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Characterizing the Geometry of Envy Human Behaviour Using Game Theory Model with Two Types of Homogeneous Players

A. S. Mousa, R. I. Rajab, A. A. Pinto

Abstract—An envy behavioral game theoretical model with two types of homogeneous players is considered in this paper. The strategy space of each type of players is a discrete set with only two alternatives. The preferences of each type of players is given by a discrete utility function. All envy strategies that form Nash equilibria and the corresponding envy Nash domains for each type of players have been characterized. We use geometry to construct two dimensional envy tilings where the horizontal axis reflects the preference for players of type one, while the vertical axis reflects the preference for the players of type two. The influence of the envy behavior parameters on the Cartesian position of the equilibria has been studied, and in each envy tiling we determine the envy Nash equilibria. We observe that there are 1024 combinatorial classes of envy tilings generated from envy chromosomes: 256 of them are being structurally stable while 768 are with bifurcation. Finally, some conditions for the disparate envy Nash equilibria are stated.

Keywords—Game theory, Nash Equilibrium, envy Nash Equilibrium, geometric tilings, bifurcation thresholds.

I. Introduction

MODELING the behavior of players using game theory has been studied intensively by economists and scientists. Ajzen [1] constructed the main goal in Planned Behavior or Reasoned Action theories to understand and predict the way players turn intentions into behaviors. In 2010, Brida et al. [2] studied the characteristics of individuals that might affect their decisions in a game theory model. One year later, Almeida et al. [3] developed a game theoretical model for reasoned action based on the works of Cownley and Wooders [4] where different types of players were included. In [5] Mousa et al. presented a dichotomous decision model, where players choose between two alternative decisions and can influence the decisions of the others. Soeiro et al. [6] presented a game theoretical model to study the effects of societies behaviour on the market shares and characterized all possible strategies that form Nash equilibria. In [7], Mousa and Pinto show that the pure Nash equilibria can be either cohesive (all players with the same preferences make the same decision) or can be disparate (there are players with the same preferences who make an opposite decisions). For further readings in this context, we refer the reader to [8] and [9].

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In this paper, we will study the influence of the envy behavior for players over the utility function of other type by extending the pure Nash equilibria studied in the game decision model [5]. We will characterize all the pure envy strategies that form Nash equilibria and determine the corresponding envy Nash domains. *Pure envy strategies* means the cohesive or the disparate envy strategies. The disparate pure envy Nash equilibria can explain the conflict decisions that divide a community. For a given level of an envy behaviour, we construct the corresponding geometric tiling in the cartesian xy- plane, where the horizontal axis represents the relative preference of players with type t_1, and the vertical axis represents the relative preference of players with type t_2. Noting that the envy Nash domains form the decision tiles, we show that there are 1024 combinatorial classes of envy decision tilings, generated from the horizontal envy chromosomes for players of type t_1 and vertical envy chromosomes for players of type t_2, which demonstrates the high complexity of human envy behaviour. Furthermore, we found 256 combinatorial classes of tilings that are being structurally stable while 768 combinatorial classes have either single or double or degenerate bifurcations. We will show that the tilings give a full geometrical characterization of the envy Nash equilibria.

Possible extension to the decision game model would be to include some kind of stochastic pattern with diffusion and solve an optimization problem in a continuous time framework (see [10], [11]).

This paper is organized as follows. In Section II we review the decision model presented in [5]. In section III we study the influence of the envy behavior for both types of players over the utility functions of each other and characterize the cohesive envy Nash equilibria. In Section IV we study the geometric classes of pure envy Nash equilibria domains. In Section V we study the disparate envy Nash equilibria. We conclude in Section VI.

II. Review of the Decision Model

In this section we review the decision model formulated in [5]. Let $\mathbf{T} = \{t_1, t_2\}$ be set with two types of players, $I_1 = \{1, \ldots, n_1\}$ be the set of all players with type t_1, $I_2 = \{1, \ldots, n_2\}$ be the set of all players with type t_2, and $I = I_1 \sqcup I_2$ be the disjoint union. Each player $i \in \mathbf{I}$ is assumed to make one decision $d \in \mathbf{D} = \{Y, N\}$.

Let $\mathcal{L}$ be the *preference decision matrix* whose coordinates

ω_p^d indicate how much player with type t_p likes or dislikes to make decision d

$$\mathcal{L} = \begin{pmatrix} \omega_1^Y & \omega_1^N \\ \omega_2^Y & \omega_2^N \end{pmatrix} .$$

The preference decision matrix indicates, for each type of players, the decision the players prefer, i.e. the players taste type.

Let $\mathcal{N}_d$ be the *preference neighbors matrix* whose coordinates α_{pq}^d indicate how much player with type t_p likes or dislikes that player with type t_q makes decision d

$$\mathcal{N}_d = \begin{pmatrix} \alpha_{11}^d & \alpha_{12}^d \\ \alpha_{21}^d & \alpha_{22}^d \end{pmatrix} .$$

The preference neighbors matrix indicates for each type of players whom they prefer or not to be with in each decision, i.e. the players crowding type.

We describe the (pure) decision of the players by a *(pure) strategy map* $S : \mathbf{I} \to \mathbf{D}$ that associates to each player $i \in \mathbf{I}$ its decision $S(i) \in \mathbf{D}$. Let $\mathbf{S}$ be the space of all strategies S. Given a strategy S, let $\mathcal{O}_S$ be the *strategic decision matrix* whose coordinates $l_p^d = l_p^d(S)$ indicate the number of players with type t_p who make decision d, so $\mathcal{O}_S$ is defined by

$$\mathcal{O}_S = \begin{pmatrix} l_1^Y & l_1^N \\ l_2^Y & l_2^N \end{pmatrix} .$$

We denote by $(l_1, l_2) = (l_1^Y(S), l_2^Y(S))$ the *strategic decision vector* associated with strategy S, where l_1 (resp. $n_1 - l_1$) is the number of players with type t_1 who make the decision Y (resp. N) and l_2 (resp. $n_2 - l_2$) is the number of players with type t_2 who make the decision Y (resp. N). The set $\mathbf{O}$ of all possible *strategic decision vectors* is given by

$$\mathbf{O} = \{0, \ldots, n_1\} \times \{0, \ldots, n_2\} .$$

Let $U_1 : \mathbf{D} \times \mathbf{O} \to \mathbb{R}$ be the *utility function* of player with type t_1 who makes decision Y (resp. N) defined by

$$\begin{aligned} U_1(Y; l_1, l_2) &= \omega_1^Y + \alpha_{11}^Y (l_1 - 1) + \alpha_{12}^Y l_2 \\ U_1(N; l_1, l_2) &= \omega_1^N + \alpha_{11}^N (n_1 - l_1 - 1) + \alpha_{12}^N (n_2 - l_2) . \end{aligned} \tag{1}$$

Let $U_2 : \mathbf{D} \times \mathbf{O} \to \mathbb{R}$ be the *utility function* of player with type t_2 who makes decision Y (resp. N) defined by

$$\begin{aligned} U_2(Y; l_1, l_2) &= \omega_2^Y + \alpha_{22}^Y (l_2 - 1) + \alpha_{21}^Y l_1 \\ U_2(N; l_1, l_2) &= \omega_2^N + \alpha_{22}^N (n_2 - l_2 - 1) + \alpha_{21}^N (n_1 - l_1) . \end{aligned} \tag{2}$$

Given a strategy $S \in \mathbf{S}$, the *utility* $U_i(S)$ of player i with type $t_{p(i)}$ is given by $U_{p(i)}(S(i); l_1^Y(S), l_2^Y(S))$.

Definition 1: Let $x = \omega_1^Y - \omega_1^N$ be the *horizontal relative decision preference* of the players with type t_1 and $y = \omega_2^Y - \omega_2^N$ be the *vertical relative decision preference* of the players with type t_2.

If $x > 0$, then players with type t_1 prefer to decide Y without taking into account the influence of the others. If $x = 0$, then players with type t_1 are indifferent to decide Y or N without taking into account the influence of the others. If $x < 0$, then players with type t_1 prefer to decide N without taking into account the influence of the others.

A strategy $S^* : \mathbf{I} \to \mathbf{D}$ is a *(pure) Nash Equilibrium* if

$$U_i(S^*) \geq U_i(S)$$

for every player $i \in \mathbf{I}$ and for every strategy $S \in \mathbf{S}$. The *Nash domain* $\mathbf{N}(S)$ of a strategy $S \in \mathbf{S}$ is the set of all pairs (x, y) for which S is a Nash Equilibrium.

Definition 2: A *cohesive strategy* is a strategy in which all players with the same type prefer to make the same decision. A *disparate strategy* is a pure strategy that is not cohesive.

Now we construct the Nash domains $\mathbf{N}(S)$ for the cohesive strategies. We observe that there are four cohesive strategies: (Y, Y) *strategy*: all players make the decision Y; (Y, N) *strategy*: all players with type t_1 make the decision Y and all players with type t_2 make the decision N; (N, Y) *strategy*: all players with type t_1 make the decision N and all players with type t_2 make the decision Y; (N, N) *strategy*: all players make the decision N.

The *Nash domain* $\mathbf{N}(Y, Y)$ is the right-upper quadrant given by

$$\mathbf{N}(Y, Y) = \{(x, y) : x \geq H(Y, Y) \quad \text{and} \quad y \geq V(Y, Y)\} , \tag{3}$$

where the *horizontal* $H(Y, Y)$ and *vertical* $V(Y, Y)$ *strategic thresholds* of the (Y, Y) strategy are

$$\begin{aligned} H(Y, Y) &= -\alpha_{11}^Y (n_1 - 1) - \alpha_{12}^Y n_2 \quad \text{and} \\ V(Y, Y) &= -\alpha_{22}^Y (n_2 - 1) - \alpha_{21}^Y n_1 . \end{aligned} \tag{4}$$

Hence, the cohesive strategy (Y, Y) is a *Nash Equilibrium* if and only if $(x, y) \in \mathbf{N}(Y, Y)$.

The *Nash domain* $\mathbf{N}(Y, N)$ is the right-lower quadrant

$$\mathbf{N}(Y, N) = \{(x, y) : x \geq H(Y, N) \quad \text{and} \quad y \leq V(Y, N)\} , \tag{5}$$

where the *horizontal* $H(Y, N)$ and *vertical* $V(Y, N)$ *strategic thresholds* of the (Y, N) strategy are

$$\begin{aligned} H(Y, N) &= -\alpha_{11}^Y (n_1 - 1) + \alpha_{12}^N n_2 \quad \text{and} \\ V(Y, N) &= \alpha_{22}^N (n_2 - 1) - \alpha_{21}^Y n_1 . \end{aligned} \tag{6}$$

Hence, the cohesive strategy (Y, N) is a *Nash Equilibrium* if and only if $(x, y) \in \mathbf{N}(Y, N)$.

The *Nash domain* $\mathbf{N}(N, Y)$ is the left-upper quadrant

$$\mathbf{N}(N, Y) = \{(x, y) : x \leq H(N, Y) \quad \text{and} \quad y \geq V(N, Y)\} , \tag{7}$$

where the *horizontal* $H(N, Y)$ and *vertical* $V(N, Y)$ *strategic thresholds* of the (N, Y) strategy are

$$\begin{aligned} H(N, Y) &= \alpha_{11}^N (n_1 - 1) - \alpha_{12}^Y n_2 \quad \text{and} \\ V(N, Y) &= -\alpha_{22}^Y (n_2 - 1) + \alpha_{21}^N n_1 . \end{aligned} \tag{8}$$

Hence, the cohesive strategy (N, Y) is a *Nash Equilibrium* if and only if $(x, y) \in \mathbf{N}(N, Y)$.

The *Nash domain* $\mathbf{N}(N, N)$ is the left-lower quadrant

$$\mathbf{N}(N, N) = \{(x, y) : x \leq H(N, N) \quad \text{and} \quad y \leq V(N, N)\} , \tag{9}$$

where the *horizontal* $H(N, N)$ and *vertical* $V(N, N)$ strategic thresholds of the (N, N) strategy are

$$\begin{aligned} H(N, N) &= \alpha_{11}^N (n_1 - 1) + \alpha_{12}^N n_2 \quad \text{and} \\ V(N, N) &= \alpha_{22}^N (n_2 - 1) + \alpha_{21}^N n_1 . \end{aligned} \tag{10}$$

Hence, the cohesive strategy (N, N) is a *Nash Equilibrium* if and only if $(x, y) \in \mathbf{N}(N, N)$.

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III. Envy Nash Equlibria

In this section we will model the influence of envy behavior created by both types of players over the utility function of each other and study how this influence changes the Cartesian position of the Nash equilibria studied in [5].

Let $\beta_i > 0$, $i = 1, 2$ be the envy parameter associated with players of type t_i. We remark that β_1 (resp. β_2) measures the influence of the envy behavior created by players with type t_1 (resp. t_2) over the utility function of players with type t_2 (resp. t_1). Furthermore, we assume that β_1 and β_2 do not depend on the decision d which has been made by players with type t_1 and t_2, respectively. However, a general framework includes such dependence could be studied in a different paper.

Let $U_1^e : \mathbf{D} \times \mathbf{O} \times \mathbb{R}^+ \to \mathbb{R}$ be the *utility function* of an envy player with type t_1 who makes decision Y (resp. N) given by

$$\begin{aligned} U_1(Y; l_1, l_2, \beta_1) &= U_1(Y; l_1, l_2) - \beta_1 U_2(Y; l_1, l_2) \qquad (11) \\ U_1(N; l_1, l_2, \beta_1) &= U_1(N; l_1, l_2) - \beta_1 U_2(N; l_1, l_2) \end{aligned}$$

and $U_2^e : \mathbf{D} \times \mathbf{O} \times \mathbb{R}^+ \to \mathbb{R}$ be the *utility function* of an envy player with type t_2 who makes decision Y (resp. N) given by

$$\begin{aligned} U_2(Y; l_1, l_2, \beta_2) &= U_2(Y; l_1, l_2) - \beta_2 U_1(Y; l_1, l_2) \qquad (12) \\ U_2(N; l_1, l_2, \beta_2) &= U_2(N; l_1, l_2) - \beta_2 U_1(N; l_1, l_2) \ , \end{aligned}$$

where the utility functions $U_i(d; l_1, l_2)$, $i = 1, 2$ and $d \in \mathbf{D}$ are as given in (1) and (1). We remark that if $\beta_1 = \beta_2 = 0$, then the envy model coincides with the decision model presented in [5].

A. Geometry of Pure Envy Nash Equilibria

In this section we will show that the horizontal and vertical relative decision preferences, the preference neighbors coordinates and together with the total number of players of each type encode all the relevant information for characterizing the cohesive envy Nash equilibria.

A strategy $S_e^* : \mathbf{I} \to \mathbf{D}$ is a *(pure) envy Nash Equilibrium* if

$$U_i(S_e^*) \geq U_i(S)$$

for every player $i \in \mathbf{I}$ and for every strategy $S \in \mathbf{S}$. The *envy Nash domain* $\mathbf{N}^e(S)$ of a strategy $S \in \mathbf{S}$ is the set of all pairs (x, y) for which S is an envy Nash Equilibrium.

Given a strategy $S_\mathbf{e} \in S$, the utility functions $U_i(S_\mathbf{e})$ of player i with type $t_{p(i)}$ is given by

$$U_{p(i)}(S_\mathbf{e}(i); l_1^Y(S_\mathbf{e}), l_2^Y(S_\mathbf{e})) \ .$$

Definition 3: An *envy cohesive strategy* is a strategy in which all players with the same type prefer to make the same decision. An *envy disparate strategy* is a pure envy strategy that is not envy cohesive strategy.

Similarly, we observe that there are four distinct envy cohesive strategies. We will now construct the envy Nash domains $\mathbf{N}^e(S_e)$ for each pure envy strategy $S_e \in \mathbf{S}$. The four envy Nash domains are $\mathbf{N}^e(Y, Y)$, $\mathbf{N}^e(Y, N)$, $\mathbf{N}^e(N, Y)$ and $\mathbf{N}^e(N, N)$.

Theorem 1: Assume that $\beta_1 \beta_2 \neq 1$.

(i) The envy cohesive strategy $S_e = (Y, Y)$ is an *envy Nash Equilibrium* if and only if $(x, y) \in \mathbf{N}^e(Y, Y)$, where the *envy Nash domain* $\mathbf{N}^e(Y, Y)$ is the right-upper quadrant given by

$$\mathbf{N}^e(Y, Y) = \{(x, y) \in \mathbb{R}^2 : x \geq H^e(Y, Y), y \geq V^e(Y, Y)\} \qquad (13)$$

and the *horizontal envy* $H^e(Y, Y)$ and *vertical envy* $V^e(Y, Y)$ *strategic thresholds* of the (Y, Y) pure envy strategy are given by

$$\begin{aligned} H^e(Y, Y) &= H(Y, Y) \\ &\quad + \beta_1 \left(\frac{\alpha_{22}^N - \alpha_{21}^N + \beta_2(\alpha_{11}^N - \alpha_{12}^N)}{1 - \beta_1 \beta_2} \right) \qquad (14) \\ V^e(Y, Y) &= V(Y, Y) \\ &\quad + \beta_2 \left(\frac{\beta_1(\alpha_{22}^N - \alpha_{21}^N) + \alpha_{11}^N - \alpha_{12}^N}{1 - \beta_1 \beta_2} \right), \end{aligned}$$

where the *horizontal* $H(Y, Y)$ and *vertical* $V(Y, Y)$ *strategic thresholds* are as given in (4).

(ii) The envy cohesive strategy $S_e = (Y, N)$ is an *envy Nash Equilibrium* if and only if $(x, y) \in \mathbf{N}^e(Y, N)$, where the *envy Nash domain* $\mathbf{N}^e(Y, N)$ is the right-lower quadrant

$$\mathbf{N}^e(Y, N) = \{(x, y) \in \mathbb{R}^2 : x \geq H^e(Y, N), y \leq V^e(Y, N)\} \qquad (15)$$

and the *horizontal envy* $H^e(Y, N)$ and *vertical envy* $V^e(Y, N)$ *strategic thresholds* of the (Y, N) pure envy strategy are given by

$$\begin{aligned} H^e(Y, N) &= H(Y, N) \\ &\quad + \beta_1 \left(\frac{\beta_2(\alpha_{11}^N + \alpha_{12}^Y) - (\alpha_{22}^Y + \alpha_{21}^N)}{1 - \beta_1 \beta_2} \right) \qquad (16) \\ V^e(Y, N) &= V(Y, N) \\ &\quad + \beta_2 \left(\frac{(\alpha_{11}^N + \alpha_{12}^Y) - \beta_1(\alpha_{22}^Y + \alpha_{21}^N)}{1 - \beta_1 \beta_2} \right), \end{aligned}$$

where the *horizontal* $H(Y, N)$ and *vertical* $V(Y, N)$ *strategic thresholds* are as given in (6).

(iii) The envy cohesive strategy $S_e = (N, Y)$ is an *envy Nash Equilibrium* if and only if $(x, y) \in \mathbf{N}^e(N, Y)$, where the *envy Nash domain* $\mathbf{N}^e(N, Y)$ is the left-upper quadrant

$$\mathbf{N}^e(N, Y) = \{(x, y) \in \mathbb{R}^2 : x \leq H^e(N, Y), y \geq V^e(N, Y)\} \qquad (17)$$

and the *horizontal envy* $H^e(N, Y)$ and *vertical envy* $V^e(N, Y)$ *strategic thresholds* of the (N, Y) pure envy strategy are given by

$$\begin{aligned} H^e(N, Y) &= H(N, Y) \\ &\quad + \beta_1 \left(\frac{\alpha_{22}^N + \alpha_{21}^Y - \beta_2(\alpha_{11}^Y + \alpha_{12}^N)}{1 - \beta_1 \beta_2} \right) \qquad (18) \\ V^e(N, Y) &= V(N, Y) \\ &\quad + \beta_2 \left(\frac{\beta_1(\alpha_{22}^N + \alpha_{21}^Y) - (\alpha_{11}^Y + \alpha_{12}^N)}{1 - \beta_1 \beta_2} \right), \end{aligned}$$

where the *horizontal* $H(N, Y)$ and *vertical* $V(N, Y)$ *strategic thresholds* are as given in (8).

(iv) The envy cohesive strategy $S_e = (N, N)$ is an *envy Nash Equilibrium* if and only if $(x, y) \in \mathbf{N}^e(N, N)$, where the *envy Nash domain* $\mathbf{N}^e(N, N)$ is the left-lower quadrant

$$\mathbf{N}^e(N,N) = \{(x,y) \in \mathbb{R}^2 : x \le H^e(N,N), y \le V^e(N,N)\} \tag{19}$$

and the *horizontal envy* $H^e(N, N)$ and *vertical envy* $V^e(N, N)$ *strategic thresholds* of the (N, N) pure envy strategy are given by

$$\begin{aligned} H^e(N,N) &= H(N,N) \\ &+ \beta_1 \left(\frac{\alpha^Y_{21} - \alpha^Y_{22} + \beta_2(\alpha^Y_{12} - \alpha^Y_{11})}{1 - \beta_1\beta_2} \right) \end{aligned} \tag{20}$$

$$\begin{aligned} V^e(N,N) &= V(N,N) \\ &+ \beta_2 \left(\frac{\beta_1(\alpha^Y_{21} - \alpha^Y_{22}) + \alpha^Y_{12} - \alpha^Y_{11}}{1 - \beta_1\beta_2} \right), \end{aligned}$$

where the *horizontal* $H(N, N)$ and *vertical* $V(N, N)$ *strategic thresholds* are as given in (10).

Proof: We will prove case (i) and the proof of the other cases follow similarly. The coherent envy strategy $S_e = (Y, Y)$ is Nash Equilibrium if and only if the following inequalities hold

$$\begin{aligned} U_1(Y^e; n_1, n_2, \beta_1) &\ge U_1(N^e; n_1 - 1, n_2, \beta_1) \\ U_2(Y^e; n_1, n_2, \beta_2) &\ge U_2(N^e; n_1, n_2 - 1, \beta_2) \ . \end{aligned} \tag{21}$$

Substituting the envy utility functions from (11) and (12) in (21) and rearrange the terms we obtain

$$\begin{aligned} \omega^Y_1 - \omega^N_1 &\ge -\alpha^Y_{11}(n_1 - 1) - \alpha^Y_{12} n_2 \\ &+\beta_1 \left(\frac{\alpha^N_{22} - \alpha^N_{21} + \beta_2(\alpha^N_{11} - \alpha^N_{12})}{1 - \beta_1\beta_2} \right) \\ \omega^Y_2 - \omega^N_2 &\ge -\alpha^Y_{22}(n_2 - 1) - \alpha^Y_{21} n_1 \\ &+\beta_2 \left(\frac{\beta_1(\alpha^N_{22} - \alpha^N_{21}) + \alpha^N_{11} - \alpha^N_{12}}{1 - \beta_1\beta_2} \right) . \end{aligned}$$

This, respectively, simplifies to

$$x \ge H^e(Y,Y) \quad \text{and} \quad y \ge V^e(Y,Y) \ .$$

Hence, the envy cohesive strategy $S_e = (Y, Y)$ is an *envy Nash Equilibrium* if and only if $(x, y) \in \mathbf{N}^e(Y, Y)$. ■

As a result of Theorem 1, we conclude the following: the envy cohesive strategy $S_e = (Y, N)$ is an *envy Nash Equilibrium* if and only if $(x, y) \in \mathbf{N}^e(Y, N)$, the envy cohesive strategy $S_e = (N, Y)$ is an *envy Nash Equilibrium* if and only if $(x, y) \in \mathbf{N}^e(N, Y)$, and the envy cohesive strategy $S_e = (N, N)$ is an *envy Nash Equilibrium* if and only if $(x, y) \in \mathbf{N}^e(N, N)$.

Now we will study the influence of the envy parameters created by both types of players on the location of Nash equilibria. More precisely, when a certain Nash Equilibrium strategy can be envy Nash Equilibrium by comparing the Nash domains $\mathbf{N}(S)$ with the envy Nash domains $\mathbf{N}^e(S)$ for a given strategy $S \in \mathbf{S}$.

Lemma 1: Given a strategy $S \in \mathbf{S}$. If $S = (Y, Y)$ is a Nash Equilibrium, then it is an envy Nash Equilibrium if and only if

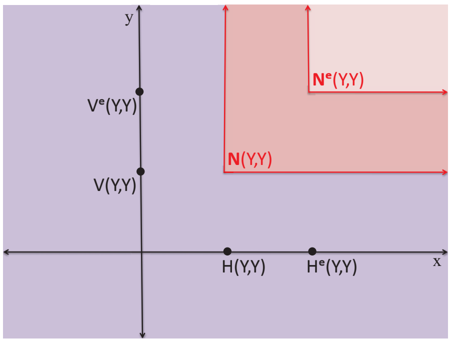
$$\begin{aligned} \beta_1(\alpha^N_{22} - \alpha^N_{21}) &< \alpha^N_{12} - \alpha^N_{11} \quad \text{and} \\ \beta_2(\alpha^N_{11} - \alpha^N_{12}) &< \alpha^N_{21} - \alpha^N_{22} \ . \end{aligned}$$

Proof: The proof follows from the definitions of the envy Nash domain $\mathbf{N}^e(Y, Y)$ given in (13) and the Nash domain $\mathbf{N}(Y, Y)$ given in (3) when $\mathbf{N}(Y, Y) \subset \mathbf{N}^e(Y, Y)$. ■

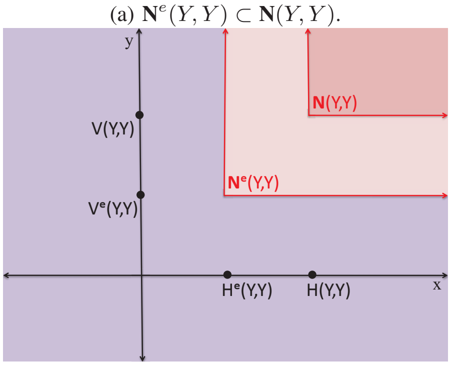
Hence, if players with type t_1 (resp. t_2) like more being with players with type t_2 (resp. t_1) than being together making decision N (means $\alpha^N_{11} < \alpha^N_{12}$ and $\alpha^N_{22} < \alpha^N_{21}$), then $\mathbf{N}(Y, Y) \subset \mathbf{N}^e(Y, Y)$ holds (see Fig. 1b) and the following inequalities hold

$$\beta_1 > 0 > \frac{\alpha^N_{12} - \alpha^N_{11}}{\alpha^N_{22} - \alpha^N_{21}} \quad \text{and} \quad \beta_2 > 0 > \frac{\alpha^N_{22} - \alpha^N_{21}}{\alpha^N_{12} - \alpha^N_{11}} \ . \tag{22}$$

On the other hand, if players with type t_1 (resp. t_2) like more being together than being with players with type t_2 (resp. t_1) making decision N (means $\alpha^N_{11} > \alpha^N_{12}$ and $\alpha^N_{22} > \alpha^N_{21}$), then $\mathbf{N}^e(Y, Y) \subset \mathbf{N}(Y, Y)$ (see Fig. 1a).



(a) $\mathbf{N}^e(Y, Y) \subset \mathbf{N}(Y, Y)$.



(b) $\mathbf{N}(Y, Y) \subset \mathbf{N}^e(Y, Y)$.

Fig. 1 The geometry of envy pure Nash domain $\mathbf{N}^e(Y, Y)$

We remark that Lemma 1 provides some properties for the Nash domains $\mathbf{N}^e(Y, Y)$ and $\mathbf{N}(Y, Y)$:

(i) $\mathbf{N}^e(Y, Y) = \mathbf{N}(Y, Y)$ if and only if $\alpha^N_{11} = \alpha^N_{12}$ and $\alpha^N_{22} = \alpha^N_{21}$, which means the equilibria coincide.

(ii) $\mathbf{N}^e(Y, Y) \subset \mathbf{N}(Y, Y)$ if and only if

$$\begin{aligned} \beta_1(\alpha^N_{22} - \alpha^N_{21}) &> \alpha^N_{12} - \alpha^N_{11} \quad \text{and} \\ \beta_2(\alpha^N_{11} - \alpha^N_{12}) &> \alpha^N_{21} - \alpha^N_{22} \ . \end{aligned}$$

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(iii) the Nash domains $\mathbf{N}^e(Y,Y)$ and $\mathbf{N}(Y,Y)$ overlaps in the otherwise cases.

Lemma 2: Given a strategy $S \in \mathbf{S}$. If $S = (Y,N)$ is a Nash Equilibrium, then it is an envy Nash Equilibrium if and only if

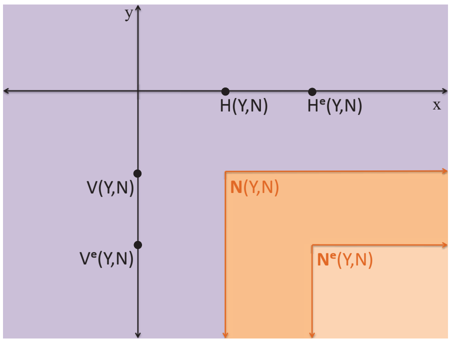
$$\begin{aligned} \beta_1(\alpha_{22}^Y + \alpha_{21}^N) &< \alpha_{11}^N + \alpha_{12}^Y \quad \text{and} \\ \beta_2(\alpha_{11}^N + \alpha_{12}^Y) &< \alpha_{22}^Y + \alpha_{21}^N \ . \end{aligned}$$

Proof: The proof follows from the definitions of the envy Nash domain $\mathbf{N}^e(Y,N)$ given in (15) and the Nash domain $\mathbf{N}(Y,N)$ given in (5) when $\mathbf{N}(Y,N) \subset \mathbf{N}^e(Y,N)$. ■

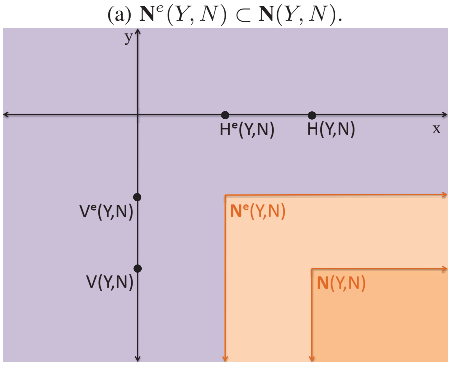
Hence, if $\alpha_{11}^N + \alpha_{12}^Y < 0$ and $\alpha_{22}^Y + \alpha_{21}^N < 0$, then $\mathbf{N}(Y,N) \subset \mathbf{N}^e(Y,N)$ (see Fig. 2b) and the following inequalities hold

$$\beta_1 > \frac{\alpha_{11}^N + \alpha_{12}^Y}{\alpha_{22}^Y + \alpha_{21}^N} > 0 \text{ and } \beta_2 > \frac{\alpha_{22}^Y + \alpha_{21}^N}{\alpha_{11}^N + \alpha_{12}^Y} > 0 \ . \quad (23)$$

On the other hand, if $\alpha_{11}^N + \alpha_{12}^Y > 0$ and $\alpha_{22}^Y + \alpha_{21}^N > 0$, then $\mathbf{N}^e(Y,N) \subset \mathbf{N}(Y,N)$ (see Fig. 2a).



(a) $\mathbf{N}^e(Y,N) \subset \mathbf{N}(Y,N)$.



(b) $\mathbf{N}(Y,N) \subset \mathbf{N}^e(Y,N)$.

Fig. 2 The geometry of envy pure Nash domain $\mathbf{N}^e(Y,N)$

We remark that Lemma 2 provides some properties for the Nash domains $\mathbf{N}^e(Y,N)$ and $\mathbf{N}(Y,N)$:

(i) $\mathbf{N}^e(Y,N) \subset \mathbf{N}(Y,N)$ if and only if

$$\begin{aligned} \beta_1(\alpha_{22}^Y + \alpha_{21}^N) &> \alpha_{11}^N + \alpha_{12}^Y \quad \text{and} \\ \beta_2(\alpha_{11}^N + \alpha_{12}^Y) &> \alpha_{22}^Y + \alpha_{21}^N \ . \end{aligned}$$

(ii) $\mathbf{N}^e(Y,N) = \mathbf{N}(Y,N)$ if and only if $\alpha_{11}^N = -\alpha_{12}^Y$ and $\alpha_{22}^Y = -\alpha_{21}^N$, which means the equilibria coincide.

(iii) the Nash domains $\mathbf{N}^e(Y,N)$ and $\mathbf{N}(Y,N)$ overlaps in the otherwise cases.

Lemma 3: Given a strategy $S \in \mathbf{S}$. If $S = (N,Y)$ is a Nash Equilibrium, then it is an envy Nash Equilibrium if and only if

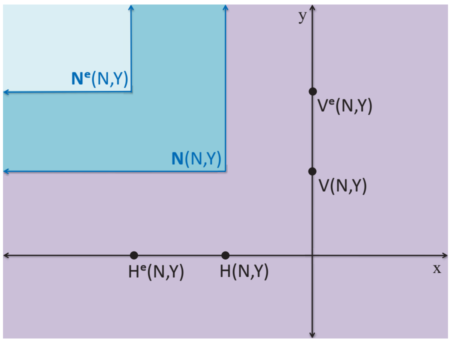
$$\begin{aligned} \beta_1(\alpha_{22}^N + \alpha_{21}^Y) &< \alpha_{11}^Y + \alpha_{12}^N \quad \text{and} \\ \beta_2(\alpha_{11}^Y + \alpha_{12}^N) &< \alpha_{22}^N + \alpha_{21}^Y \ . \end{aligned}$$

Proof: The proof follows from the definitions of the envy Nash domain $\mathbf{N}^e(N,Y)$ given in (17) and the Nash domain $\mathbf{N}(N,Y)$ given in (7) when $\mathbf{N}(N,Y) \subset \mathbf{N}^e(N,Y)$. ■

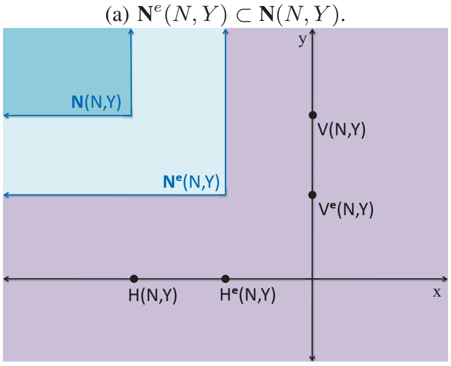
Hence, if $\alpha_{11}^Y + \alpha_{12}^N < 0$ and $\alpha_{22}^N + \alpha_{21}^Y < 0$, then $\mathbf{N}(N,Y) \subset \mathbf{N}^e(N,Y)$ (see Fig. 3b) and the following inequalities hold

$$\beta_1 > \frac{\alpha_{11}^Y + \alpha_{12}^N}{\alpha_{22}^N + \alpha_{21}^Y} > 0 \text{ and } \beta_2 > \frac{\alpha_{22}^N + \alpha_{21}^Y}{\alpha_{11}^Y + \alpha_{12}^N} > 0 \ . \quad (24)$$

On the other hand, if $\alpha_{11}^Y + \alpha_{12}^N > 0$ and $\alpha_{22}^N + \alpha_{21}^Y > 0$, then $\mathbf{N}^e(N,Y) \subset \mathbf{N}(N,Y)$ (see Fig. 3a).



(a) $\mathbf{N}^e(N,Y) \subset \mathbf{N}(N,Y)$.



(b) $\mathbf{N}(N,Y) \subset \mathbf{N}^e(N,Y)$.

Fig. 3 The geometry of envy pure Nash domain $\mathbf{N}^e(N,Y)$

We remark that Lemma 3 provides some properties for the Nash domains $\mathbf{N}^e(N,Y)$ and $\mathbf{N}(N,Y)$:

(i) $\mathbf{N}^e(N,Y) \subset \mathbf{N}(N,Y)$ if and only if

$$\begin{aligned} \beta_1(\alpha_{22}^N + \alpha_{21}^Y) &> \alpha_{11}^Y + \alpha_{12}^N \quad \text{and} \\ \beta_2(\alpha_{11}^Y + \alpha_{12}^N) &> \alpha_{22}^N + \alpha_{21}^Y \ . \end{aligned}$$

(ii) $\mathbf{N}^e(N,Y) = \mathbf{N}(N,Y)$ if and only if $\alpha_{11}^Y = -\alpha_{12}^N$ and $\alpha_{22}^N = -\alpha_{21}^Y$, which means the equilibria coincide.

(iii) the Nash domains $\mathbf{N}^e(N,Y)$ and $\mathbf{N}(N,Y)$ overlaps in the otherwise cases.

Lemma 4: Given a strategy $S \in \mathbf{S}$. If $S = (N,N)$ is a Nash Equilibrium, then it is an envy Nash Equilibrium if and only if

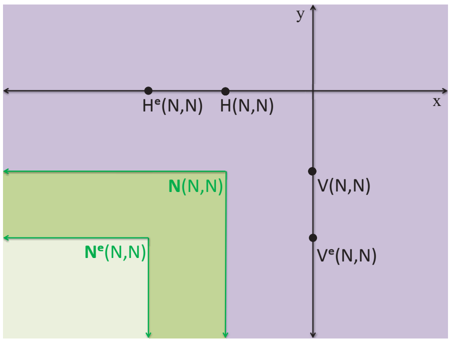
$$\begin{array}{rcl} \beta_1(\alpha^Y_{22} - \alpha^Y_{21}) & < & \alpha^Y_{12} - \alpha^Y_{11} \quad \text{and} \\ \beta_2(\alpha^Y_{11} - \alpha^Y_{12}) & < & \alpha^Y_{21} - \alpha^Y_{22}\ . \end{array}$$

Proof: The proof follows from the definitions of the envy Nash domain $\mathbf{N}^e(N,N)$ given in (19) and the Nash domain $\mathbf{N}(N,N)$ given in (9) when $\mathbf{N}(N,N) \subset \mathbf{N}^e(N,N)$. ∎

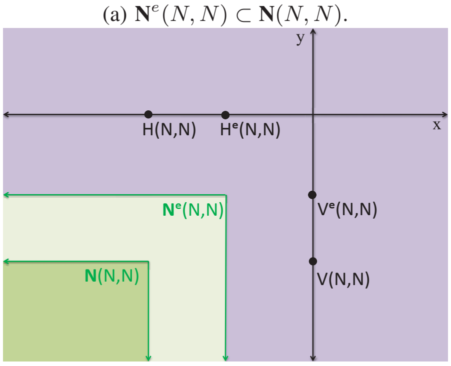
Hence, if players with type t_1 (resp. t_2) like more being with players with type t_2 (resp. t_1) than being together making decision Y (means $\alpha^Y_{11} < \alpha^Y_{12}$ and $\alpha^Y_{22} < \alpha^Y_{21}$), then $\mathbf{N}(N,N) \subset \mathbf{N}^e(N,N)$ (see Fig. 4b) and the following inequalities hold

$$\beta_1 > 0 > \frac{\alpha^Y_{12} - \alpha^Y_{11}}{\alpha^Y_{22} - \alpha^Y_{21}} \text{ and } \beta_2 > 0 > \frac{\alpha^Y_{22} - \alpha^Y_{21}}{\alpha^Y_{12} - \alpha^Y_{11}}\ . \tag{25}$$

On the other hand, if players with type t_1 (resp. t_2) like more being together than being with players with type t_2 (resp. t_1) making decision Y (means $\alpha^Y_{11} > \alpha^Y_{12}$ and $\alpha^Y_{22} > \alpha^Y_{21}$), then $\mathbf{N}^e(N,N) \subset \mathbf{N}(N,N)$ (see Fig. 4a).



(a) $\mathbf{N}^e(N,N) \subset \mathbf{N}(N,N)$.



(b) $\mathbf{N}(N,N) \subset \mathbf{N}^e(N,N)$.

Fig. 4 The geometry of envy pure Nash domain $\mathbf{N}^e(N,N)$

We remark that Lemma 4 provides some properties for the Nash domains $\mathbf{N}^e(N,N)$ and $\mathbf{N}(N,N)$:

(i) $\mathbf{N}^e(N,N) \subset \mathbf{N}(N,N)$ if and only if

$$\begin{array}{rcl} \beta_1(\alpha^Y_{22} - \alpha^Y_{21}) & > & \alpha^Y_{12} - \alpha^Y_{11} \quad \text{and} \\ \beta_2(\alpha^Y_{11} - \alpha^Y_{12}) & > & \alpha^Y_{21} - \alpha^Y_{22}\ . \end{array}$$

(ii) $\mathbf{N}^e(N,N) = \mathbf{N}(N,N)$ if and only if $\alpha^Y_{11} = \alpha^Y_{12}$ and $\alpha^Y_{22} = \alpha^Y_{21}$, which means the equilibria coincide.

(iii) the Nash domains $\mathbf{N}^e(N,N)$ and $\mathbf{N}(N,N)$ overlaps in the otherwise cases.

IV. Geometric Classes of Envy Tilings

The representations of the Nash domains $\mathbf{N}(Y,Y)$, $\mathbf{N}(Y,N)$, $\mathbf{N}(N,Y)$, $\mathbf{N}(N,N)$ and the envy Nash domains $\mathbf{N}^e(Y,Y)$, $\mathbf{N}^e(Y,N)$, $\mathbf{N}^e(N,Y)$, $\mathbf{N}^e(N,N)$ in the cartesian $xy-$plan determine an *envy decision tiling*. These tilings characterize geometrically all Nash equilibria strategies.

Definition 4: Let $A_{tt'} = \alpha^Y_{tt'} + \alpha^N_{tt'}$, for $t,t' \in \{1,2\}$, be the coordinates of the *influence matrix*.
If $A_{tt'} > 0$, then players with type t' have a *positive influence* over the utility function of the players with type t. If $A_{tt'} = 0$, then players with type t' are *indifferent* for the utility function of the players with type t. If $A_{tt'} < 0$, then players with type t' have a *negative influence* over the utility function of the players with type t.

Definition 5: Let $\mathcal{B}(n_1,n_2)$ be the *balanced threshold weight matrix* whose coordinates are given by

$$\begin{array}{rcl} B_{11}(n_1,n_2) & = & A_{11}(n_1-1) - A_{12}n_2, \\ B_{12}(n_1,n_2) & = & A_{11}(n_1-1) + A_{12}n_2, \\ B_{21}(n_1,n_2) & = & A_{22}(n_2-1) + A_{21}n_1, \\ B_{22}(n_1,n_2) & = & A_{22}(n_2-1) - A_{21}n_1. \end{array}$$

The signs of the coordinates of the *influence matrix* and *balanced threshold weight matrix* determine a certain order for the horizontal and vertical strategic thresholds.

Definition 6: An envy decision tiling is *structurally stable* if all horizontal and vertical thresholds are pairwise distinct. An envy decision tiling is a *bifurcation* if there are at least two horizontal thresholds coincide or there are at least two vertical thresholds coincide. Two envy decision tilings are *combinatorial equivalent*, if the lexicographic orders of the horizontal and vertical thresholds along the axis are the same in both tilings.

We call the pair of horizontal and vertical braids the *envy human decision chromosomes* as they play a central role to determine the human decision behavior, see Fig. 5 where we show only the horizontal braid of envy human decision chromosomes for players with type t_1 (the vertical braid of envy human decision chromosomes for players with type t_2 follows similarly to Fig. 5). Each pair of lines transversal to the horizontal and vertical braids, respectively, determines a unique envy decision tiling, up to combinatorial equivalence, and vice-versa. We observe that there are 1024 combinatorial classes of envy decision tilings, and 256 of them are being structurally stable and 768 combinatorial classes of bifurcation decision tilings.

In Fig. 5, note that pink circles ● represent the horizontal envy threshold $H^e(N,N)$, black circles ● represent the horizontal envy threshold $H^e(N,Y)$, green circles ● represent the horizontal threshold $H(N,N)$, blue circles ● represent the horizontal threshold $H(N,Y)$, orange circles ● represent the horizontal threshold $H(Y,N)$, red circles ● represent the horizontal threshold $H(Y,Y)$, yellow circles ● represent the horizontal envy threshold $H^e(Y,N)$, gray circles ● represent the horizontal envy threshold $H^e(Y,Y)$, light green arrows ↔ represent the occurrence of four times of horizontal (resp. vertically) bifurcations, and light orange arrows ↔ represent

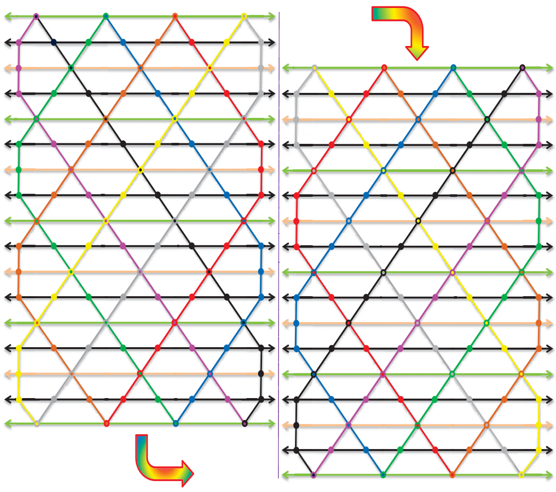


Fig. 5 Horizontal braid of envy human decision chromosomes for players with type t_1

the occurrence of three times of horizontal (resp. vertically) bifurcations.

In Figs. 6, 7 and 8, we present three envy decisions tilings where

(i) regions with cohesive uniqueness Nash equilibria domains $U(Y,Y) \subset \mathbf{N}(Y,Y)$, $U(Y,N) \subset \mathbf{N}(Y,N)$, $U(N,Y) \subset \mathbf{N}(N,Y)$ and $U(N,N) \subset \mathbf{N}(N,N)$ colored red, orange, blue and green, respectively;

(ii) regions with cohesive uniqueness envy Nash equilibria domains $U^e(Y,Y) \subset \mathbf{N}^e(Y,Y)$, $U^e(Y,N) \subset \mathbf{N}^e(Y,N)$, $U^e(N,Y) \subset \mathbf{N}^e(N,Y)$ and $U^e(N,N) \subset \mathbf{N}^e(N,N)$ colored light red, light orange, light blue and light green, respectively;

(iii) regions with neither cohesive Nash equilibria nor envy Nash equilibria colored purple;

(iv) regions with two, three, four, five, six, seven and eight cohesive Nash equilibria colored yellow, brown, pink, light yellow, gray, chartreuse and rainbow, respectively.

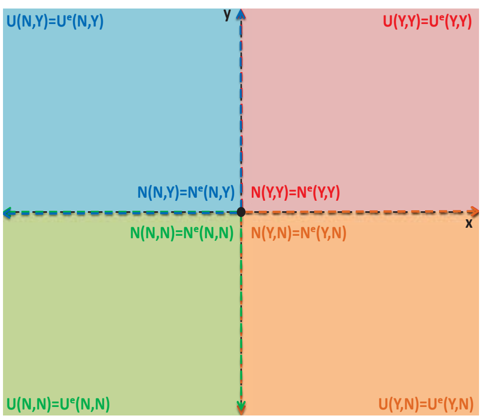


Fig. 6 Envy Nash equilibria domain when $A_{11} = A_{12} = A_{21} = A_{22} = 0$

In Fig. 6, for every relative decision preferences x and y, there is only one cohesive Nash equilibrium and one envy Nash equilibrium, except along the horizontal and vertical axises where there are two cohesive Nash equilibria and two envy Nash equilibria, and at the origin where there are four cohesive Nash equilibria and four envy Nash equilibria.

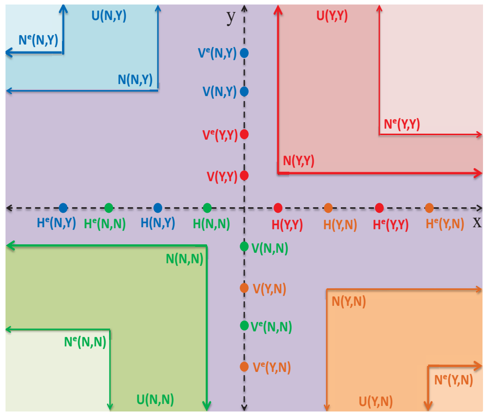


Fig. 7 Simple Distribution of cohesive envy Nash equilibria domains

In Fig. 7, we show one possible tiling with simple distribution of the cohesive envy Nash equilibria domains when $\alpha_{11}^d > \alpha_{12}^d$, $\alpha_{22}^d > \alpha_{21}^d$, $\alpha_{11}^d + \alpha_{12}^{d'} > 0$, $\alpha_{22}^{d'} + \alpha_{21}^d > 0$ for $d \neq d' \in \{Y, N\}$ and $A_{11} < 0$, $A_{22} < 0$, $A_{12} > 0$, $A_{21} > 0$ and $B_{12} < 0$, $B_{21} < 0$. We show that there is an unbounded region colored purple with neither cohesive Nash equilibrium nor cohesive envy Nash equilibrium.

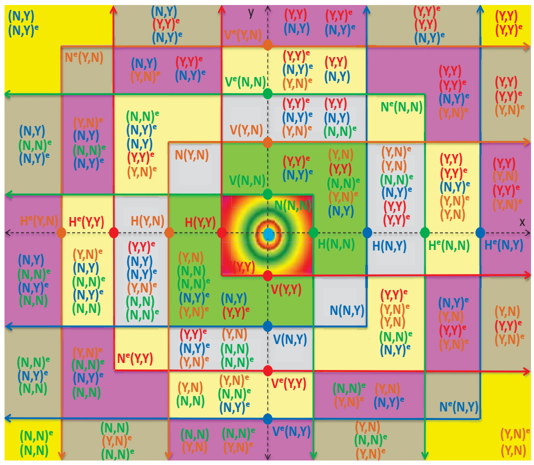


Fig. 8 The complexity of envy Nash equilibria domain

In Fig. 8, we show the high complexity of distributing the cohesive envy Nash eqilibria domains when $\alpha_{11}^d < \alpha_{12}^d$, $\alpha_{22}^d < \alpha_{21}^d$, $\alpha_{11}^d + \alpha_{12}^{d'} < 0$, $\alpha_{22}^d + \alpha_{21}^{d'} < 0$ for $d \neq d' \in \{Y, N\}$ and $A_{11} > 0$, $A_{22} > 0$, $A_{12} < 0$, $A_{21} < 0$ and $B_{12} > 0$, $B_{21} > 0$. We show that there are regions with two, three, four, five, six, seven and eight cohesive Nash equilibria colored yellow, brown, pink, light yellow, gray, chartreuse and rainbow, respectively.

V. Disparate Envy Nash Equilibria

In this section we will study the disparate envy Nash equilibria.

Definition 7: The *strategic envy set* (l_1, l_2) is the set of all pure strategies $S \in \mathbf{S}$ with $l_1(S) = l_1$ and $l_2(S) = l_2$. The *cohesive strategic envy set* (l_1, l_2) is the set all pure strategies $S \in \mathbf{S}$ with $l_1 \in \{0, n_1\}$ and $l_2 \in \{0, n_2\}$. The *disparate strategic envy set* (l_1, l_2) is the set all pure strategies that are not cohesive strategic envy set.

We observe that a cohesive strategic envy set has a single strategy but the disparate strategic envy set has more than one strategy.

Since players with the same type are identical (homogenous), a strategy to be a Nash Equilibrium depends only upon the number of players of each type who decide either Y or N, and not upon the player who is making the decision.

Definition 8: The *pure envy Nash Equilibrium (set)* (l_1, l_2) is strategic envy set whose strategies are Nash equilibria. The *(pure) envy Nash domain* $N^e(l_1, l_2)$ is the set of all pairs (x, y) for which the strategic envy set (l_1, l_2) is a Nash Equilibrium set.

The pure envy Nash Equilibrium set (l_1, l_2) is *cohesive* if $l_1 \in \{0, n_1\}$ and $l_2 \in \{0, n_2\}$. The pure envy Nash Equilibrium set (l_1, l_2) is *disparate* if $l_1 \notin \{0, n_1\}$ or $l_2 \notin \{0, n_2\}$.

Lemma 5: Let (l_1, l_2) be an envy Nash Equilibrium set.

(i) If $A_{11} > \beta_1 A_{21}$, then $l_1 \in \{0, n_1\}$.

(ii) If $A_{22} > \beta_2 A_{12}$, then $l_2 \in \{0, n_2\}$.

Furthermore, if $A_{11} > \beta_1 A_{21}$ and $A_{22} > \beta_2 A_{12}$, then (l_1, l_2) is cohesive envy Nash Equilibrium set.

Hence, if the players with a given type have a high positive influence over the utility of the players with the same type such that $A_{11} > \beta_1 A_{21}$ and $A_{22} > \beta_2 A_{12}$, then there are no disparate Nash equilibria.

Proof: Suppose, by contradiction, that the envy strategy (l_1, l_2) is a Nash Equilibrium for $l_1 \in \{1, \ldots, n_1 - 1\}$. Hence, the following two inequalities hold

$$\begin{aligned} U_1(Y; l_1, l_2, \beta_1) &\geq U_1(N; l_1 - 1, l_2, \beta_1) \quad \text{and} \\ U_1(N; l_1, l_2, \beta_1) &\geq U_1(Y; l_1 + 1, l_2, \beta_1). \end{aligned}$$

By rearranging the terms in the previous inequalities, we obtain $A_{11} \leq \beta_1 A_{21}$ which contradicts that $A_{11} > \beta_1 A_{21}$. Hence, Lemma 5 (i) holds. The proof of the other cases follow similarly to the proof of the first case. ■

VI. Conclusion

We have presented an envy behavioral game theoretical model for two homogenous types of players. We have characterized all envy strategies that form Nash equilibria and determined the corresponding envy Nash domains for each type of players. We have compared between the Nash domains and the envy Nash domains. We have studied the geometric envy tilings and showed that there are 1024 combinatorial classes of envy decision tilings, 256 of them are being structurally stable while 768 have bifurcation. We have stated some conditions for which the disparate envy strategic set is a Nash Equilibrium.

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