)$ and $\dot{x}_2(t)$ are quite similar as shown in Fig. 6 to Fig. 9.

In conclusion, the numerical solution of minimum direct jerk and energy problem are exactly the same solution; however, the solutions from problem of minimum energy yield the similar solution especially in term of tendency. This can be concluded that the specify cost functions used in this paper of minimum direct jerk and combining direct jerk and energy do not have any effect to the solution of minimum direct jerk at all. According to the cost function of minimum direct jerk and energy, the factors of both terms called jerk and energy are equal to one. This might be the reason; therefore, these factors can be varied and adjusted in the future work.

REFERENCES

[1] S. K. Agrawal and B.C. Fabien, *Optimization of Dynamic Systems*. Boston: Kluwer Academic Publishers, 1999.
[2] HG. Bock, "Numerical Solution of Nonlinear Multipoint Boundary Value Problems with Application to Optimal Control," *ZAMM*, pp. 58, 1978.
[3] JJ. Craig, *Introduction to Robotic: Mechanics and Control*. Addision-Wesley Publishing Company, 1986.
[4] WS. Mark, *Robot Dynamics and Control*. University of Illinois at Urbana-Champaign, 1989.
[5] TR. Kane and DA. Levinson, *Dynamics: Theory and Applications*. McGraw-Hill Inc, 1985.
[6] T. Veeraklaew, Extensions of Optimization Theory and New Computational Approaches for Higher-order Dynamic systems [Dissertation]. The University of Delaware, 2000.
[7] T. Veeraklaew, N. Phatthana-im and S. Heama, *Comparison between Minimum Direct and Indirect Jerks of Linear Dynamic Systems*. Proceeding of world academy of science and Technology, Vol. 27, Feb 2008.

Lt. Col. Tawiwat V., Assistant Professor, was obtained his Ph.D. (Mechanical Engineering) from the University of Delaware, USA in 1999. Now, he is working as Assistant Professor at Mechanical Engineering Department Chulachomklao Royal Military Academy (CRMA) Nakhon-nayok 26001 Thailand. He has done lots of research in the area of dynamic optimization.