

# Bose-Einstein Condensation in Neutral Many Bosonic System

M. Al-Sugheir, M. Sakhreya, G. Alna'washi, F. Al-Dweri

**Abstract**—In this work, the condensation fraction and transition temperature of neutral many bosonic system are studied within the static fluctuation approximation (SFA). The effect of the potential parameters such as the strength and range on the condensate fraction was investigated. A model potential consisting of a repulsive step potential and an attractive potential well was used. As the potential strength or the core radius of the repulsive part increases, the condensation fraction is found to be decreased at the same temperature. Also, as the potential depth or the range of the attractive part increases, the condensation fraction is found to be increased. The transition temperature is decreased as the potential strength or the core radius of the repulsive part increases, and it increases as the potential depth or the range of the attractive part increases.

**Keywords**—About four key words or phrases in alphabetical order, separated by commas

## I. INTRODUCTION

IN this work, Bose-Einstein condensation of neutral many-bosonic system was studied. A previous study [1] on a neutral many-bosonic system was performed using a potential step model. In this work we shall go further toward a more realistic potential. In general, the interaction potential in neutral many-bosonic systems has two terms; a short range repulsive part which arises from the Pauli exclusion principle, and a long range attractive part which arises from phonon-phonon interaction. These two parts are modulated in our work by a repulsive step potential model of strength  $A$  and core radius  $r_1$ , and the attractive part is approximated to a potential well of depth  $B$  ranging from  $r_1$  to  $r_2$ . An extended system of  $N$  particles in three dimensions each of mass  $m$ , and occupying a volume  $\Omega$  is considered. The effects of potential parameters on condensate fraction of the system are investigated. The static fluctuation approximation (SFA) [1]-[4], a relatively new many-body approach in condensed matter, is adopted in this study. It is based on a replacement of the square of the local-field operator by its mean value. This means that the true quantum mechanical spectrum of this operator is replaced by a distribution around its mean value. In the SFA, the temperature is inserted smoothly from basic principles in quantum statistical mechanics, while most of the conventional many body theories were established at ground state.

M. K. Al-Sugheir, G. Alna'washi, and F. Al-Dweri are with the Department of Physics, The Hashemite University, Zarqa, Jordan (phone: +962-777792110; fax: +962 (5) 3826613; e-mail: [msugh@hu.edu.jo](mailto:msugh@hu.edu.jo)).

G. Alna'washi is with the Department of Physics, The Hashemite University, Zarqa, Jordan (phone: +962-779133453; fax: +962 (5) 3826613; e-mail: [alnawashi@hu.edu.jo](mailto:alnawashi@hu.edu.jo)).

F. Al-Dweri is with the Department of Physics, The Hashemite University, Zarqa, Jordan (phone: +962-796647410; fax: +962 (5) 3826613; e-mail: [faldweri@ugr.es](mailto:faldweri@ugr.es)).

M. Sakhreya is with the Department of Physics, The Hashemite University, Zarqa, Jordan.

Also, it is relatively simple compared with other many-body approaches because it is not based on Green's functions or Feynman diagrams, and in principle it is applicable at least to any arbitrary system, either weakly- or strongly-interacting [2]. The fascinating phenomena in neutral many-bosonic systems such as Bose-Einstein condensation (BEC) motivates us to shed more light on these systems, especially in the last decade after the first realization of Bose-Einstein condensation (BEC) in trapped atomic gases systems observed experimentally [5]-[7]. Many techniques, ranging from simple mean-field approach to more advanced approaches, have been used to study many-bosonic systems. These include a single-particle Green-function method [8], a density-functional approach [9], exact quantum Monte Carlo techniques [10], and gauge-theoretic formulation [11]. Most of these techniques are formulated at the ground state. They have plagued with difficulties and applicability at finite temperature.

The Bose-Einstein condensation and transition temperature in liquid  $^4\text{He}$  are challenging phenomena. London [12], in 1938, suggested that the existence of transition temperature in liquid  $^4\text{He}$  might indicate the existence of Bose-Einstein condensation. Deeney *et al.* [13] found that the equilibrium density of liquid  $^4\text{He}$  has a maximum value at the transition temperature. The graphical representation of the partition function was used [14] to study a one-dimensional Bose gas, where the interaction potential was assumed to be a repulsive delta function. This method has some advantages; it can be applied to confined and infinite systems, it has wide applicability, and it can be used to calculate the thermodynamic properties.

Since the discovery of BEC in dilute atomic gases, theoretical studies of this and related phenomenon in such systems have grown exponentially [10]. The starting point of most of the theoretical studies of BEC in confined gases is the so-called Gross-Pitaevskii equation (GPE), which is a mean-field Schrödinger equation for a system of bosons interacting through a two-body interaction described by  $\delta$ -function. The GPE equation can only be solved numerically. The first "pure" BEC was created by Eric Cornell and Carl Wieman in 1995 [6]. After that, an independent effort led by Wolfgang Ketterle created a condensation in sodium-23 [5] and [7].

Following this Introduction, the SFA formalism is presented in Section Two for neutral many-bosonic system. The so-called long-range equation and a closed system of nonlinear integral equations for a model potential are derived. In Section Three, the results are presented and discussed. Finally, in Section Four, the article is concluded with a general summary.

## II. STATIC FLUCTUATION APPROXIMATION FORMALISM

In the SFA we assume that the Hamiltonian can be expressed as a linear combination of the local-field operator

$\hat{E}_{\vec{k}}$  and the number of particles operator  $\hat{n}_{\vec{k}} = \hat{b}_{\vec{k}}^{\dagger} \hat{b}_{\vec{k}}$ , such that [2]

$$\hat{H} = \sum_{\vec{k}} \hat{E}_{\vec{k}} \hat{n}_{\vec{k}}, \quad (1)$$

The local-field operator is assumed to be hermitian and commutes with the creation operator  $\hat{b}_{\vec{k}}^{\dagger}$  and annihilation operator  $\hat{b}_{\vec{k}}$ .

In the mean field approximation, the local-field operator  $\hat{E}_{\vec{k}}$  is replaced with its mean value:

$$\hat{E}_{\vec{k}} \cong \langle \hat{E}_{\vec{k}} \rangle. \quad (2)$$

In the present approximation, SFA, the square of the local field operator is replaced by its mean value [2]:

$$\hat{E}_{\vec{k}}^2 \cong \langle \hat{E}_{\vec{k}}^2 \rangle = \varphi_{\vec{k}}^2. \quad (3)$$

The physical meaning of that is that the true quantum-mechanical spectrum of the local-field operator is replaced by a distribution around its mean value.

The desired long-range equation (generation equation) for many-bosonic system is given by [2]:

$$\langle \hat{n}_{\vec{k}} \hat{A} \rangle = \eta_0(\vec{k}) \langle \hat{A} \rangle + \eta_1(\vec{k}) \langle \Delta \hat{E}_{\vec{k}} \hat{A} \rangle, \quad (4)$$

where

$$\eta_0(\vec{k}) \equiv \frac{1}{2} \left[ \frac{1}{\exp(\beta(\langle \hat{E}_{\vec{k}} \rangle + \varphi_{\vec{k}})) - 1} + \frac{1}{\exp(\beta(\langle \hat{E}_{\vec{k}} \rangle - \varphi_{\vec{k}})) - 1} \right],$$

$$\eta_1(\vec{k}) \equiv \frac{1}{2\varphi_{\vec{k}}} \left[ \frac{1}{\exp(\beta(\langle \hat{E}_{\vec{k}} \rangle + \varphi_{\vec{k}})) - 1} - \frac{1}{\exp(\beta(\langle \hat{E}_{\vec{k}} \rangle - \varphi_{\vec{k}})) - 1} \right],$$

$\hat{A}$  is an arbitrary operator commute with the creation and annihilation operators, and  $\beta = \frac{1}{k_B T}$ , where  $k_B$  is the Boltzmann's constant and  $T$  is the absolute temperature.

Now, we are in a position to derive the closed set of nonlinear integral equations. We start by setting  $\hat{A} = 1$  in (4):

$$\langle \hat{n}_{\vec{k}} \rangle = \eta_0(\vec{k}) + \eta_1(\vec{k}) \langle \Delta \hat{E}_{\vec{k}} \rangle. \quad (5)$$

From the symmetry of the fluctuations in the local-field operator:  $\langle \Delta \hat{E}_{\vec{k}} \rangle = 0$ ; this yields to

$$\langle \hat{n}_{\vec{k}} \rangle = \eta_0(\vec{k}). \quad (6)$$

It is more convenient to rewrite the long-range equation in terms of the fluctuations of occupation number operator:

$$\Delta \hat{n}_{\vec{k}} = \hat{n}_{\vec{k}} - \langle \hat{n}_{\vec{k}} \rangle. \quad (7)$$

Thus

$$\langle \Delta \hat{n}_{\vec{k}} \hat{A} \rangle = \eta_1(\vec{k}) \langle \Delta \hat{E}_{\vec{k}} \hat{A} \rangle. \quad (8)$$

To obtain the pair correlation function, we simply put  $\hat{A} = \Delta \hat{n}_{\vec{q}}$  in (8), where  $\vec{k} = \vec{q}$ :

$$\langle \Delta \hat{n}_{\vec{k}} \Delta \hat{n}_{\vec{q}} \rangle_c = \eta_1(\vec{k}) \langle \Delta \hat{E}_{\vec{k}} \Delta \hat{n}_{\vec{q}} \rangle_c \quad (9)$$

The quadratic fluctuations in the occupation numbers can not be calculated from the long-range equation, where  $\Delta \hat{n}_{\vec{k}}$

does not commute with  $\hat{b}_{\vec{k}}^{\dagger}$  and  $\hat{b}_{\vec{k}}$ . It is given by [2]:

$$\langle (\Delta \hat{n}_{\vec{k}})^2 \rangle = \langle \hat{n}_{\vec{k}}^2 \rangle - \langle \hat{n}_{\vec{k}} \rangle^2; \quad (10)$$

$$\langle (\Delta \hat{n}_{\vec{k}})^2 \rangle = \langle \hat{n}_{\vec{k}} \rangle (1 + \langle \hat{n}_{\vec{k}} \rangle) + 2\eta_1(\vec{k}) \langle \Delta \hat{E}_{\vec{k}} \Delta \hat{n}_{\vec{k}} \rangle \quad (11)$$

Finally, to obtain the fluctuations in the local-field operator,

$$\Delta \hat{E}_{\vec{k}}^2 = \varphi_{\vec{k}}^2, \text{ we set } \hat{A} = \Delta \hat{E}_{\vec{k}} \text{ in (8):}$$

$$\eta_1(\vec{k}) \varphi_{\vec{k}}^2 = \langle \Delta \hat{E}_{\vec{k}} \Delta \hat{n}_{\vec{k}} \rangle. \quad (12)$$

The total Hamiltonian describing neutral many-bosonic system in second quantization can be written as [16]

$$\hat{H} = \sum_{\vec{k}} \frac{\hbar^2 k^2}{2m} \hat{b}_{\vec{k}}^{\dagger} \hat{b}_{\vec{k}} + \frac{1}{2} \sum_{\vec{k}} V(k) \hat{\rho}_{\vec{k}} \hat{\rho}_{-\vec{k}}, \quad (13)$$

where

$$V(k) = \int V(r) \exp(i\vec{k} \cdot \vec{r}) d\vec{r}; \quad \hat{\rho}_{\vec{k}} \equiv \frac{1}{\Omega} \sum_{\vec{q}} \hat{b}_{\vec{k}+\vec{q}}^{\dagger} \hat{b}_{\vec{q}}, \quad (14)$$

$\Omega$  is the normalization volume of the system, and  $V(k)$  the Fourier transform of the pair potential  $V(r)$ .

In our calculations we are interested in the grand Hamiltonian rather than the Hamiltonian; this is defined as

$$\hat{H}_g = \sum_{\vec{k}} \left( \frac{\hbar^2 k^2}{2m} - \mu \right) \hat{b}_{\vec{k}}^{\dagger} \hat{b}_{\vec{k}} + \frac{1}{2} \sum_{\vec{k}} V(k) \hat{\rho}_{\vec{k}} \hat{\rho}_{-\vec{k}}. \quad (15)$$

Here  $\mu$  is the chemical potential which is the energy required to add a particle to the system or to remove a particle from the system.

The local-field operator  $\hat{E}_{\vec{k}}$ , is given by [2] and [17]

$$\hat{E}_{\vec{k}} = \left[ \hat{b}_{\vec{k}}, \left[ \hat{H}_g, \hat{b}_{\vec{k}}^{\dagger} \right] \right] = \varepsilon(k) + \frac{1}{\Omega} \sum_{\vec{q}} W(\vec{k}, \vec{q}) \hat{n}_{\vec{q}}, \quad (16)$$

where  $\varepsilon(k) = \frac{\hbar^2 k^2}{2m} - \mu$ , and  $W(\vec{k}, \vec{q}) = V(0) + V(\vec{k} - \vec{q})$ .

The variation in the local-field operator  $\Delta \hat{E}_{\vec{k}}^2$  can be calculated from (16) to be

$$\Delta \hat{E}_{\vec{k}} = \frac{1}{\Omega} \sum_{\vec{q}} W(\vec{k}, \vec{q}) \Delta \hat{n}_{\vec{q}}. \quad (17)$$

The closed set of nonlinear integral equations (6), (9), (11), (12), and (16) can be solved numerically by iteration method along with Gaussian quadrature point method. Throughout this work, a natural system of units is used such that  $\hbar = 1 = m$ , where  $m$  is the  $^4\text{He}$  atomic mass, the conversion factor being  $\frac{\hbar^2}{m} = 12.120048 \text{ K} \cdot \text{\AA}^2$ .

### III. RESULTS AND DISCUSSION

A previous study [1] approximated the interaction potential to a step potential model. In the present work, the potential is approximated to a repulsive part represented by a potential

step of strength  $A$  and core radius  $r_1$  and an attractive part represented by a potential well of depth  $B$  and range  $r_2$ ,

$$V(r) = \begin{cases} A & r < r_1 \\ B & r_1 < r < r_2 \\ 0 & \text{otherwise} \end{cases} \quad (18)$$

The role of potential parameters on the condensate fraction is examined in this work.

The effect of the strength of the repulsive potential on the condensate fraction is studied and shown in Fig. 1. It is found that the condensate fraction decreases as the potential strength increases. In the low temperature limit the effect of the potential is almost not observed and the system behaves like as an ideal Bose gas. The transition from the ground state to the excited state occurs smoothly as the potential strength is weak. This transition occurs drastically as the potential strength has strong values.

The transition temperature,  $T_c$ , the temperature where the condensate fraction vanishes, decreases as the potential strength increases. Table 1 shows the transition temperature as determined from Fig. 1 for different values of potential strength.

For ideal  $^4\text{He}$  gas the transition temperature is 3.14 K. In the interacting system the condensate fraction depends on the temperature and potential strength. It is very clear from Fig. 1 that the effect of the potential strength appears preciously in the vicinity of the transition temperature.

Bose-Einstein condensation occurs if the thermal wave length (De-Broglie wave length) is greater than the interparticle spacing. All of the above results are consistent with this condition. As the potential strength increases, the interparticle spacing increases and the probability of overlapping between the De-Broglie waves of the particle system decreases. This means that the condensate fraction decreases, as well as the transition temperature. Also, from pure quantum mechanics language, the repulsive potential may excite the system and the probability of excitation will increase as the temperature or potential strength increases.

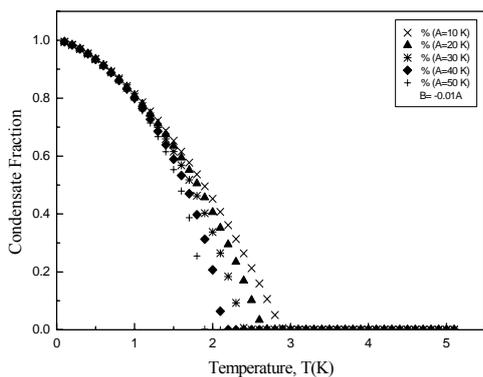


Fig. 1 The condensate fraction as a function of temperature  $T$  at  $B = -0.01 A$ ,  $r_1 = 1.5 \text{ \AA}$ ,  $r_2 = 3.0 \text{ \AA}$ ,  $N = 10^6$ , and density number  $n = 0.02165 \text{ (\AA}^{-3}\text{)}$ , and different values of the potential strength  $A$

TABLE I

THE TRANSITION TEMPERATURE  $T_c$  AT  $B = -0.01 A$ ,  $r_1 = 1.5 \text{ \AA}$ ,  $r_2 = 3.0 \text{ \AA}$ , TOTAL NUMBER OF PARTICLES IN THE SYSTEM  $N = 10^6$ ,  $n = 0.02165 \text{ (\AA}^{-3}\text{)}$ , AND DIFFERENT VALUES OF THE POTENTIAL STRENGTH  $A$

$A$ (K)	10.0	20.0	30.0	40.0	50.0
$T_c$ (K)	2.89	2.70	2.41	2.20	1.89

To investigate the role of the potential well, we repeat the previous calculations at the same values of the repulsive potential strength but at a different value of the potential depth of the well,  $B = -0.1 A$ . The results of the condensate fraction as a function of temperature are shown in Fig. 2 as well as the transition temperature in Table 2. It is very clear that as the attractive potential increases the condensate fraction and transition temperature increase. Fig. 2 indicates that the depth of the attractive potential plays a crucial role and is dominant over the repulsive potential. The transition temperature in this case is greater than that for an ideal  $^4\text{He}$  system. This means that the system will remain in the ground state at relatively high temperature. As the depth of the attractive potential decreases, the interparticle spacing and the probability of overlapping between the De-Broglie waves of the particles increase.

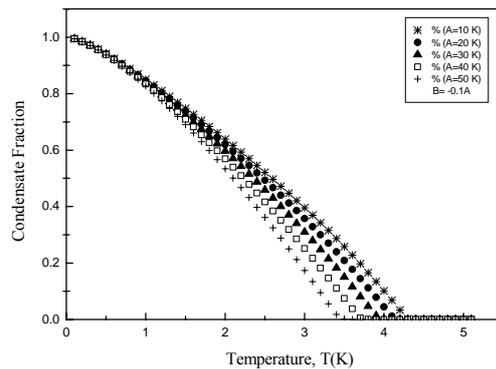


Fig. 2 The condensate fraction as a function of temperature  $T$  at  $B = -0.1 A$ ,  $r_1 = 1.5 \text{ \AA}$ ,  $r_2 = 3.0 \text{ \AA}$ ,  $N = 10^6$ ,  $n = 0.02165 \text{ (\AA}^{-3}\text{)}$ , and different values of the potential strength  $A$

TABLE II

THE TRANSITION TEMPERATURE  $T_c$  AT  $B = -0.1 A$ ,  $r_1 = 1.5 \text{ \AA}$ ,  $r_2 = 3.0 \text{ \AA}$ ,  $N = 10^6$ ,  $n = 0.02165 \text{ (\AA}^{-3}\text{)}$ , AND DIFFERENT VALUES OF POTENTIAL STRENGTH  $A$

$A$ (K)	10.0	20.0	30.0	40.0	50.0
$T_c$ (K)	4.30	4.11	3.91	3.69	3.51

We studied the effect of the repulsive and attractive potential strength on the condensate fraction. To complete our study, we should explore the role of the core radius of the repulsive part of the potential and the range of the attractive potential well.

Fig. 3 shows the condensate fraction of the system at different values of core radius,  $r_1$ , and Table 3 lists the corresponding values of the transition temperature. The behavior of the results resembles that in Fig. 1. The condensate fraction and the transition temperature decrease by increasing the strength of the repulsive part of the potential or increasing the core radius. By increasing the core radius of the repulsive potential the interparticle spacing increases.

Fig. 4 shows the results of condensate fraction at different values of attractive potential range,  $r_2$ , and the determined values of transition temperatures tabulated in Table 4. The condensate fraction and transition temperature increase by increasing the range of the attractive potential as expected. In a system of a deep potential well or a large range, the transition temperature of the system will be increase.

To make sure that the used value of  $N = 10^6$  satisfies the thermodynamic limit, the condensate fraction at different values of  $N$  is calculated as shown in Fig. 5. The results are close to each other which indicate that we are working almost in the thermodynamic limit.

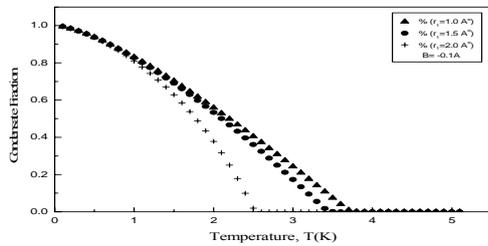


Fig. 3 The condensate fraction as a function of temperature  $T$  at  $A = 10.0\text{K}$ ,  $B = -0.1 A$ ,  $r_2 = 3.0 \text{ \AA}$ ,  $N = 10^6$ ,  $n = 0.02165 (\text{ \AA}^{-3})$ , and different values of the core radius of the repulsive part  $r_1$

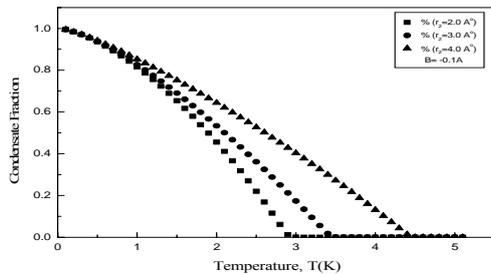


Fig. 4 The condensate fraction as a function of temperature  $T$  at  $A = 10.0 \text{ K}$ ,  $B = -0.1 A$ ,  $r_1 = 1.5 \text{ \AA}$ ,  $N = 10^6$ ,  $n = 0.02165 (\text{ \AA}^{-3})$ , and different values of the range of the attractive part  $r_2$

TABLE III

THE TRANSITION TEMPERATURE  $T_c$  AT  $A = 10.0 \text{ K}$ ,  $B = -0.1 A$ ,  $r_2 = 3.0 \text{ \AA}$ ,  $N = 10^6$ ,  $n = 0.02165 (\text{ \AA}^{-3})$ , AND DIFFERENT VALUES OF CORE RADIUS OF THE REPULSIVE PART  $r_1$

$r_1 (\text{ \AA})$	1.0	1.5	2.0
$T_c (\text{ K})$	3.71	3.5	2.61

Finally, the effect of the density number of particles on condensate fraction is investigated. Fig. 6 shows the results, and Table 5 list the corresponding transition temperature. The condensate fraction and transition temperature increase as the system become more condensed. In this case the interparticle spacing decreases and becomes shorter than the thermal wavelength. We conclude that in condensed systems the required temperature to hold the system in excited state is increased.

TABLE IV

THE TRANSITION TEMPERATURE  $T_c$  AT  $A = 10.0 \text{ K}$ ,  $B = -0.1 A$ ,  $r_1 = 1.5 \text{ \AA}$ ,  $N = 10^6$ ,  $n = 0.02165 (\text{ \AA}^{-3})$ , AND DIFFERENT VALUES OF THE RANGE OF THE ATTRACTIVE PART  $r_2$

$r_2 (\text{ \AA})$	2.00	3.00	4.00
$T_c (\text{ K})$	3.01	3.50	4.51

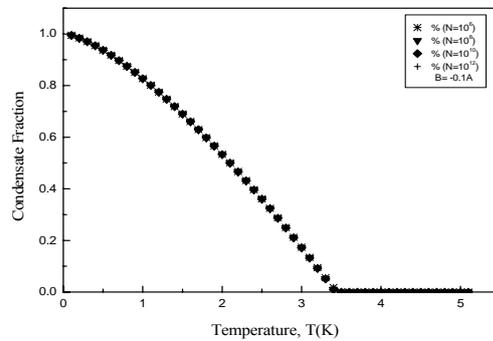


Fig. 5 The condensate fraction as a function of temperature  $T$  at  $A = 10.0 \text{ K}$ ,  $B = -0.1 A$ ,  $r_1 = 1.5 \text{ \AA}$ ,  $r_2 = 3.0 \text{ \AA}$ ,  $n = 0.02165 (\text{ \AA}^{-3})$ , and different values of the total number of particles  $N$

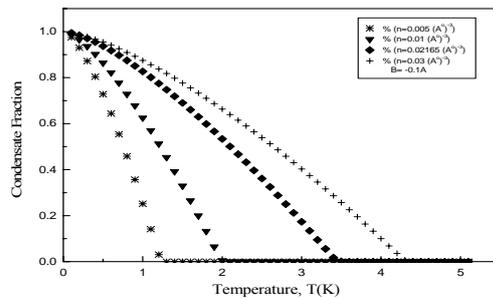


Fig. 6 The condensate fraction as a function of temperature  $T$  at  $A = 10.0 \text{ K}$ ,  $B = -0.1 A$ ,  $r_1 = 1.5 \text{ \AA}$ ,  $r_2 = 3.0 \text{ \AA}$ ,  $N = 10^6$ , and different values of the number density  $n$

TABLE V

THE TRANSITION TEMPERATURE  $T_c$  AT  $A = 10.0 \text{ K}$ ,  $B = -0.1 A$ ,  $r_1 = 1.5 \text{ \AA}$ ,  $r_2 = 3.0 \text{ \AA}$ ,  $N = 10^6$ , AND DIFFERENT VALUES OF NUMBER DENSITY  $n$

$n (\text{ \AA}^{-3})$	0.005	0.01	0.02165	0.03
$T_c (\text{ K})$	1.30	2.02	3.50	4.33

## IV. CONCLUSION

It was found that the condensate fraction and the transition temperature depend on the strength, the depth, the core radius of the repulsive part, and the range of the attractive part. As the potential strength or the core radius of the repulsive part increases, the condensate fraction and the transition temperature decrease. On the other hand, as the potential depth or the range of the attractive part increases, the condensation fraction and the transition temperature increase.

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