International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:3, No:11, 2009

Bounds On The Second Stage Spectral Radius Of Graphs

S.K.Ayyaswamy, S.Balachandran and K.Kannan

Abstract—Let G be a graph of order n. The second stage adjacency matrix of G is the symmetric $n \times n$ matrix for which the ij^{th} entry is 1 if the vertices v_i and v_j are of distance two; otherwise 0. The sum of the absolute values of this second stage adjacency matrix is called the second stage energy of G. In this paper we investigate a few properties and determine some upper bounds for the largest eigenvalue.

Keywords—Second stage spectral radius; Irreducible matrix; Derived graph.

I. Introduction

Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$. The second stage adjacency matrix is denoted by $A_2(G)$ and the second stage energy by $E_2(G)$. As it is symmetrical it will be an adjacency matrix for some graph G' which we call the derived graph of G. If Δ' is the maximum degree of G' then clearly $\Delta' \leq \Delta$. Irreducibility of the adjacency matrix is related to the property of connectedness[2]. Hence $A_2(G)$ is irreducible if and only if the derived graph G' is connected. Proposition 2.1 guarantees plenty of graphs for which their derived graphs are connected, for example, the Peterson graph whose derived graph is a 6-regular graph. In this paper we consider only those graphs for which $A_2(G)$ is irreducible.

II. Some properties

The derived graph of any odd cycle $C_{2m-1} =< v_1, v_2, ..., v_{2m-1} >$ is the odd cycle $C_{2m-1} =< v_1, v_3, v_5, ..., v_{2m-1}, v_2, ..., v_{2m-2} >$. This motivates to enunciate the following proposition:

Proposition 2.1. Let G be a graph having $C_{2m-1} =< v_1, v_2, ..., v_{2m-1} >$ as an induced subgraph for some $m \geq 3$. If (i)$\Delta \leq n-2$ and
(ii) for every $u \in V(G) - V(C_{2m-1})$, there exist at least one $v_j \notin N(u)$, $j \in \{1, 2, , ..., 2m-1\}$,then the derived graph is connected.

Proof: As mentioned above the induced subgraph $< v_1, v_2, ..., v_{2m-1} >$ is connected in G'. Choose any vertex $u \neq v_i$ for all $i = 1, 2, ..., 2m-1$ and let v_j be a vertex in C_{2m-1} which is not in $N(u)$.
Case 1. $N(u) \cap \{v_1, v_2, ..., v_{2m-1}\} = \phi$.
Let $u = u_1 u_2 ... u_r = v_i$ be the shortest path from u to C_{2m-1} of length r.
Case 1.1. r is even, then we have
$d(u = u_1, u_3) = d(u_3, u_5) = ... = d(u_{r-2}, u_r = v_i) = 2$ and so the derived graph has the path $uu_3u_5...u_{r-2}v_i$.
Case 1.2. r is odd, then we have $uu_3u_5...u_{r-1}v_{i+1}$ is a path in the derived graph.
Case 2. $N(u) \cap \{v_1, v_2, ..., v_{2m-1}\} \neq \phi$.
Choose a vertex $v_k \in N(u) \cap \{v_1, v_2, ..., v_{2m-1}\}$ such that v_k is nearest to v_j. If $k = j \pm 1$, then $d(u, v_j) = 2$. Otherwise v_l is of distance two from u where $l = k \pm 1$,. ie., $d(u, v_l) = 2$.

Proposition 2.2. Let G be a r-regular graph with order n such that $n = 2r+1$. Then the derived graph G' of G is also r-regular.
Proof: Clearly r is even. Choose any vertex v_i. Let v_k be a vertex such that $v_k \in N(v_i)$.
Claim: $d(v_k, v_i) = 2$. Otherwise, $v_k \notin N(v_j)$ for all $v_j \in N(v_i)$. This implies $deg(v_k) \leq (2r+1)-(r+2) = r-1$, which is a contradiction since $deg(v_k) = r$.
Remark: Converse of the above proposition is not true. For example, consider any odd cycle other than C_5. It is 2-regular and its derived graph being an odd cycle is also 2-regular. But $n \neq 2r+1$.

Proposition 2.3. The derived graph of circulant graph is a circulant graph.
Proof: Let G be a circulant graph formed by the set $S \subseteq \{1, 2, ..., n\}$. Then $i \in S$ if and only if $n - i \in S$ [1]. Consider a vertex v_i. Let $v_k \in D(v_i)$. Then there exists a vertex v_j such that v_j is adjacent to v_i and v_k. Then by the definition of circulant graph, v_{n-k} is also adjacent to v_j and so $v_{n-k} \in D(v_i)$. Thus, G' is formed by a set $S' \subseteq \{1, 2, ..., n\}$ such that $k \in S'$ if and only if $n - k \in S'$ and hence G' is also circulant.

Proposition 2.4. Given any positive integer n of the form p^r where p is a prime number and r is a positive integer, there exists a graph G for which the second stage energy is $2(p-1)p^{r-1}$.
Proof: Let G be the complement of the circulant graph H formed by the set $S = \{\alpha_1, \alpha_2, ..., \alpha_k\}$ where α_i's are all numbers less than n and prime to n. Then the derived graph of G is the circulant graph H whose energy is $2(p-1)p^{r-1}$ [1]. Hence $E_2(G) = E(H) = 2(p-1)p^{r-1}$.

Theorem 2.5. Let $D(v_i) = \{v_j : d(v_i, v_j) = 2\}$. Then for each fixed
$i = 1, 2, ..., n$, $|D(v_i)| = S_1 - S_2$, where

S.K.Ayyaswamy is with Department of Mathematics, Thanjavur, INDIA e-mail:sjcayya@yahoo.co.in
S.Balachandran is with Department of Mathematics, Thanjavur, INDIA e-mail:bala_maths@rediffmail.com
K.Kannan is with Department of Mathematics, Thanjavur, INDIA e-mail:kannan@maths.sastra.edu

$S_1 = \sum_{v_j adjtov_i andv_j nonpendent} |N(v_j)| - \sum_{v_j adjtov_i andv_j nonpendent} |N[v_i] \cap N(v_j)|$ and $S_2 = \sum_{v_k \in D(v_i)} (l_k - 1)$, where l_k is the number of vertices which are adjacent to both v_i and v_k.

Proof: If we take any vertex v_j adjacent to v_i, then all members of $N(v_j)$ need not be in $D(v_i)$; because some neighbours of v_j may be neighbours of v_i and so v_j can contribute only $|N(v_j)| - |N[v_i] \cap N(v_j)|$ number of members to $D(v_i)$. Similarly for all other neighbours of v_i. Therefore, the total number of members contributed by the neighbours of v_i is

$\sum_{v_j adjtov_i andv_j nonpendent} \{|N(v_j)| - |N[v_i] \cap N(v_j)|\}$, which can also be written as

$S_1 = \sum_{v_j adjtov_i andv_j nonpendent} |N(v_j)| - \sum_{v_j adjtov_i andv_j nonpendent} |N[v_i] \cap N(v_j)|$.

Among these S_1 members, some may appear more than once. For example, a member v_k of $D(v_i)$ may have neighbours $v_1, v_2, ..., v_{l_k}$ which all are in turn neighbours of v_i also. Thus, v_k is repeated say l_k times in S_1. But it should be taken only once. Thus we get the required result.

Corollary 2.6. If the second stage adjacency matrix is irreducible, then

$|D(v_i)| \leq 2m - 2d_i - \delta + \epsilon_{F_i}$ where ϵ_{F_i} is the number of pendent vertices adjacent to v_i

Proof: We observe that v_i is included as many times as $d_i - \epsilon_{F_i}$ in $\sum_{v_j adjtov_i andv_j nonpendent} |N[v_i] \cap N(v_j)|$.

Hence $\sum_{v_j adjtov_i andv_j nonpendent} |N[v_i] \cap N(v_j)| \geq d_i - \epsilon_{F_i}$. Therefore

$$D(v_i) \leq S_1 \leq \sum_{v_j adjtov_i andv_j nonpendent} |N(v_j)| - d_i + \epsilon_{F_i} \qquad (1)$$

Since the second stage adjacency matrix is irreducible, for each vertex v_i, there is atleast one vertex v_k which is non adjacent to v_i. Therefore

$$\sum_{v_j adjtov_i andv_j nonpendent} |N(v_j)| \leq 2m - d_i - \delta \qquad (2)$$

Combining (1) and (2), we get $D(v_i) \leq 2m - 2d_i - \delta + \epsilon_{F_i}$.

III. Bounds for the largest eigenvalue

Theorem 3.1. Let G be a graph with minimum degree $\delta \geq 1$ and maximum degree Δ, then

$\rho(G) \leq \sqrt{2\Delta(m + n - \delta - 1) - 4m + \delta(2 - \delta) + A}$, where $A = \epsilon_F(2\Delta + \delta + 1)$ and ϵ_F is the number of pendent vertices of G.

Proof:

Proof: Let $D(v_i) = \{v_j : d(v_i, v_j) = 2\}$. Let $D_1(v_i) = \{v_j : d(v_i, v_j) \neq 2\}$ and let $D_1'(v_i) = D_1(v_i) - \{v_i\}$. Let $x = (x_1, x_2, ..., x_n)^T$ be the unit eigenvector corresponding to $\rho(G)$. Then $\rho(G)x_i = \sum_{j=1}^n a_{ij}x_j$. By Cauchy- Schwarz inequality,

$\rho^2(A)x_i^2 = (\sum_{j=1}^n a_{ij}(a_{ij}x_j))^2$.

$\leq \sum_{j=1}^n a_{ij}^2 \sum_{j=1}^n (a_{ij}x_j)^2$

$\leq (2m - (2d_i + \delta - \epsilon_{F_i})) \sum_{j \in D(v_i)} x_j^2$, by using corollary 2.6.

Hence

$\rho(G)^2 = \sum_{i=1}^n \rho(G)^2 x_i^2$

$\leq \sum_{i=1}^n (2m - (2d_i + \delta - \epsilon_{F_i})) \sum_{j \in D(v_i)} x_j^2$

$= \sum_{i=1}^n (2m - (2d_i + \delta - \epsilon_{F_i}))(1 - \sum_{j \in D_1(v_i)} x_j^2)$

$= \sum_{i=1}^n (2m - (2d_i + \delta - \epsilon_{F_i})) - \sum_{i=1}^n (2m - (2d_i + \delta - \epsilon_{F_i})) \sum_{j \in D_1(v_i)} x_j^2$

$$= 2mn - 4m - n\delta + \epsilon_F - \sum_{i=1}^n (2m - (2d_i + \delta - \epsilon_{F_i})) \sum_{j \in D_1(v_i)} x_j^2 \qquad (3)$$

In (3), we estimate, $-\sum_{i=1}^n (2m - (2d_i + \delta - \epsilon_{F_i})) \sum_{j \in D_1(v_i)} x_j^2$

$$= -\sum_{i=1}^n 2m \sum_{j \in D_1(v_i)} x_j^2 + \sum_{i=1}^n (2d_i + \delta - \epsilon_{F_i}) \sum_{j \in D_1(v_i)} x_j^2 \qquad (4)$$

Now, consider

$\sum_{i=1}^n (2d_i + \delta - \epsilon_{F_i}) \sum_{j \in D_1(v_i)} x_j^2$

$= \sum_{i=1}^n (2d_i + \delta - \epsilon_{F_i}) x_i^2 + \sum_{i=1}^n (2d_i + \delta - \epsilon_{F_i}) \sum_{j \in D_1'(v_i)} x_j^2$

$= \sum_{i=1}^n 2d_i x_i^2 + \sum_{i=1}^n \delta x_i^2 - \sum_{i=1}^n \epsilon_{F_i} x_i^2 + \sum_{i=1}^n 2d_i \sum_{j \in D_1'(v_i)} x_j^2 + \sum_{i=1}^n \delta \sum_{j \in D_1'(v_i)} x_j^2$

$- \sum_{i=1}^n \epsilon_{F_i} \sum_{j \in D_1'(v_i)} x_j^2$

$\leq \sum_{i=1}^n 2d_i x_i^2 + \delta \sum_{i=1}^n x_i^2 + \sum_{i=1}^n 2d_i \sum_{j \in D_1'(v_i)} x_j^2 + \sum_{i=1}^n \delta \sum_{j \in D_1'(v_i)} x_j^2$

$\leq 2\sum_{i=1}^n d_i x_i^2 + \delta + 2\Delta \sum_{i=1}^n \sum_{j \in D_1'(v_i)} x_j^2 + \delta \sum_{i=1}^n \sum_{j \in D_1'(v_i)} x_j^2$

$= 2\sum_{i=1}^n d_i x_i^2 + \delta + 2\Delta \sum_{i=1}^n (n - (d_i - \epsilon_{F_i}) - 1)x_i^2 + \delta \sum_{i=1}^n (n - (d_i - \epsilon_{F_i}) - 1)x_i^2$

$= 2\sum_{i=1}^n d_i x_i^2 + \delta + 2\Delta \sum_{i=1}^n (n - d_i - 1)x_i^2 + 2\Delta \sum_{i=1}^n \epsilon_{F_i} x_i^2 + \delta \sum_{i=1}^n (n - d_i - 1)x_i^2 + \delta \sum_{i=1}^n \epsilon_{F_i} x_i^2$

$\leq 2\sum_{i=1}^n d_i x_i^2 + \delta + 2\Delta \sum_{i=1}^n (n - d_i - 1)x_i^2 + 2\Delta\epsilon_F \sum_{i=1}^n x_i^2 + \delta \sum_{i=1}^n (n - d_i - 1)x_i^2 + \delta\epsilon_F$

$= 2\sum_{i=1}^n d_i x_i^2 + \delta + 2\Delta \sum_{i=1}^n (n - d_i - 1)x_i^2 + 2\Delta\epsilon_F + \delta \sum_{i=1}^n (n - d_i - 1)x_i^2 + \delta\epsilon_F$

$= \Delta(2\sum_{i=1}^n d_i x_i^2 + 2\sum_{i=1}^n (n - d_i - 1)x_i^2) - (2\Delta - 2)\sum_{i=1}^n d_i x_i^2 + \delta + 2\Delta\epsilon_F$

$+ \delta(\sum_{i=1}^n d_i x_i^2 + \sum_{i=1}^n (n - d_i - 1)x_i^2) - \delta \sum_{i=1}^n d_i x_i^2 + \delta\epsilon_F$

$= \Delta(2n - 2) - (2\Delta - 2)\sum_{i=1}^n d_i x_i^2 + \delta + 2\Delta\epsilon_F + \delta(n - 1) - \delta \sum_{i=1}^n d_i x_i^2 + \delta\epsilon_F$

$\leq 2\Delta(n - 1) - 2(\Delta - 1)\delta + \delta + 2\Delta\epsilon_F + \delta(n - 1) - \delta\delta + \delta\epsilon_F$

$= 2\Delta(n - 1) - 2\delta(\Delta - 1) + \delta + \delta(n - 1) - \delta^2 + 2\Delta\epsilon_F + \delta\epsilon_F$

$= 2\Delta(n - 1) + \delta(-2(\Delta - 1) + 1 + (n - 1) - \delta) + \epsilon_F(2\Delta + \delta)$

$$= 2\Delta(n - 1) + \delta(n - 2(\Delta - 1) - \delta) + \epsilon_F(2\Delta + \delta) \qquad (5)$$

In a similar fashion, we have $-\sum_{i=1}^n 2m \sum_{j \in D_1(v_i)} x_j^2$

$= -\sum_{i=1}^n 2m x_i^2 - \sum_{i=1}^n 2m \sum_{j \in D_1'(v_i)} x_j^2$

$= -2m - 2m \sum_{i=1}^n \sum_{j \in D_1'(v_i)} x_j^2$

$= -2m - 2m \sum_{i=1}^n (n - (d_i - \epsilon_{F_i}) - 1)x_i^2$

$= -2m - 2m \sum_{i=1}^n (n - d_i - 1)x_i^2 - 2m \sum_{i=1}^n \epsilon_{F_i} x_i^2$

$\leq -2m - 2m\sum_{i=1}^{n}(n - d_i - 1)x_i^2$
$= -2m - 2m\sum_{i=1}^{n} nx_i^2 + 2m\sum_{i=1}^{n} d_i x_i^2 + 2m\sum_{i=1}^{n} x_i^2$
$= -2m - 2mn + 2m\sum_{i=1}^{n} d_i x_i^2 + 2m$

$$\leq -2mn + 2m\Delta \quad (6)$$

From (3),(4),(5),(6), we get,
$\rho(G)^2 \leq (2mn - 4m - n\delta + \epsilon_F) + (2\Delta(n-1) + \delta(n-2(\Delta-1)-\delta) + \epsilon_F(2\Delta+\delta)) - 2mn + 2m\Delta$
$= -4m - n\delta + \epsilon_F + 2\Delta(n-1) + \delta(n-2(\Delta-1)-\delta) + \epsilon_F(2\Delta+\delta) + 2m\Delta$
$= -4m + \epsilon_F + 2\Delta(n-1) - \delta(2(\Delta-1)+\delta) + \epsilon_F(2\Delta+\delta) + 2m\Delta$
$= -4m + 2m\Delta + 2\Delta(n-1) - \delta(2(\Delta-1)+\delta) + \epsilon_F(2\Delta+\delta+1)$
$= -4m + 2m\Delta + 2n\Delta - 2\Delta - 2\delta\Delta + 2\delta - \delta^2 + \epsilon_F(2\Delta+\delta+1)$
$= -4m + 2\Delta(m+n-\delta-1) + (2\delta - \delta^2) + \epsilon_F(2\Delta+\delta+1)$
Hence
$\rho(G) \leq \sqrt{2\Delta(m+n-\delta-1) - 4m + \delta(2-\delta) + A}$, where $A = \epsilon_F(2\Delta+\delta+1)$. ■

Let B be an $n \times n$ matrix and let $S_i(B)$ denote the i^th row sum of B, ie., $S_i(B) = \sum_{j=1}^{n} B_{ij}$, where $1 \leq i \leq m$.

Lemma 3.2. Let G be a connected n-vertex graph and A_2 its second stage adjacency matrix, with spectral radius ρ. Let P be any polynomial. If A_2 is irreducible, then,
$min_{v\in V(G)} S_v(P(A_2)) \leq P(\rho) \leq max_{v\in V(G)} S_v(P(A_2))$

Moreover, if the row sums of $P(A_2)$ are not all equal then both inequalities are strict.

Proof: Since A_2 is irreducible, the proof is just analogous to that of Lemma 2.2 in [4].

Lemma 3.3. For each fixed i=1,2, . . . ,n,
$S_{v_i}(A_2^2) = |D(v_i)| + \sum_{i\neq j}\sum_k |\{v_k : d(v_k, v_i) = 2 \;\; and \;\; d(v_k, v_j) = 2\}|$
Proof: ij^th entry in b_{ij} in $A_2^2 = \sum_{i=1}^{n} a_{ik}a_{kj}$
Case 1. Let $i = j$, then $b_{ii} = \sum_{k=1}^{n} a_{ik}a_{ki}$

$$= |D(v_i)| \quad (7)$$

Case 2. Let $i \neq j$, $a_{ik}a_{kj} = 1$ if and only if $a_{ik} = 1$ and $a_{kj} = 1$
$a_{ik}a_{kj} = 1$ if and only if $d(v_k, v_i) = 2$ and $d(v_k, v_j) = 2$. Therefore

$$b_{ij} = \sum_k |\{v_k : d(v_k, v_i) = 2 \; and \; d(v_k, v_j) = 2\}| \quad (8)$$

$S_{v_i}(A_2^2) = b_{ii} + \sum_{i\neq j} b_{ij}$
$= |D(v_i)| + B \; where$
$B = \sum_{i\neq j}\sum_k |\{v_k : d(v_k, v_i) = 2 \;\; and \;\; d(v_k, v_j) = 2\}|$, using (7) and (8).

Let G be a simple graph with n vertices and m edges. Let $\delta = \delta(G)$ be the minimum degree of vertices of G and $\rho(G)$ be the spectral be the spectral radius of the adjacency matrix A of G. Then in [6] it is proved that,
$\rho(G) \leq (\delta - 1 + \sqrt{(\delta+1)^2 + 4(2m - \delta n)})/2$.
Corresponding to the above result, we have the following theorem for the second stage matrix.

Theorem 3.4. Let G be a simple graph with n vertices and m edges. Let $\Delta = \Delta(G)$ be maximum degree of vertices of G and $\rho(G)$ be the spectral radius of the second stage adjacency matrix A_2 of G. Then $\rho(G) \leq (1 + \sqrt{4(n-1)\Delta})/2$. Proof: Since $S_{v_i}(A_2^2) = |D(v_i)| + \sum_{i\neq j}\sum_k |\{v_k : d(v_k, v_i) = 2 and d(v_k, v_j) = 2\}|$
$S_{v_i}(A_2^2) - S_{v_i}(A_2) = \sum_{i\neq j}\sum_k |\{v_k : d(v_k, v_i) = 2 and d(v_k, v_j) = 2\}|$
$\leq (n-1)\Delta$. As this holds for every vertex $v \in V(G)$. Lemma 3.2 implies that $\rho(G)^2 - \rho(G) \leq (n-1)\Delta$. Solving the quadratic inequality, we obtain $\rho(G) \leq (1 + \sqrt{4(n-1)\Delta})/2$.

For a non regular graph, many upper bounds for the largest eigenvalue of adjacency matrix are found. One such upper bound is
$\lambda_1 \leq \Delta - (1/2n(n\Delta - 1)\Delta^2$ [5]. In the following theorem we find a similar upper bound for our second stage concept.

Theorem 3.5. If G is connected and not regular, then
$\lambda_1 \leq \Delta - (1/4\Delta^2 n(2m - 3\delta + \epsilon_F))$.
Proof: Let x be a positive unit eigenvector of $A_2(G)$ corresponding to λ_1. We have that $\lambda_1 = \lambda_1\|x\|^2$
$= \lambda_1 \sum_{v_i\in V} x_i^2$
$= 2\sum_{d(v_i,v_j)=2} x_i x_j$
Since the maximum degree of G is Δ and G is not regular, we have
$\Delta = \Delta\|x\|^2 > \sum_{v_i\in V} |D_i| x_i^2$
Thus, $\Delta - \lambda_1 > \sum_{v_i\in V} |D_i| x_i^2 - 2\sum_{d(v_i,v_j)=2} x_i x_j$
$= \sum_{v_i\in V}\sum_{v_j\in D(v_i)} x_i^2 - 2\sum_{d(v_i,v_j)=2} x_i x_j$
$= \sum_{d(v_i,v_j)=2}(x_i^2 + x_j^2 - 2x_i x_j)$
$= \sum_{d(v_i,v_j)=2}(x_i - x_j)^2$

From Cauchy-schwarz inequality and $|D(v_i)| \leq 2m - 2d_i - \delta + \epsilon_{F_i}$, it follows that $\sum_{d(v_i,v_j)=2}(x_i - x_j)^2 \geq (1/|D(v_i)|)(\sum_{d(v_i,v_j)=2}|x_i - x_j|)^2$
$\geq (1/2m - 2d_i - \delta + \epsilon_{F_i})(\sum_{d(v_i,v_j)=2}|x_i - x_j|)^2$
$\geq (1/2m - 3\delta + \epsilon_F)(\sum_{d(v_i,v_j)=2}|x_i - x_j|)^2$

Let u and v be the vertices of derived graph G such that $x_u = max_{v_i\in V} x_i$ and $x_v = min_{v_i\in V} x_i$ and let $u = w_0 w_1 ... w_k = v$ be a path between u and v in the derived graph G. Then
$\sum_{\{v_i,v_j\}\in E} |x_i - x_j| \geq \sum_{l=0}^{k-1} x_{w_l} - x_{w_{l+1}}$
$\geq \sum_{l=0}^{k-1}(x_{w_l} - x_{w_{l+1}})$
$= x_{w_0} - x_{w_k}$
$= x_u - x_v$.

We have $\Delta - \lambda_1 > (1/2m - 3\delta + \epsilon_F)(x_u - x_v)^2$. It remains to estimate $x_u - x_v$. Since $\sum_{v_i\in V} x_i^2 = 1$, we have $x_u \geq 1/\sqrt{n}$ and $x_v \leq 1/\sqrt{n}$. There are three cases to consider.
Case Ia: $x_u \geq 1/\sqrt{n} + c$. Then $x_v < 1/\sqrt{n}$ and $\Delta - \lambda_1 > (c^2/2m - \delta + \epsilon_F)$
Case Ib: $x_v \leq 1/\sqrt{n} - c$. Then $x_u > 1/\sqrt{n}$ and again $\Delta - \lambda_1 > (c^2/2m - \delta + \epsilon_F)$ Case II : $1/\sqrt{n} - c < x_v < x_u < 1/\sqrt{n} + c$. Then $x_i \in (1/\sqrt{n} - c, 1/\sqrt{n} + c)$. Then $x_i \in (1/\sqrt{n} - c, 1/\sqrt{n} + c)$ holds for each $v_i \in V$, and by choosing $s \in V'$ with $d_s < \Delta' - 1$, which is regular, we get
$\lambda_1(1/\sqrt{n} - c) < \lambda_1 x_s$

$= \sum_{\{t:\{s,t\}\in E'\}} x_t < (\Delta' - 1)(1/\sqrt{n} + c)$
where $\Delta' = maxD(v_i)$, $i = 1, 2, ..., n$ and E' is the edge set of G' which implies,
$\lambda_1 < (\Delta' - 1)(1 + c\sqrt{n}/1 - c\sqrt{n})$. In order for the expression on the RHS to be useful, it must be less than Δ', which is satisfied for $c < 1/(2\Delta' - 1)\sqrt{n})$. Put $c = 1/2\Delta'\sqrt{n}$ in cases Ia and Ib, we get,

$\Delta - \lambda_1 > 1/(2m - 3\delta + \epsilon_F)4(\Delta')^2 n$

$\lambda_1 < \Delta - (1/(2m - 3\delta + \epsilon_F)4(\Delta')^2 n)$

While in $\lambda_1 < (\Delta' - 1)(1 + \sqrt{n}/2\Delta'\sqrt{n}/1 - \sqrt{n}/2\Delta'\sqrt{n})$

$$< (\Delta' - 1)(2\Delta' + 1/2\Delta' - 1)$$
$$= (2\Delta^2 + \Delta' - 2\Delta' - 1/2\Delta' - 1)$$
$$= \Delta' - (1/2\Delta' - 1)$$

This implies $\lambda_1 < \Delta' - (1/2\Delta' - 1)$

$$< \Delta - (1/4(\Delta')^2(2m - 3\delta + \epsilon_F))$$
$$< \Delta - (1/4(\Delta)^2 n(2m - 3\delta + \epsilon_F))$$

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