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Bifurcation analysis in a two-neuron system with different time delays

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Abstract—In this paper, we consider a two-neuron system with time-delayed connections between neurons. By analyzing the associated characteristic transcendental equation, its linear stability is investigated and Hopf bifurcation is demonstrated. Some explicit formulae for determining the stability and the direction of the Hopf bifurcation periodic solutions bifurcating from Hopf bifurcations are obtained by using the normal form theory and center manifold theory. Some numerical simulation results are given to support the theoretical predictions. Finally, main conclusions are given.

Keywords—Two-neuron system; Delay; Stability; Hopf bifurcation.

I. Introduction

In recent years, the dynamics (including stable, unstable, persistent and oscillatory behavior) of neural networks has become very popular since Marcus and Westervelt [13] proposed a neural network with delay in the 1980s. Great attention has been paid to the dynamics properties of the neural networks models which have significant physical background. Many excellent and interesting results have been obtained, for example, Gopalsamy and Leng [4] investigated the globally asymptotical stability of the following two simple neuron models with discrete or distributed delays:

$$\dot{x}(t) = -x(t) + a\tanh[x(t) - bx(t-\tau) - c], \tag{1}$$

$$\dot{x}(t) = -x(t) + a\tanh[x(t) - b\int_0^{+\infty} F(s)x(t-s)ds - c], \tag{2}$$

where a denotes the range of the continuous variable $x(.)$, while b can be considered as a measure of the inhibitory influence from the past history, c is a off-set constant, τ is the time delay, and $F(.)$ is a kernel function.

In 1999, Liao et al. [11,12] studied the stability of Eq.(2) with a weak kernel and a strong kernel, respectively. In 2001, Liao et al. [10] discussed the Hopf bifurcation and chaos of Eq.(1). In 1996, Gopalsamy and Leung [3] studied the Hopf bifurcation of the following neural network of two neurons constituting an activator inhibitor assembly modeled by the delay differential system

$$\begin{cases} \dot{x}(t) = -x(t) + a\tanh[c_1 y(t-\tau)], \\ \dot{y}(t) = -y(t) + a\tanh[-c_2 x(t-\tau)], \end{cases} \tag{3}$$

where a, c_1, c_2 and τ are positive constants, y denotes the activating potential of x, and x is the inhibiting potential.

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In 1998, Gopalsamy et al. [5] considered an analogue of model (3) containing continuously distributed delays in following form

$$\begin{cases} \dot{x}(t) = -x(t) + a\tanh[\int_{-\infty}^{t} k(t-s)y(s)ds], \\ \dot{y}(t) = -y(t) + a\tanh[\int_{-\infty}^{t} k(t-s)x(s)ds], \end{cases} \tag{4}$$

where a is a positive constant and the delay kernel k is assumed to satisfy the condition: $k : [0, +\infty) \to [0, +\infty)$, $k\int_0^{\infty} k(s)ds = 1; \int_0^{\infty} sk(s)ds < +\infty$.

In 1986 and 1987, Babcock and Westervelt [1, 2] studied the rich dynamics including under-damped ringing transients, stable and unstable limit cycles of the following two-neuron network model

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + a_1\tanh[x_2(t-\tau_1)], \\ \dot{x}_2(t) = -x_2(t) + a_2\tanh[x_1(t-\tau_2)], \end{cases} \tag{5}$$

In 1997, Olien and Belair [14] investigated stability and Hopf bifurcation of the following system with two delays

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + a_{11}f(x_1(t-\tau_1)) + a_{12}f(x_2(t-\tau_2)), \\ \dot{x}_2(t) = -x_2(t) + a_{21}f(x_1(t-\tau_1)) + a_{22}f(x_2(t-\tau_2)), \end{cases} \tag{6}$$

under the assumptions: $\tau_1 = \tau_2, a_{11} = a_{22} = 0$.

In 2001 and 2004, Liao et al. [8,9] investigated Local Hopf bifurcation of the following two-neuron system with distributed delays in the time domain and frequency domain, respectively.

$$\begin{cases} \dot{x}_1^*(t) = -x_1^*(t) + a_1^* f[(x_2^*(t) - b_2\int_0^{\infty} F(r)x_2^*(t-r)dr - c_1], \\ \dot{x}_2^*(t) = x_2^*(t) + a_2^* f[(x_1^*(t) - b_1\int_0^{\infty} F(r)x_1^*(t-r)dr - c_2], \end{cases} \tag{7}$$

Recently, Fenghua Tu et al. [16] investigated the local and global stability of the following two-neuron system with time-delayed connections between neurons:

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + a_1 g[x_2(t) - b_2 x_2(t-\tau)], \\ \dot{x}_2(t) = -x_2(t) + a_2 g[x_1(t) - b_1 x_1(t-\sigma)], \end{cases} \tag{8}$$

where "." denote the derivative with t, a_1, a_2, b_1 and b_2 are arbitrary real numbers, $x_1(t), x_2(t)$ represent the mean soma potential of the neuron while a_1 and a_2 correspond to the range of the continuous variable x_1 and x_2, respectively. The b_1 and b_2 denote the measure of the inhibitory influence of the past history. The term x_1 and x_2 in the argument of the f and g function denote local feedbacks. For more detail knowledge, one can see [16].

By using the following transformation

$$\begin{cases} y_1(t) = x_1(t) - b_1 x_1(t-\sigma), \\ y_2(t) = x_2(t) - b_2 x_2(t-\tau), \end{cases} \tag{9}$$

we can transform system (8) to

$$\begin{cases} \dot{y}_1(t) = -y_1(t) + a_1 g(y_2(t)) - a_1 b_1 g[y_2(t-\sigma)], \\ \dot{y}_2(t) = -y_2(t) + a_2 f(y_1(t)) - a_2 b_2 f[y_1(t-\tau)]. \end{cases} \quad (10)$$

In this paper, we will study the stability, local Hopf bifurcation for system (10). To the best of our knowledge, it is the first time to deal with the research of Hopf bifurcation for model (10) under the assumption $\sigma \neq \tau$.

We would like to point out that it is easy to study the local stability and Hopf bifurcation of system (10) under the assumption: $\sigma = \tau$ as many previous similar work. While in most cases, $\sigma \neq \tau$. Considering the factor, we investigate the model (10) with $\sigma \neq \tau$ as a complementarity.

The remainder of the paper is organized as follows. In Section 2, we investigate the stability of the equilibrium and the occurrence of local Hopf bifurcations. In Section 3, the direction and stability of the local Hopf bifurcation are established. In Section 4, numerical simulations are carried out to illustrate the validity of the main results. Some main conclusions are drawn in Section 5.

II. Stability of the equilibrium and local Hopf bifurcations

In this section, we shall study the stability of the equilibrium and the existence of local Hopf bifurcations.

Throughout the paper, we assume that the following condition

$$(H1) \quad f(0) = 0, g(0) = 0$$

holds. It is easy to see that system (10) has an equilibrium point $E_0(0,0)$.

The linearization of Eq. (10) at $(0,0)$ is

$$\begin{cases} \dot{y}_1(t) = -y_1(t) + a_1 g^{'}(0) y_2(t) - a_1 b_1 g^{'}(0) y_2(t-\sigma), \\ \dot{y}_2(t) = -y_2(t) + a_2 f^{'}(0) y_1(t) - a_2 b_2 f^{'}(0) y_1(t-\tau) \end{cases} \quad (11)$$

whose characteristic equation is

$$\lambda^2 + 2\lambda + m_1 + m_2 e^{-\lambda\sigma} + m_3 e^{-\lambda\tau} + m_4 e^{-\lambda(\sigma+\tau)} = 0, \quad (12)$$

where

$$\begin{aligned} m_1 &= 1 - a_1 a_2 f^{'}(0) g^{'}(0), \\ m_2 &= a_1 a_2 b_1 f^{'}(0) g^{'}(0), \\ m_3 &= a_1 a_2 b_2 f^{'}(0) g^{'}(0), \\ m_4 &= -a_1 a_2 b_1 b_2 f^{'}(0) g^{'}(0). \end{aligned}$$

In order to investigate the distribution of roots of the transcendental equation (12), the following Lemma is useful.

Lemma 2.1. [15] *For the transcendental equation*

$$\begin{aligned} & P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m}) \\ &= \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\ &+ \left[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}\right] e^{-\lambda\tau_1} + \cdots \\ &+ \left[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\right] e^{-\lambda\tau_m} = 0, \end{aligned}$$

as $(\tau_1, \tau_2, \tau_3, \cdots, \tau_m)$ vary, the sum of orders of the zeros of $P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m})$ in the open right half plane can change, and only a zero appears on or crosses the imaginary axis.

In the sequel, we consider three cases.

Case (a). $\sigma = \tau = 0$, (12) becomes

$$\lambda^2 + 2\lambda + m_1 + m_2 + m_3 + m_4 = 0. \quad (13)$$

A set of necessary and sufficient conditions that all roots of (12) have a negative real part is given in the following form:

$$(H2) \quad m_1 + m_2 + m_3 + m_4 > 0.$$

Then the equilibrium point $E_0(0,0)$ is locally asymptotically stable when the condition $(H2)$ holds.

Case (b). $\sigma = 0, \tau > 0$, (12) becomes

$$\lambda^2 + 2\lambda + m_1 + m_2 + (m_3 + m_4)e^{-\lambda\tau} = 0. \quad (14)$$

For $\omega > 0, i\omega$ be a root of (14), then it follows that

$$\begin{cases} (m_3 + m_4)\cos\omega\tau = \omega^2 - (m_1 + m_2), \\ (m_3 + m_4)\sin\omega\tau = 2\omega \end{cases} \quad (15)$$

which leads to

$$\omega^4 + [4 - 2(m_1 + m_2)]\omega^2 + (m_1 + m_2)^2 - (m_3 + m_4)^2 = 0. \quad (16)$$

It is easy to see that if the condition

$$(H3) \quad m_1 + m_2 < 2, \ |m_1 + m_2| > |m_3 + m_4|$$

holds, then Eq.(16) has no positive roots. Hence, all roots of (16) have negative real parts when $\tau \in [0, +\infty)$ under the conditions $(H2)$ and $(H3)$.

If $(H2)$ and

$$(H4) \quad m_1 + m_2 < 2, |m_1 + m_2| < |m_3 + m_4|$$

hold, then (15) has a unique positive root ω_0^2. Substituting ω_0^2 into (15), we obtain

$$\tau_n = \frac{1}{\omega_0}\Big\{\arcsin\frac{2\omega_0}{m_3 + m_4} + 2n\pi\Big\}, n = 0, 1, 2, \cdots. \quad (17)$$

If $(H2)$ and

$$(H5) \quad m_1 + m_2 > 2, \ |m_1 + m_2| > |m_3 + m_4|,$$
$$[4 - 2(m_1 + m_2)]^2 > 4[(m_1 + m_2)^2 - (m_3 + m_4)^2]$$

hold, then (16) has two positive roots ω_+^2 and ω_-^2. Substituting $\omega_\pm^2$ into (15), we obtain

$$\tau_k = \frac{1}{\omega_\pm}\Big\{\arcsin\frac{2\omega_\pm}{m_3 + m_4} + 2k\pi\Big\}, k = 0, 1, 2, \cdots. \quad (18)$$

Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be a root of (16) near $\tau = \tau_n$ and $\alpha(\tau_n) = 0, \ \omega(\tau_n) = \omega_0$. Due to functional differential equation theory, for every $\tau_n, n = 0, 1, 2, \cdots$, there exists $\varepsilon > 0$ such that $\lambda(\tau)$ is continuously differentiable in τ for $|\tau - \tau_n| < \varepsilon$. Substituting $\lambda(\tau)$ into (14) and taking derivative with respect to τ, we have

$$\left(\frac{d\lambda}{d\tau_2}\right)^{-1} = \frac{2e^{\lambda\tau}}{m_3 + m_4} + \frac{2e^{\lambda\tau}}{\lambda(m_3 + m_4)} - \frac{\tau}{\lambda} \quad (19)$$

which leads to

$$\begin{aligned}
&\left[\frac{d(Re\lambda(\tau))}{d\tau}\right]^{-1}_{\tau=\tau_n}\\
&= Re\Big\{\frac{2e^{\lambda\tau}}{m_3+m_4}\Big\}\Big|_{\tau=\tau_n} + Re\Big\{\frac{2e^{\lambda\tau}}{\lambda(m_3+m_4)}\Big\}\Big|_{\tau=\tau_n}\\
&= \frac{2\cos\omega_0\tau_n}{m_3+m_4} + \frac{2\sin\omega_0\tau_n}{(m_3+m_4)\omega_0}\\
&= \frac{2}{(m_3+m_4)^2}[\omega_0^2-(m_1+m_2)+2] > 0.
\end{aligned}$$

Noting that

$$sign\Big\{\frac{d(Re\lambda)}{d\tau}\Big\}\Big|_{\tau=\tau_n} = sign\Big\{Re\left(\frac{d\lambda}{d\tau}\right)\Big\}\Big|_{\tau=\tau_n} = 1,$$

we have

$$\frac{d(Re\lambda)}{d\tau}\Big|_{\tau=\tau_n} > 0.$$

Similarly, we can obtain

$$\frac{d(Re\lambda)}{d\tau}\Big|_{\tau=\tau_k^+} > 0,\ \frac{d(Re\lambda)}{d\tau}\Big|_{\tau=\tau_k^-} < 0.$$

According to above analysis and the Corollary 2.4 in Ruan and Wei [15], we have the following results.

Lemma 2.2. *For $\sigma = 0$, assume that $(H1)$ and $(H2)$ are satisfied. Then the following conclusions hold:*
(i) If $(H3)$ holds, then the equilibrium $E_0(0,0)$ of system (10) is asymptotically stable for all $\tau \geq 0$.
(ii) If $(H4)$ holds, then the equilibrium $E_0(0,0)$ of system (10) is asymptotically stable for $\tau < \tau_0$ and unstable for $\tau > \tau_0$. Furthermore, system (1.10) undergoes a Hopf bifurcation at the equilibrium $E_0(0,0)$ when $\tau = \tau_0$.
(iii) If $(H5)$ holds, then there is a positive integer m such that the equilibrium $E_0(0,0)$ is stable when $\tau \in [0,\tau_0^+) \cup (\tau_0^-,\tau_1^+)\cup\cdots\cup(\tau_{m-1}^-,\tau_m^+)$, and unstable when $\tau \in [\tau_0^+,\tau_0^-)\cup(\tau_1^+,\tau_1^-)\cup\cdots\cup(\tau_m^+,\tau_m^-)\cup(\tau_m^+,\infty)$. Furthermore, system (1.1) undergoes a Hopf bifurcation at the equilibrium $E_0(0,0)$ when $\tau = \tau_k^{\pm}, k = 0,1,2,\cdots$.

Case (c). $\sigma > 0, \tau > 0$. We consider Eq.(12) with τ in its stable interval. Regarding σ as a parameter. Without loss of generality, we consider system (10) under the assumptions $(H1),(H2)$ and $(H4)$. Let $i\omega(\omega > 0)$ be a root of (12), then we can obtain

$$k_1\omega^4 + k_2\omega^3 + k_3\omega^2 + k_4\omega + k_5 = 0, \tag{20}$$

where

$$\begin{aligned}
k_1 &= (m_2 + m_4\cos\omega\tau)^2 + (m_4\sin\omega\tau)^2,\\
k_2 &= 4m_4\sin\omega\tau(m_2+m_4\cos\omega\tau)\\
&\quad -4m_4\sin\omega\tau(m_2+m_4\cos\omega\tau),\\
k_3 &= 2m_4\sin\omega\tau[(m_2+m_4\cos\omega\tau)m_3\sin\omega\tau\\
&\quad -m_4^2\sin^2\omega\tau(m_1+m_3\cos\omega\tau)]\\
&\quad -2(m_2+m_4\cos\omega\tau)[m_3m_4\sin^2\omega\tau\\
&\quad +(m_2+m_4\cos\omega\tau)(m_1+m_3\cos\omega\tau)]\\
&\quad +4m_4^2\sin^2\omega\tau + 4(m_2+m_4\cos\omega\tau)^2,\\
k_4 &= -4m_4\sin\omega\tau[m_3m_4\sin^2\omega\tau(m_2+m_4\cos\omega\tau)\\
&\quad (m_1+m_3\cos\omega\tau)] - 4(m_2+m_4)\cos\omega\tau\\
&\quad [m_3\sin\omega\tau(m_2+m_4\cos\omega\tau)\\
&\quad -m_4\sin\omega\tau(m_1+m_3\cos\omega\tau)],\\
k_5 &= [m_3m_4\sin^2\omega\tau + (m_2+m_4\cos\omega\tau)(m_1+m_3\cos\omega\tau)]^2\\
&\quad +[m_3\sin\omega\tau(m_2+m_4\cos\omega\tau)\\
&\quad -m_4\sin\omega\tau(m_1+m_3\cos\omega\tau)]^2\\
&\quad -[(m_2+m_4\cos\omega\tau)^2 + (m_4\sin\omega\tau)^2]^2.
\end{aligned}$$

Denote

$$H(\omega) = k_1\omega^4 + k_2\omega^3 + k_3\omega^2 + k_4\omega + k_5. \tag{21}$$

Assume that

$$(H6)\ \ |m_1+m_3| < |m_2+m_4|.$$

It is easy to check that $H(0) < 0$ if (H6) holds and $\lim_{\omega\to+\infty} H(\omega) = +\infty$. We can obtain that (20) has finite positive roots $\omega_1,\omega_2,\cdots,\omega_n$. For every fixed $\omega_i, i = 1,2,3,\cdots,k$, there exists a sequence $\{\sigma_i^j | j = 1,2,3,\cdots\}$, such that (20) holds. Let

$$\sigma_0 = \min\{\sigma_i^j | i = 1,2,\cdots,k; j = 1,2,\cdots\}. \tag{22}$$

When $\sigma = \sigma_0$, Eq.(12) has a pair of purely imaginary roots $\pm i\omega^*$ for $\tau \in [0,\tau_0)$.

In the following, we assume that

$$(H7)\ \ \left[\frac{d(Re\lambda)}{d\sigma}\right]_{\lambda=i\omega^*} \neq 0.$$

Thus, by the general Hopf bifurcation theorem for FDEs in Hale [6], we have the following result on the stability and Hopf bifurcation in system (10).

Theorem 2.1. *For system (10), assume that $(H1),(H2),(H4),(H6)$ and $(H7)$ are satisfied, and $\tau \in [0,\tau_0)$, then the equilibrium $E_0(0,0)$ is asymptotically stable when $\sigma \in (0,\sigma_0)$, and system (1) undergoes a Hopf bifurcation at the equilibrium $E_0(0,0)$ when $\sigma = \sigma_0$.*

III. Direction and Stability of the Hopf Bifurcation

In the previous section, we obtained conditions for Hopf bifurcation to occur when $\sigma = \sigma_0$. In this section, we shall derive the explicit formulae for determining the direction, stability, and period of these periodic solutions bifurcating from the equilibrium $E_0(0,0)$ at these critical value of σ, by using techniques from normal form and center manifold theory [7]. Throughout this section, we always assume that system (10) undergoes Hopf bifurcation at the equilibrium $E_0(0,0)$ for $\sigma = \sigma_0$, and then $\pm i\omega^*$ is corresponding purely imaginary roots of the characteristic equation at the equilibrium $E_0(0,0)$.

Without loss of generality, we assume that $\tau^* < \sigma_0$, where $\tau^* \in (0,\tau_0)$. For convenience, let $\bar{y}_i(t) = y_i(\tau t)(i = 1,2)$ and $\sigma = \sigma_0 + \mu$, where σ_0 is defined by (2.14) and $\mu \in R$, drop the bar for the simplification of notations, then system (10) can be written as an FDE in $C = C([-1,0],R^2)$ as

$$\dot{y}(t) = L_\mu(y_t) + F(\mu, y_t), \tag{23}$$

where $y(t) = (y_1(t), y_2(t))^T \in C$ and $y_t(\theta) = y(t+\theta) = (y_1(t+\theta), y_2(t+\theta))^T \in C$, and $L_\mu : C \to R, F : R \times C \to R$ are given by

$$\begin{aligned} L_\mu\phi &= (\sigma_0+\mu)B\begin{pmatrix}\phi_1(0)\\ \phi_2(0)\end{pmatrix} \quad (24)\\ &\quad +(\sigma_0+\mu)C\begin{pmatrix}\phi_1\left(-\frac{\tau^*}{\sigma_0}\right)\\ \phi_1\left(-\frac{\tau^*}{\sigma_0}\right)\end{pmatrix}\\ &\quad +(\sigma_0+\mu)D\begin{pmatrix}\phi_1(-1)\\ \phi_2(-1)\end{pmatrix} \quad (25)\end{aligned}$$

and

$$F(\mu,\phi) = (\sigma_0+\mu)(f_1, f_2)^T, \quad (26)$$

respectively, where $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta))^T \in C$,

$$\begin{aligned} B &= \begin{pmatrix}-1 & a_1 g^{'}(0)\\ a_2 f^{'}(0) & -1\end{pmatrix}, \ C = \begin{pmatrix}0 & 0\\ -a_2 b_2 g^{'}(0) & 0\end{pmatrix},\\ D &= \begin{pmatrix}0 & -a_1 b_1 g^{'}(0)\\ 0 & 0\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned} f_1 &= l_1\phi_2(0)^2 + l_2\phi_2^3(0) + l_3\phi_2^2(-1) + l_4\phi_2^3(-1) + h.o.t.,\\ f_2 &= k_1\phi_1^2(0)^2 + k_2\phi_1^3(0) + k_3\phi_1^2\left(-\frac{\tau^*}{\sigma}\right) + k_4\phi_1^3\left(-\frac{\tau^*}{\sigma}\right)\\ &\quad +\text{h.o.t.},\end{aligned}$$

where

$$\begin{aligned} l_1 &= \frac{a_1 g^{''}(0)}{2!}, l_2 = \frac{a_1 g^{'''}(0)}{3!}, l_3 = -\frac{a_1 b_1 g^{''}(0)}{2!},\\ l_4 &= -\frac{a_1 b_1 g^{'''}(0)}{2!}, k_1 = \frac{a_2 f^{''}(0)}{2!}, k_2 = \frac{a_2 f^{'''}(0)}{3!},\\ k_3 &= -\frac{a_2 b_2 f^{''}(0)}{2!}, k_4 = -\frac{a_2 b_2 f^{'''}(0)}{2!}.\end{aligned}$$

From the discussion in Section 2, we know that if $\mu = 0$, then system (23) undergoes a Hopf bifurcation at the equilibrium $E_0(0,0)$ and the associated characteristic equation of system (23) has a pair of simple imaginary roots $\pm i\omega^*\sigma_0$.

By the representation theorem, there is a matrix function with bounded variation components $\eta(\theta,\mu), \theta \in [-1,0]$ such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta), \qquad for \quad \phi \in C. \quad (27)$$

In fact, we can choose

$$\eta(\theta,\mu) = \begin{cases}(\sigma_0+\mu)(B+C+D), & \theta = 0,\\ (\sigma_0+\mu)(C+D), & \theta \in \left[-\frac{\tau^*}{\sigma_0}, 0\right),\\ (\sigma_0+\mu)D, & \theta \in \left(-1, -\frac{\tau^*}{\sigma_0}\right),\\ 0, & \theta = -1.\end{cases} \quad (28)$$

For $\phi \in C([-1,0], R^2)$, define

$$A(\mu)\phi = \begin{cases}\frac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0,\\ \int_{-1}^{0} d\eta(s,\mu)\phi(s), & \theta = 0\end{cases} \quad (29)$$

and

$$R\phi = \begin{cases}0, & -1 \le \theta < 0,\\ F(\mu,\phi), & \theta = 0.\end{cases} \quad (30)$$

Then (23) is equivalent to the abstract differential equation

$$\dot{y_t} = A(\mu)y_t + R(\mu)y_t, \quad (31)$$

where $y_t(\theta) = y(t+\theta), \theta \in [-1,0]$.

For $\psi \in C([-1,0], (R^2)^*)$, define

$$A^*\psi(s) = \begin{cases}-\frac{d\psi(s)}{ds}, & s \in (0,1],\\ \int_{-1}^{0} d\eta^T(t,0)\psi(-t), & s = 0.\end{cases}$$

For $\phi \in C([-1,0], R^2)$ and $\psi \in C([0,1], (R^2)^*)$, define the bilinear form

$$< \psi, \phi >= \overline{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \psi(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi,$$

where $\eta(\theta) = \eta(\theta,0)$, the $A = A(0)$ and A^* are adjoint operators. By the discussions in the Section 2, we know that $\pm i\omega^*\sigma_0$ are eigenvalues of $A(0)$, and they are also eigenvalues of A^* corresponding to $i\omega^*\sigma_0$ and $-i\omega^*\sigma_0$ respectively. By direct computation, we can obtain

$$q(\theta) = (1,\alpha)^T e^{i\omega^*\sigma_0\theta}, q^*(s) = M(1,\alpha^*,)e^{i\omega^*\sigma_0 s}, M = \frac{1}{K},$$

where

$$\begin{aligned} \alpha &= \frac{a_2 f^{'}(0) - a_2 b_2 g^{'}(0) e^{-i\omega^*\tau^*}}{1 + i\omega^*},\\ \alpha^* &= \frac{1 - i\omega^*}{a_2 f^{'}(0) - a_2 b_2 g^{'}(0) e^{-i\omega^*\tau^*}},\\ K &= 1 + \bar{\alpha}\alpha^* + a_2 b_2 g^{'}(0)\tau^* e^{-i\omega^*\tau^*} + \sigma_0 a_1 b_1 g^{'}(0)\bar{\alpha^*} e^{i\omega^*\sigma_0}.\end{aligned}$$

Furthermore, $< q^*(s), q(\theta) >= 1$ and $< q^*(s), \bar{q}(\theta) >= 0$.

Next, we use the same notations as those in Hassard [7] and we first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let y_t be the solution of Eq. (23) when $\mu = 0$.

Define

$$z(t) =< q^*, u_t >, W(t,\theta) = y_t(\theta) - 2Re\{z(t)q(\theta)\}. \quad (32)$$

on the center manifold C_0, and we have

$$W(t,\theta) = W(z(t), \bar{z}(t), \theta), \quad (33)$$

where

$$W(z(t),\bar{z}(t),\theta) = W(z,\bar{z}) = W_{20}\frac{z^2}{2} + W_{11}z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + \cdots, \quad (34)$$

and z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$. Noting that W is also real if y_t is real, we consider only real solutions. For solutions $y_t \in C_0$ of (23),

$$\begin{aligned} \dot{z}(t) &= i\omega^*\sigma_0 z + \bar{q}^*(\theta)F(0, W(z,\bar{z},\theta) + 2Re\{zq(\theta)\}\\ &\stackrel{\text{def}}{=} i\omega^*\sigma_0 z + \bar{q}^*(0)F_0.\end{aligned}$$

That is

$$\dot{z}(t) = i\omega^*\sigma_0 z + g(z,\bar{z}),$$

where

$$g(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots.$$

Hence, we have

$$\begin{aligned}
&g(z,\bar{z}) = \bar{q}^*(0)F_0(z,\bar{z}) = F(0,y_t) = \\
&\bar{M}\sigma_0\left[l_1\alpha^2 + l_3\alpha^2 e^{-2i\omega^*\sigma_0} + \bar{\alpha}^*(k_1 + k_3 e^{-i\omega^*\tau^*})\right] z^2 \\
&+\bar{M}\sigma_0\left[2l_1|\alpha|^2 + l_3|\alpha|^2 + \bar{\alpha}^*(k_1+k_3)\right] z\bar{z} \\
&+\bar{M}\sigma_0\left[l_1\bar{\alpha}^2 + l_3\bar{\alpha}^2 e^{2i\omega^*\sigma_0} + \bar{\alpha}^*(k_1 + k_3 e^{2i\omega^*\tau^*})\right]\bar{z}^2 \\
&+\bar{M}\sigma_0\Bigg\{l_1\left[W_{20}^{(2)}(0)\bar{\alpha} + 2W_{11}^{(2)}(0)\alpha\right] + 3l_3\alpha^2\bar{\alpha} \\
&+l_3\left[W_{20}^{(2)}(-1)\bar{\alpha}e^{i\omega^*\sigma_0} + 2W_{11}^{(2)}(-1)\alpha e^{-i\omega^*\sigma_0}\right] \\
&+3l_4\alpha^2 e^{-i\omega^*\sigma_0} + \bar{\alpha}^*\left[k_1\left(W_{20}^{(1)}(0)\bar{\alpha} + 2W_{11}^{(1)}(0)\right)\right. \\
&+3k_2 + k_3\left(W_{20}^{(1)}(-\frac{\tau^*}{\sigma_0})e^{i\omega^*\tau^*}\bar{\alpha} + 2W_{11}^{(1)}(-\frac{\tau^*}{\sigma_0})\right. \\
&\left.\left.e^{-i\omega^*\tau^*}\bar{\alpha}\right) + 3k_4 e^{-i\omega^*\tau^*}\bar{\alpha}\right]\Bigg\} z^2\bar{z} + \text{h.o.t.}.
\end{aligned}$$

And we obtain

$$\begin{aligned}
g_{20} &= 2\bar{M}\sigma_0\left[l_1\alpha^2 + l_3\alpha^2 e^{-2i\omega^*\sigma_0} + \bar{\alpha}^*(k_1 + k_3 e^{-i\omega^*\tau^*})\right], \\
g_{11} &= \bar{M}\sigma_0\left[2l_1|\alpha|^2 + 2l_3|\alpha|^2 + \bar{\alpha}^*(2k_1 + 2k_3)\right], \\
g_{02} &= 2\bar{M}\sigma_0\left[2l_1|\alpha|^2 + l_3|\alpha|^2 + \bar{\alpha}^*(k_1+k_3)\right], \\
g_{21} &= 2\bar{M}\sigma_0\Bigg\{l_1\left[W_{20}^{(2)}(0)\bar{\alpha} + 2W_{11}^{(2)}(0)\alpha\right] + 3l_3\alpha^2\bar{\alpha} \\
&\quad +l_3\left[W_{20}^{(2)}(-1)\bar{\alpha}e^{i\omega^*\sigma_0} + 2W_{11}^{(2)}(-1)\alpha e^{-i\omega^*\sigma_0}\right] \\
&\quad +3l_4\alpha^2 e^{-i\omega^*\sigma_0} + \bar{\alpha}^*\left[k_1\left(W_{20}^{(1)}(0)\bar{\alpha} + 2W_{11}^{(1)}(0)\right)\right. \\
&\quad +3k_2 + k_3\left(W_{20}^{(1)}(-\frac{\tau^*}{\sigma_0})e^{i\omega^*\tau^*}\bar{\alpha} + 2W_{11}^{(1)}(-\frac{\tau^*}{\sigma_0})\right. \\
&\quad \left.\left.e^{-i\omega^*\tau^*}\bar{\alpha}\right) + 3k_4 e^{-i\omega^*\tau^*}\bar{\alpha}\right]\Bigg\}.
\end{aligned}$$

For unknown $W_{20}^{(i)}(\theta), W_{11}^{(i)}(\theta), (i=1,2)$ in g_{21}, we still need to compute them.
Form (30), (31), we have

$$\begin{aligned}
W' &= \begin{cases} AW - 2Re\{\bar{q}^*(0)Fq(\theta)\}, & -1 \le \theta < 0, \\ AW - 2Re\{\bar{q}^*(0)Fq(\theta)\} + F, & \theta = 0 \end{cases} \\
&= AW + H(z,\bar{z},\theta), \qquad (35)
\end{aligned}$$

where

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots. \quad (36)$$

Comparing the coefficients, we obtain

$$(AW - 2i\sigma_0\omega^*)W_{20} = -H_{20}(\theta), \quad (37)$$

$$AW_{11}(\theta) = -H_{11}(\theta), \quad (38)$$

$$\cdots\cdots.$$

And we know that for $\theta \in [-1, 0)$,

$$\begin{aligned}
H(z,\bar{z},\theta) &= -\bar{q}^*(0)f_0 q(\theta) - q^*(0)\bar{f}_0\bar{q}(\theta) \\
&= -g(z,\bar{z})q(\theta) - \bar{g}(z,\bar{z})\bar{q}(\theta). \qquad (39)
\end{aligned}$$

Comparing the coefficients of (38) with (35) gives that

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta). \quad (40)$$

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \quad (41)$$

From (36),(39) and the definition of A , we get

$$\dot{W}_{20}(\theta) = 2i\omega^*\sigma_0 W_{20}(\theta) + g_{20}q(\theta) + \bar{g_{02}}\bar{q}(\theta). \quad (42)$$

Noting that $q(\theta) = q(0)e^{i\omega^*\sigma_0\theta}$, we have

$$W_{20}(\theta) = \frac{ig_{20}}{\omega^*\sigma_0}q(0)e^{i\omega^*\sigma_0\theta} + \frac{i\bar{g}_{02}}{3\omega^*\sigma_0}\bar{q}(0)e^{-i\omega^*\sigma_0\theta} + E_1 e^{2i\omega^*\sigma_0\theta}, \quad (43)$$

where $E_1 = (E_1^{(1)}, E_1^{(2)}) \in R^2$ is a constant vector.
Similarly, from (37), (40) and the definition of A, we have

$$\dot{W}_{11}(\theta) = g_{11}q(\theta) + \bar{g_{11}}\bar{q}(\theta), \quad (44)$$

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega^*\sigma_0}q(0)e^{i\omega^*\sigma_0\theta} + \frac{i\bar{g}_{11}}{\omega^*\sigma_0}\bar{q}(0)e^{-i\omega^*\sigma_0\theta} + E_2. \quad (45)$$

where $E_2 = (E_2^{(1)}, E_2^{(2)}) \in R^2$ is a constant vector

In what follows, we shall seek appropriate E_1,E_2 in (42), (43), respectively. It follows from the definition of A and (39), (40) that

$$\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega^*\sigma_0 W_{20}(0) - H_{20}(0) \quad (46)$$

and

$$\int_{-1}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0), \quad (47)$$

where $\eta(\theta) = \eta(0,\theta)$.
From (36), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g_{02}}\bar{q}(0) + 2\sigma_0(H_1, H_2)^T, \quad (48)$$

$$H_{11}(0) = -g_{11}q(0) - \bar{g_{11}}(0)\bar{q}(0) + 2\sigma_0(P_1, P_2)^T, \quad (49)$$

where

$$\begin{aligned}
H_1 &= l_1\alpha^2 + l_3\alpha^2 e^{-2i\omega^*\sigma_0}, \\
H_2 &= k_1 + k_3 e^{-i\omega^*\tau^*}, \\
P_1 &= l_1|\alpha|^2 + l_3|\alpha|^2, \\
P_2 &= k_1 + k_3.
\end{aligned}$$

Noting that

$$\left(i\omega^*\sigma_0 I - \int_{-1}^{0} e^{i\omega^*\sigma_0\theta}d\eta(\theta)\right)q(0) = 0,$$

$$\left(-i\omega^*\sigma_0 I - \int_{-1}^{0} e^{-i\omega^*\sigma_0\theta}d\eta(\theta)\right)\bar{q}(0) = 0$$

and substituting (42) and (46) into (45), we have

$$\left(2i\omega^*\sigma_0 I - \int_{-1}^{0} e^{2i\omega^*\sigma_0\theta}d\eta(\theta)\right)E_1 = 2\sigma_0(H_1, H_2)^T.$$

That is

$$\begin{pmatrix} 2i\omega^* - 1 & a_1 b_1 g'(0)e^{-2i\omega^*\sigma_0} - a_1 g'(0) \\ a_2 b_2 g'(0)e^{-2i\omega^*\sigma_0} - a_2 f'(0) & 2i\omega^* + 1 \end{pmatrix}$$

$\times E_1 = 2(H_1, H_2)^T$. It follows that

$$E_1^{(1)} = \frac{\Delta_{11}}{\Delta_1},\ E_1^{(2)} = \frac{\Delta_{12}}{\Delta_1}, \tag{50}$$

where $\Delta_1 = \det$

$$\begin{pmatrix} 2i\omega^* - 1 & a_1b_1g^{'}(0)e^{-2i\omega^*\sigma_0} - a_1g^{'}(0) \\ a_2b_2g^{'}(0)e^{-2i\omega^*\sigma_0} - a_2f^{'}(0) & 2i\omega^* + 1 \end{pmatrix},$$

$$\begin{aligned} \Delta_{11} &= 2\det\begin{pmatrix} H_1 & a_1b_1g^{'}(0)e^{-2i\omega^*\sigma_0} - a_1g^{'}(0) \\ H_2 & 2i\omega^* + 1 \end{pmatrix}, \\ \Delta_{12} &= 2\det\begin{pmatrix} 2i\omega^* - 1 & H_1 \\ a_2b_2g^{'}(0)e^{-2i\omega^*\sigma_0} - a_2f^{'}(0) & H_2 \end{pmatrix}. \end{aligned}$$

Similarly, substituting (43) and (48) into (46), we have

$$\left(\int_{-1}^{0} d\eta(\theta)\right) E_2 = 2\sigma_0(P_1, P_2)^T.$$

That is

$$\begin{pmatrix} -1 & a_1g^{'}(0) - a_1b_1g^{'}(0) \\ a_2f^{'}(0) - a_2b_2g^{'}(0) & -1 \end{pmatrix} E_2$$

$= 2(-P_1, -P_2)^T$. It follows that

$$E_2^{(1)} = \frac{\Delta_{21}}{\Delta_2},\ E_2^{(2)} = \frac{\Delta_{22}}{\Delta_2}, \tag{51}$$

where

$$\begin{aligned} \Delta_2 &= \det\begin{pmatrix} -1 & a_1g^{'}(0) - a_1b_1g^{'}(0) \\ a_2f^{'}(0) - a_2b_2g^{'}(0) & -1 \end{pmatrix} \\ \Delta_{21} &= 2\det\begin{pmatrix} -P_1 & a_1g^{'}(0) - a_1b_1g^{'}(0) \\ -P_2 & -1 \end{pmatrix}, \\ \Delta_{22} &= 2\det\begin{pmatrix} -1 & -P_1 \\ a_2f^{'}(0) - a_2b_2g^{'}(0) & -P_2 \end{pmatrix}. \end{aligned}$$

From (42),(44),(49),(50), we can calculate g_{21} and derive the following values:

$$\begin{aligned} c_1(0) &= \frac{i}{2\omega^*\sigma_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{Re\{c_1(0)\}}{Re\{\lambda^{'}(\sigma_0)\}}, \\ \beta_2 &= 2Re(c_1(0)), \\ T_2 &= -\frac{Im\{c_1(0)\} + \mu_2 Im\{\lambda^{'}(\sigma_0)\}}{\omega^*\sigma_0}. \end{aligned}$$

These formulaes give a description of the Hopf bifurcation periodic solutions of (23) at $\sigma = \sigma_0$ on the center manifold. From the discussion above, we have the following result:

Theorem 3.1. *The periodic solution is supercritical (subcritical) if $\mu_2 > 0$ ($\mu_2 < 0$); the bifurcating periodic solutions are orbitally asymptotically stable with asymptotical phase (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$); the periodic of the bifurcating periodic solutions increase (decrease) if $T_2 > 0$ ($T_2 < 0$).*

IV. Numerical Examples

In this section, we present some numerical results of system (10) to verify the analytical predictions obtained in the previous section. From Section 3, we may determine the direction of a Hopf bifurcation and the stability of the bifurcation periodic solutions. Let us consider the following system:

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + 0.75\tanh[x_2(t) - 0.4x_2(t-\tau)], \\ \dot{x}_2(t) = -x_2(t) + 2\tanh[x_1(t) - 0.2x_1(t-\sigma)]. \end{cases} \tag{52}$$

By using the transformation

$$\begin{cases} y_1(t) = x_1(t) - 0.2x_1(t-\sigma), \\ y_2(t) = x_2(t) - 0.4x_2(t-\tau), \end{cases} \tag{53}$$

we can transform system (51) to

$$\begin{cases} \dot{y}_1(t) = -y_1(t) + 0.75\tanh(y_2(t)) - 0.75\times 0.2\tanh(y_2(t-\sigma)], \\ \dot{y}_2(t) = -y_2(t) + 2\tanh(y_1(t)) - 2\times 0.4\tanh(y_1(t-\tau)), \end{cases} \tag{54}$$

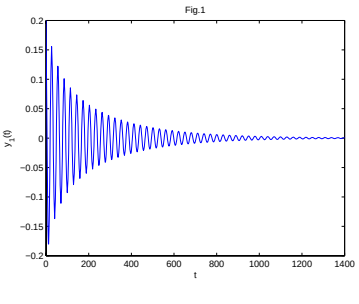
which has an equilibrium $E_0(0,0)$. When $\sigma = 0$, then we can easily obtain that $(H2)$ and $(H4)$ are satisfied. Take $n = 0$ for example, by some computation by means of Matlab 7.0, we get $\tau_0 \approx 4.9782$. From Lemma 2.2, we know that the transversal condition is satisfied. Thus the equilibrium $E_0(0,0)$ is asymptotically stable for $\tau < \tau_0 \approx 4.9782$ and unstable for $\tau > \tau_0 \approx 4.9782$ which is shown in Fig.1-3. When $\tau = \tau_0 \approx 4.9782$, Eq.(53) undergoes a Hopf bifurcation at the equilibrium $E_0(0,0)$, i.e., an small amplitude periodic solution occurs around $E_0(0,0)$ when $\sigma = 0$ and τ is close to $\tau_0 \approx 4.9782$ which is shown in Fig.4-6.

Let $\tau = 3.6 \in (0, 4.9782)$ and choose σ as a parameter. We have $\sigma_0 \approx 0.6488$. Then the equilibrium is asymptotically when $\sigma \in [0, \sigma_0)$. The Hopf bifurcation value of Eq.(51) is $\sigma_0 \approx 0.6488$. By the algorithm derived in Section 3, we can obtain

$$\lambda^{'}(\sigma_0) \approx 0.7032 - 5.0014i, c_1(0) \approx -4.1013 - 2.3121i,$$

$$\mu_2 \approx 0.3422, \beta_2 \approx -5.2451, T_2 \approx 9.2304.$$

Furthermore, it follows that $\mu_2 > 0$ and $\beta_2 < 0$. Thus, the equilibrium $E_0(0,0)$ is stable when $\sigma < \sigma_0$ as is illustrated by the computer simulations (see Fig.7-9). When σ passes through the critical value σ_0, the equilibrium $E_0(0,0)$ loses its stability and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcations from the equilibrium $E_0(0,0)$. Since $\mu_2 > 0$ and $\beta_2 < 0$, the direction of the Hopf bifurcation is $\sigma > \sigma_0$, and these bifurcating periodic solutions from $E_0(0,0)$ at σ_0 are stable, which are depicted in Fig.10-12.



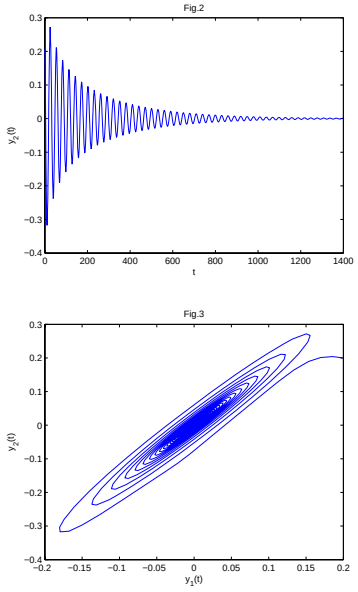


Fig.1-3. Behavior and phase portrait of system (53) with $\sigma = 0, \tau = 4.9 < \tau_0 \approx 4.9782$. The equilibrium $E_0(0,0)$ is asymptotically stable. The initial value is (0.2,0.2).

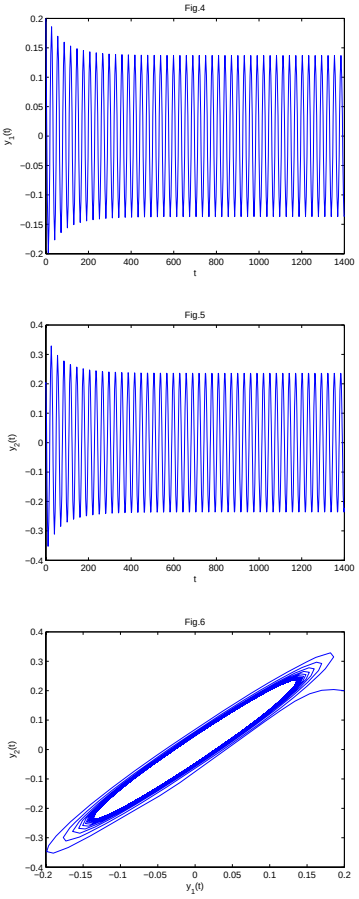


Fig.4-6. Behavior and phase portrait of system (53) with $\sigma = 0, \tau = 5.2 > \tau_0 \approx 4.9782$. Hopf bifurcation occurs from the equilibrium $E_0(0,0)$. The initial value is (0.2,0.2).

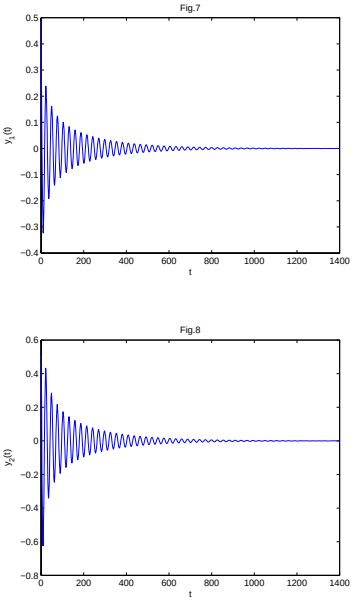
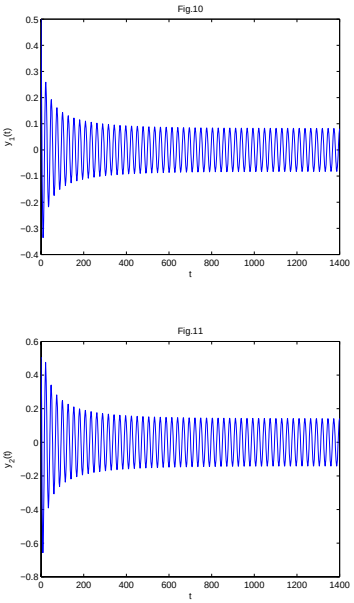


Fig.7-9. Behavior and phase portrait of system (53) with $\tau = 3.6, \sigma = 0.5 < \sigma_0 \approx 0.6488$. The equilibrium $E_0(0,0)$ is asymptotically stable. The initial value is (0.2,0.2).



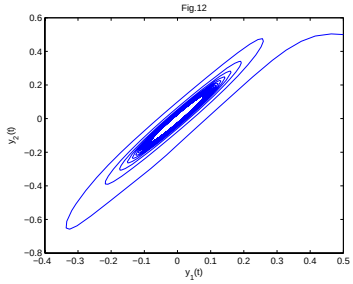


Fig.10-12. Behavior and phase portrait of system (53) with $\tau = 3.6, \sigma = 0.8 > \sigma_0 \approx 0.6488$. Hopf bifurcation occurs from the equilibrium $E_0(0,0)$. The initial value is (0.2,0.2).

V. Conclusions

In this paper, we have investigated local stability of the equilibrium $E_0(0,0)$ and local Hopf bifurcation of a two-neuron system with time-delayed connections between neurons. We have showed that if the conditions $(H1), (H2), (H4), (H6)$ and $(H7)$ are satisfied, and $\tau \in [0, \tau_0)$, then the equilibrium $E_0(0,0)$ is asymptotically stable when $\sigma \in (0, \sigma_0)$, as the delay σ increases, the equilibrium $E_0(0,0)$ loses its stability and a sequence of Hopf bifurcations occur at the equilibrium $E_0(0,0)$, i.e., a family of periodic orbits bifurcates from the the equilibrium $E_0(0,0)$. At last, the direction of Hopf bifurcation and the stability of the bifurcating periodic orbits are discussed by applying the normal form theory and the center manifold theorem. Some numerical simulations verifying our theoretical results is also carried out.

References

[1] K. L. Babcock and R. M. Westervelt, *Dynamics of simple electric neural networks*. Physica D 28 (1987) 305-316.
[2] K. L. Babcock and R. M. Westervelt, *Dynamics of simple electric neural networks with added intertial*. Physica D 23 (1986) 464-469.
[3] K. Gopalsamy and I. Leung, *Delay-induced periodicity in a neural network of excitation and inhibition*. Physica D 89 (1996) 395-426.
[4] K. Gopalsamy and I. Leung, *Converge under dynamical thresholds with delays*. IEEE Trans. Neural Netw. 8 (1997) 341-348.
[5] K. Gopalsamy, I. Leung and P. Liu, *Global Hopf bifurcation in a neural netlet*. Appl. Math. Comput. 94 (1998) 171-192.
[6] J. Hale, *Theory of Functional Differential Equation*. Springer-Verlag, 1977.
[7] B. Hassard, D. Kazarino and Y. Wan, *Theory and applications of Hopf bifurcation*. Cambridge: Cambridge University Press, 1981.
[8] X. F. Liao, S. W. Li and G. R. Chen, *Bifurcation analysis on a two-neuron system with distributed dedays in the domain*. Neural Netw. 17 (2004) 545-561.
[9] X. F. Liao, K. W. Wong and Z. F. Wu, *Bifurcation analysis on a two-neuron system with distributed dedays*. Physica D 149 (2001) 123-141.
[10] X. F. Liao, K. W. Wong, C. S. Leung and Z. F. Wu, *Hopf bifurcation and chaos in a single delayed neuron equation with nonmonotonic activation function*. Chaos, Solitons and Fractals 12 (2001) 1535-1547.
[11] X. F. Liao, Z. F. Wu and J. B. Yu, *Hopf bifurcation analysis of a neural systems with a continuously distributed delay*. Proceeding of the International Symposium on Signal Processing and Intelligent System. Guangzhou, China, (1999) 102-106.
[12] X. F. Liao, Z. F. Wu and J. B. Yu, *Stability switches and bifurcation analysis of a neural network with continuous delay*. IEEE Trans. syst. man cyber. A 29 (1999) 692-696.
[13] C. M. Marcus and R. M. Westervelt, *Stability of analog networks with delay*. Phys. Rev. A 39 (1989) 347-359.
[14] L. Olien and J. Belair, *Bifurcation, stability and monotonicity properties of a delayed neural model*. Physica D 102 (1997) 349-363.
[15] S. Ruan and J. Wei, *On the zero of some transcendential functions with applications to stability of delay differential equations with two delays*. Dyn. Contin. Discrete Impuls. Syst. Ser. A 10 (2003) 863-874.
[16] F. H. Tu, X. F. Liao and W. Zhang, *Delay-dependent asymptotic stability of a two-neuron system with different time delays*. Chaos, Solitons and Frctals 28 (2006) 437-447.

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