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Banach lattices with weak Dunford-Pettis property

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Abstract—We introduce and study the class of weak almost Dunford-Pettis operators. As an application, we characterize Banach lattices with the weak Dunford-Pettis property. Also, we establish some sufficient conditions for which each weak almost Dunford-Pettis operator is weak Dunford-Pettis. Finally, we derive some interesting results.

Keywords—eak almost Dunford-Pettis operator, almost Dunford-Pettis operator, weak Dunford-Pettis operator.eak almost Dunford-Pettis operator, almost Dunford-Pettis operator, weak Dunford-Pettis operator.W

I. Introduction and notation

As many Banach spaces do not have the Dunford-Pettis property, a weak notion is introduced, called the weak Dunford-Pettis property. A Banach space (respectively, Banach lattice) E has the Dunford-Pettis (respectively, weak Dunford-Pettis) property if every weakly compact operator defined on E (and taking their values in a Banach space F) is Dunford-Pettis (respectively, almost Dunford-Pettis, that is, the sequence $(\|T(x_n)\|)$ converges to 0 for every weakly null sequence (x_n) consisting of pairwise disjoint elements in E [5]). It is obvious that if E has the Dunford-Pettis property, then it has the weak Dunford-Pettis property.

On the other hand, whenever Aliprantis-Burkinshaw [1] and Kalton-Saab [4] studied the domination property of Dunford-Pettis operators, they used the class of weak Dunford-Pettis operators which satisfies the domination property [4]. Let us recall from [2] that an operator T from a Banach space X into another Y is called *weak Dunford-Pettis* if the sequence $(f_n(T(x_n)))$ converges to 0 whenever (x_n) converges weakly to 0 in X and (f_n) converges weakly to 0 in Y. Alternatively, T is weak Dunford-Pettis if T maps relatively weakly compact sets of X into Dunford-Pettis sets of Y (see Theorem 5.99 of [2]). A norm bounded subset A of a Banach lattice E is said to be *Dunford-Pettis set* if every weakly null sequence (f_n) of E' converges uniformly to zero on the set A, that is, $\sup_{x\in\mathbf{A}}|f_n(x)|\to 0$ (see Theorem 5.98 of [2]).

In [3], we introduced a new class of sets we call almost Dunford-Pettis set. A norm bounded subset A of a Banach lattice E is said to be *almost Dunford-Pettis set* if every disjoint weakly null sequence (f_n) of E' converges uniformly to zero on the set A, that is, $\sup_{x\in\mathbf{A}}|f_n(x)|\to 0$.

As weak Dunford-Pettis operators, we introduce a new class of operators that we call *weak almost Dunford-Pettis* operator. An operator T from a Banach space X into a Banach lattice F is said to be *weak almost Dunford-Pettis* if T maps relatively weakly compact sets of X into almost Dunford-Pettis sets of F. The latter class of operators differs from that of weak Dunford-Pettis operators. In fact, the first one is defined between Banach spaces while the second one is defined from a Banach space into a Banach lattice.

On the other hand, since each Dunford-Pettis set in a Banach lattice is almost Dunford-Pettis, then the class of weak almost Dunford-Pettis operators contains strictly that of weak Dunford-Pettis operators, that is, every weak Dunford-Pettis operator is weak almost Dunford-Pettis. But a weak almost Dunford-Pettis operator is not necessary weak Dunford-Pettis. In fact, for Wnuk (see [5], Example 1, p. 231)), the Lorentz space $\wedge(\omega,1)$ has the weak Dunford-Pettis property but does not have the Dunford-Pettis property, and then its identity operator is weak almost Dunford-Pettis (because each relatively weakly compact set in a Banach lattice has the weak Dunford-Pettis property is an almost Dunford-Pettis set, see Theorem 2.8 of [3]), but it is not weak Dunford-Pettis.

The objective of this paper is to study the class of weak almost Dunford-Pettis operators. Also, we derive the following interesting consequences: some characterizations of this class of operators, some characterizations of the weak Dunford-Pettis property, the coincidence of this class of operators with that of weak Dunford-Pettis operators, the domination property of this class of operators and the duality property.

To state our results, we need to fix some notation and recall some definitions. A Banach lattice is a Banach space $(E,\|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x,y\in E$ such that $|x|\le|y|$, we have $\|x\|\le\|y\|$. Note that if E is a Banach lattice, its topological dual E', endowed with the dual norm and the dual order, is also a Banach lattice. A norm $\|\cdot\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha\downarrow 0$ in E, (x_α) converges to 0 for the norm $\|\cdot\|$ where the notation $x_\alpha\downarrow 0$ means that (x_α) is decreasing, its infimum exists and $\inf(x_\alpha)=0$.

A linear mapping T from a vector lattice E into a vector lattice F is called a lattice homomorphism, if $x\wedge y=0$ in E implies $T(x)\wedge T(y)=0$ in F. An operator $T:E\longrightarrow F$ between two Banach lattices is a bounded linear mapping. It is positive if $T(x)\ge 0$ in F whenever $x\ge 0$ in E. If $T:E\longrightarrow F$ is a positive operator between two Banach lattices, then its adjoint $T':F'\longrightarrow E'$, defined by $T'(f)(x)=f(T(x))$ for each $f\in F'$ and for each $x\in E$, is also positive. We refer the reader to [2] for unexplained terminologies on Banach lattice theory and positive operators.

II. Main results

Recall from [5] that an operator from a Banach lattice E into a Banach space X is said to be almost Dunford-Pettis if the sequence $(\|T(x_n)\|)$ converges to 0 for every weakly null sequence (x_n) consisting of pairwise disjoint elements in E.

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The following result gives a characterizations of weak almost Dunford-Pettis operators from a Banach space into a Banach lattice in term of weakly compact operators and the adjoint of almost Dunford-Pettis operators.

Theorem 2.1: For an operator T from a Banach space X into a Banach lattice F, the following statements are equivalent:

1) T is weak almost Dunford-Pettis operator.
2) If S is a weakly compact operator from an arbitrary Banach space Z into X, then the adjoint of the operator product $T \circ S$ is almost Dunford-Pettis.
3) If S is a weakly compact operator from ℓ^1 into X, then the adjoint of the operator product $T \circ S$ is almost Dunford-Pettis.
4) For all weakly null sequence $(x_n)_n \subset X$, and for all disjoint weakly null sequence $(f_n)_n \subset F'$ it follows that $f_n(T(x_n)) \to 0$.

Proof: (1) $\Rightarrow$ (2) Let (f_n) be a disjoint weakly null sequence in F', we have to prove that $((T \circ S)'(f_n))$ converges to 0 for the norm of Z'. If not, then there exist a sequence (z_n) in the closed unit ball B_Z of Z, a subsequence of $((T \circ S)'(f_n))$ (which we shall denote by $((T \circ S)'(f_n))$ again), and some $\varepsilon > 0$ satisfying $|f_n(T(S(z_n)))| > \varepsilon$ for all n. Since S is weakly compact, the set $A = \{S(z_1), S(z_2), ...\}$ is relatively weakly compact subset of E, and then the set $T(A)$ is an almost Dunford-Pettis (because T carries weakly relatively compact sets of X to almost Dunford-Pettis sets of F). Hence we obtain

$$|f_n(T(S(z_n)))| \leq \sup_{x \in T(A)} |f_n(x)| \to 0.$$

Then $|f_n(T(S(z_n)))| \to 0$, which is impossible with $|f_n(T \circ S(x_n))| > \varepsilon$ for all n. Thus, the sequence $((T \circ S)'(f_n))$ converges to 0 for the norm of Z', and so the adjoint $(T \circ S)'$ is almost Dunford-Pettis.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (4) Let (f_n) be a disjoint weakly null sequence in F', and let (x_n) be a weakly null sequence in X. Consider the operator $S : l^1 \to X$ defined by
$S((\lambda_i)_{i=1}^{\infty}) = \sum_{i=1}^{\infty} \lambda_i x_i$ for each $(\lambda_i)_{i=1}^{\infty} \in l^1$.

Then S is weakly compact (Theorem 5.26 of [2]), and so by our hypothesis $(T \circ S)' = S' \circ T'$ is an almost Dunford-Pettis operator. Thus $\|(T \circ S)'(f_n)\| \to 0$ and the desired conclusion follows from the inequality

$$\begin{aligned} |f_n(T(x_n))| &= |f_n(T(S(e_n)))| \\ &\leq \sup_{(\lambda_i) \in B_{l^1}} |f_n(T(S((\lambda_i)_{i=1}^{\infty})))| \\ &= \|(T \circ S)'(f_n)\| \end{aligned}$$

for each n, where $(e_i)_{i=1}^{\infty}$ is the canonical basis of l^1.

(4) $\Rightarrow$ (1) Let W be a relatively weakly compact subset of X, and let (f_n) be a disjoint weakly null sequence in F'. If (f_n) does not converge uniformly to zero on $T(W)$, then there exist a sequence (x_n) of W, a subsequence of (f_n) (which we shall denote by (f_n) again), and some $\varepsilon > 0$ satisfying $|f_n(T(x_n))| > \varepsilon$ for all n.

Since W is weakly compact, we can assume that $x_n \to x$ weakly in X. Then $T(x_n) \to T(x)$ weakly in F and so, by our hypothesis, we have $0 < \varepsilon < |f_n(T(x_n))| \leq |f_n(T(x_n - x))| + |f_n(T(x))| \to 0$, which is impossible. Thus, (f_n) converges uniformly to zero on $T(W)$, and this shows that $T(W)$ is an almost Dunford-Pettis set. This ends the proof of the Theorem. ■

Let us recall that, an operator T from a Banach lattice E into a Banach lattice F is said to be order bounded if for each $z \in E^+$, the set $T([-z, z])$ is order bounded set in F. An operator T from a Banach lattice E into a Banach lattice F is said to be regular if it can be written as a difference of two positive operators. Note that, every regular operator is order bounded but an order bounded operator is not necessary regular (see [2], Example 1.16, p. 13).

Remark 2.2: Each order interval $[-z, z]$ of a Banach lattice E is an almost Dunford-Pettis set for each $z \in E^+$. In fact, if (f_n) be a disjoint weakly null sequence in E', then by Remark 1 of Wnuk [5], $(|f_n|)$ is a disjoint weakly null sequence in E'. Hence $\sup_{x \in [-z,z]} |f_n(x)| = |f_n|(z) \to 0$ for each $z \in E^+$. As a consequence, if $T : E \to F$ is an order bounded operator from a Banach lattice E into another F, then $T([-z, z])$ is an almost Dunford-Pettis set in F, and then $|f_n \circ T|(z) = \sup_{y \in T([-z,z])} |f_n(y)| \to 0$ for each $z \in E^+$.

We will need the following characterizations, which are just Theorem 2.4 of [3].

Theorem 2.3: [3] Let $T : E \to F$ be an order bounded operator from a Banach lattice E into another Banach lattice F, and let A be a norm bounded solid subset of E. The following statements are equivalent:

1) $T(A)$ is an almost Dunford-Pettis set.
2) $\{T(x_n), n \in N\}$ is an almost Dunford-Pettis set for each disjoint sequence (x_n) in $A^+ = A \cap E^+$.
3) $f_n(T(x_n)) \to 0$ for each disjoint sequence (x_n) in A^+ and for every disjoint weakly null sequence (f_n) of E'.

Proof: (1) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) To prove that $T(A)$ is an almost Dunford-Pettis set, it suffice to show that $\sup_{x \in \mathbf{A}} |f_n(T(x))| \to 0$ for every disjoint weakly null sequence (f_n) of F'. Otherwise, there exists a sequence $(f_n) \subset E'$ satisfying $\sup_{x \in \mathbf{A}} |f_n(T(x))| > \varepsilon$ for some $\varepsilon > 0$ and all n. For every n there exists z_n in A^+ such that $|T'(f_n)|(z_n) > \varepsilon$. Since $|T'(f_n)|(z) \to 0$ for every $z \in E^+$ (see Remark 2.2), then by an easy inductive argument shows that there exist a subsequence (y_n) of (z_n) and a subsequence (g_n) of (f_n) such that

$$|T'(g_{n+1})|(y_{n+1}) > \varepsilon \text{ and } |T'(g_{n+1})|(4^n \sum_{i=1}^{n} y_i) < \frac{1}{n}$$

for all $n \geq 1$. Put $x = \sum_{i=1}^{\infty} 2^{-i} y_i$ and $x_n = (y_{n+1} - 4^n \sum_{i=1}^{n} y_i - 2^{-n} x)^+$. By Lemma 4.35 of [2] the sequence (x_n) is disjoint. Since $0 \leq x_n \leq y_{n+1}$ for every n, and (y_{n+1}) in A^+ then $(x_n) \subset A^+$.

From the inequalities

$$\begin{aligned} |T'(g_{n+1})|(x_n) &\geq |T'(g_{n+1})|(y_{n+1} - 4^n \sum_{i=1}^{n} y_i - 2^{-n} x) \\ &\geq \varepsilon - \frac{1}{n} - 2^{-n} |T'(g_{n+1})|(x) \end{aligned}$$

we see that $|T'(g_{n+1})|(x_n) > \frac{\varepsilon}{2}$ must hold for all n sufficiently large (because $2^{-n}|T'(g_{n+1})|(x) \to 0$).

In view of $|T'(g_{n+1})|(x_n) = \sup\{|g_{n+1}(T(z))| : |z| \leq x_n\}$, for each n sufficiently large there exists some $|z_n| \leq x_n$ with $|g_{n+1}(T(z_n))| > \frac{\varepsilon}{2}$. Since (z_n^+) and (z_n^-) are both norm bounded disjoint sequence in A^+, it follows from our hypothesis that

$$\frac{\varepsilon}{2} < |g_{n+1}(T(z_n))| \leq \left|g_{n+1}\left(T(z_n^+)\right)\right| + \left|g_{n+1}\left(T(z_n^-)\right)\right| \to 0$$

which is impossible. This proves that $T(A)$ is an almost Dunford-Pettis set. ■

For order bounded operators between two Banach lattices, we give a characterization of weak almost Dunford-Pettis operators.

Theorem 2.4: Let T be an order bounded operator from a Banach lattice E into another F. Then the following assertions are equivalent:

1) T is weak almost Dunford-Pettis operator.
2) $f_n(T(x_n)) \longrightarrow 0$ for all weakly null sequence (x_n) in E consisting of pairwise disjoint terms, and for all weakly null sequence (f_n) in E' consisting of pairwise disjoint terms.

Proof: (1) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (1) Let (x_n) be a weakly null sequence in E, and let (f_n) be a disjoint weakly null sequence in F'. We have to prove that $f_n(T(x_n)) \to 0$.

Let A be the solid hull of the weak relatively compact subset $\{x_n, n \in N\}$ of E, by Theorem 4.34 of [2], $(z_n) \to 0$ weakly for each disjoint sequence (z_n) in A^+ and so, by our hypothesis, we have $g_n(T(z_n)) \to 0$ for each disjoint weakly null sequence (g_n) in F' and for each disjoint sequence (z_n) in A^+, then Theorem 2.3, implies that $T(A)$ is an almost Dunford-Pettis set, and hence $\sup_{y\in T(A)} |f_n(y)| \to 0$. Therefore,

$$|f_n(T(x_n))| \leq \sup_{x\in A} |f_n((T(x)| \leq \sup_{y\in T(A)} \|f_n(y)\| \to 0$$

holds and the proof is finished. ■

Now for positive operators between two Banach lattices, we give other characterizations of weak almost Dunford-Pettis operators.

Theorem 2.5: Let E and F be two Banach lattices. For every positive operator T from E into F, the following assertions are equivalent:

1) T is weak almost Dunford-Pettis.
2) If S is a weakly compact operator from an arbitrary Banach space Z into E, then the adjoint of the operator product $T \circ S$ is almost Dunford-Pettis.
3) If S is a weakly compact operator from ℓ^1 into E, then the adjoint of the operator product $T \circ S$ is almost Dunford-Pettis.
4) For all weakly null sequence $(x_n)_n \subset E$, and for all disjoint weakly null sequence $(f_n)_n \subset F'$ it follows that $f_n(T(x_n)) \to 0$.
5) $f_n(T(x_n)) \longrightarrow 0$ for every weakly null sequence (x_n) in E^+ and for all disjoint weakly null sequence (f_n) in F'.
6) $f_n(T(x_n)) \longrightarrow 0$ for all weakly null sequence (x_n) in E consisting of pairwise disjoint terms, and for all weakly null sequence (f_n) in F' consisting of pairwise disjoint terms.
7) For all disjoint weakly null sequences $(x_n)_n \subset E^+$, $(f_n)_n \subset (F')^+$ it follows that $f_n(T(x_n)) \longrightarrow 0$.
8) $f_n(T(x_n)) \longrightarrow 0$ for every disjoint weakly null sequence (x_n) in E^+ and for all weakly null sequence (f_n) in F'.
9) $f_n(T(x_n)) \longrightarrow 0$ for every disjoint weakly null sequence (x_n) in E^+ and for all weakly null sequence (f_n) in $(F')^+$.
10) $f_n(T(x_n)) \longrightarrow 0$ for every weakly null sequence (x_n) in E and for all weakly null sequence (f_n) in $(F')^+$.
11) $f_n(T(x_n)) \longrightarrow 0$ for every weakly null sequence (x_n) in E^+ and for all weakly null sequence (f_n) in $(F')^+$.
12) $f_n(T(x_n)) \longrightarrow 0$ for every weakly null sequence (x_n) in E^+ and for all weakly null sequence (f_n) in F'.

Proof: (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) Follows from Theorem 2.1.

(6) $\Leftrightarrow$ (4) Follows from Theorem 2.4.

(4) $\Rightarrow$ (5) Obvious.

(5) $\Rightarrow$ (6) Let (x_n) be a weakly null sequence in E consisting of pairwise disjoint elements, and let (f_n) be a weakly null sequence in F', consisting of pairwise disjoint elements, it follows from Remark 1 of Wnuk [5] that $x_n^+ \longrightarrow 0$ and $x_n^- \longrightarrow 0$ weakly in E^+. Hence by (5), $f_n(T(x_n)) = f_n(T(x_n^+)) - f_n(T(x_n^-)) \longrightarrow 0$.

(6) $\Rightarrow$ (7) Obvious.

(7) $\Rightarrow$ (8) Assume by way of contradiction that there exists a disjoint weakly null sequence $(x_n) \subset E^+$ and a weakly null sequence $(f_n) \subset F'$ such that $f_n(T(x_n)) \nrightarrow 0$. The inequality $|f_n(T(x_n))| \leq |f_n|(T(x_n))$ implies $|f_n|(T(x_n)) \nrightarrow 0$. Then there exists some $\varepsilon > 0$ and a subsequence of $|f_n|(T(x_n))$ (which we shall denote by $|f_n|(T(x_n))$ again) satisfying $|f_n|(T(x_n)) > \varepsilon \ \forall n$.

On the other hand, since $(x_n) \to 0$ weakly in E, then $T(x_n) \to 0$ weakly in F. Now an easy inductive argument shows that there exist a subsequence (z_n) of (x_n) and a subsequence (g_n) of (f_n) such that $\forall n \geq 1$

$$|g_n|(T(z_n)) > \varepsilon \text{ and } (4^n \sum\nolimits_{i=1}^{n} |g_i|)(T(z_{n+1})) < \frac{1}{n}$$

Put $h = \sum_{n=1}^{\infty} 2^{-n}|g_n|$ and $h_n = (|g_{n+1}| - 4^n\sum_{i=1}^{n}|g_i| - 2^{-n}h)^+$. By Lemma 4.35 of [2] the sequence (h_n) is disjoint. Since $0 \leq h_n \leq |g_{n+1}|$ for all $n \geq 1$ and $(g_n) \to 0$ weakly in F' then it follows from Theorem 4.34 of [2] that $(h_n) \to 0$ weakly in F'.

From the inequalities

$$\begin{aligned} h_n(T(z_{n+1})) &\geq (|g_{n+1}| - 4^n \sum\nolimits_{i=1}^{n} |g_i| - 2^{-n}h)(T(z_{n+1})) \\ &\geq \varepsilon - \frac{1}{n} - 2^{-n}h(T(z_{n+1})) \end{aligned}$$

we see that $h_n(T(z_{n+1})) > \frac{\varepsilon}{2}$ must hold for all n sufficiently large (because $2^{-n}h(T(z_{n+1})) \to 0$), which contradicts with our hypothesis (7).

(8) $\Rightarrow$ (9) Obvious.

(9) $\Rightarrow$ (10) Assume by way of contradiction that there exists a weakly null sequence $(x_n) \subset E$ and a weakly null sequence $(f_n) \subset (F')^+$ such that $f_n(T(x_n)) \nrightarrow 0$. The inequality $|f_n(T(x_n))| \leq f_n(T(|x_n|))$ implies $f_n(T(|x_n|)) \nrightarrow 0$. Then there exists some $\varepsilon > 0$ and a subsequence of $f_n(T(|x_n|))$ (which we shall denote by $f_n(T(|x_n|))$ again) satisfying $f_n(T(|x_n|)) > \varepsilon$ for all n.

On the other hand, since $(f_n) \to 0$ weakly in F', then $T'(f_n) \to 0$ weakly in E'. Now an easy inductive argument shows that there exist a subsequence (z_n) of $(|x_n|)$ and a subsequence (g_n) of (f_n) such that $\forall n \geq 1$

$$T'(g_n)(z_n) > \varepsilon \text{ and } T'(g_{n+1})(4^n \sum_{i=1}^{n} z_i) < \frac{1}{n}$$

Put $z = \sum_{n=1}^{\infty} 2^{-n} z_n$ and $y_n = (z_{n+1} - 4^n \sum_{i=1}^{n} z_i - 2^{-n} z)^+$. By Lemma 4.35 of [2] the sequence (y_n) is disjoint. Since $0 \leq y_n \leq z_{n+1}$ for all $n \geq 1$ and $(z_n) \to 0$ weakly in E, then it follows from Theorem 4.34 of [2] that $(y_n) \to 0$ weakly in E.

From the inequalities

$$\begin{aligned} T'(g_{n+1})(y_n) &\geq T'(g_{n+1})(z_{n+1} - 4^n \sum_{i=1}^{n} z_i - \frac{z}{2^n}) \\ &\geq \varepsilon - \frac{1}{n} - 2^{-n} T'(g_{n+1})(z) \end{aligned}$$

we see that $g_{n+1}(T(y_n)) = T'(g_{n+1})(y_n) > \frac{\varepsilon}{2}$ must hold for all n sufficiently large (because $2^{-n} T'(g_{n+1})(z)) \to 0$), which contradicts with our hypothesis (9).

(10) $\Rightarrow$ (11) Obvious.

(11) $\Rightarrow$ (6) Let (x_n) be a weakly null sequence in E consisting of pairwise disjoint elements, and let (f_n) be a weakly null sequence in F', consisting of pairwise disjoint elements, it follows from Remark 1 of Wnuk [5] that $|x_n| \longrightarrow 0$ in $\sigma(E, E')$, and $|f_n| \longrightarrow 0$ in $\sigma(F', F'')$. Hence by (11), $|f_n|(T(|x_n|)) \longrightarrow 0$. Now, from $|f_n(T(x_n))| \leq |f_n|(T(|x_n|))$ for each n, we derive that $f_n(T(x_n)) \longrightarrow 0$.

(12) $\Rightarrow$ (8) Obvious.

(5) $\Rightarrow$ (12) The proof is similar of the proof (7) $\Rightarrow$ (8). ■

An application of Theorem 2.5, gives other characterizations of Banach lattices with the weak Dunford-Pettis property.

Corollary 2.6: For a Banach lattice E the following statements are equivalent:

1) E has the weak Dunford-Pettis property.
2) The identity operator $Id_E : E \to E$ is weak almost Dunford-Pettis, that is, every relatively weakly compact set of E is almost Dunford-Pettis set.
3) Every weakly compact operator T from an arbitrary Banach space X to E has an adjoint $T' : E' \to X'$ which is almost Dunford-Pettis.
4) Every weakly compact operator $T : \ell^1 \to E$ has an adjoint T' which is almost Dunford-Pettis.
5) For all weakly null sequence $(x_n)_n \subset E$, and for all disjoint weakly null sequence $(f_n)_n \subset E'$ it follows that $f_n(x_n) \to 0$.
6) $f_n(x_n) \longrightarrow 0$ for every weakly null sequence $(x_n)_n$ in E^+ and for all disjoint weakly null sequence $(f_n)_n$ in E'.
7) For all disjoint weakly null sequences $(f_n)_n \subset E'$, $(x_n)_n \subset E$ it follows that $f_n(x_n) \longrightarrow 0$.
8) For all disjoint weakly null sequences $(f_n)_n \subset (E')^+$, $(x_n)_n \subset E^+$ it follows that $f_n(x_n) \longrightarrow 0$.
9) $f_n(x_n) \longrightarrow 0$ for every disjoint weakly null sequence (x_n) in E^+ and for all weakly null sequence (f_n) in E'.
10) $f_n(x_n) \longrightarrow 0$ for every disjoint weakly null sequence (x_n) in E^+ and for all weakly null sequence (f_n) in $(E')^+$.
11) $f_n(x_n) \longrightarrow 0$ for every weakly null sequence (x_n) in E and for all weakly null sequence (f_n) in $(E')^+$.
12) $f_n(x_n) \longrightarrow 0$ for every weakly null sequence $(x_n)_n$ in E^+ and for all weakly null sequence (f_n) in $(E')^+$.
13) $f_n(x_n) \longrightarrow 0$ for every weakly null sequence (x_n) in E^+ and for all weakly null sequence (f_n) in E'.

Proof: (1) $\Leftrightarrow$ (8) Follows from Proposition 1 of Wnuk [5].

(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ ... $\Leftrightarrow$ (13) Follows from Theorem 2.5. ■

The following consequence of Theorem 2.5 gives a sufficient conditions under which the class of positive weak almost Dunford-Pettis operators coincide with that of positive weak Dunford-Pettis operators.

Corollary 2.7: Let E and F be two Banach lattices. Then each positive weak almost Dunford-Pettis operator from E into F is weak Dunford-Pettis if one of the following assertions is valid:

1) The lattice operation of E are weak sequentially continuous;
2) The lattice operation of F' are weak sequentially continuous.

Proof: (1) Assume that $T : E \to F$ is a positive weak almost Dunford-Pettis operator. Let (x_n) be a weakly null sequence in E, and let (f_n) be a weakly null sequence in F'. We have to prove that $f_n(T(x_n)) \to 0$.

Since the lattice operation of E are weak sequentially continuous, then the positive sequences (x_n^+) and (x_n^-) converge weakly to zero. Thus, Theorem 2.5 (12) imply that

$$f_n\left(T(x_n^+)\right) \longrightarrow 0 \quad \textit{and} \quad f_n\left(T(x_n^-)\right) \longrightarrow 0.$$

Finally, from $f_n(T(x_n)) = f_n(T(x_n^+)) - f_n(T(x_n^-))$ for each n, we conclude that $f_n(T(x_n)) \longrightarrow 0$. This shows that T is weak Dunford-Pettis.

(2) Assume that $T : E \to F$ is a positive weak almost Dunford-Pettis operator. Let (x_n) be a weakly null sequence in E, and let (f_n) be a weakly null sequence in F'. We have to prove that $f_n(T(x_n)) \to 0$.

Since the lattice operation of F' are weak sequentially continuous, then the positive sequences (f_n^+) and (f_n^-) converge weakly to zero. Thus, Theorem 2.5 (10) imply that $f_n^+(T(x_n)) \longrightarrow 0$ and $f_n^-(T(x_n)) \longrightarrow 0$. Finally, from $f_n(T(x_n)) = f_n^+(T(x_n)) - f_n^-(T(x_n))$ for each n, we conclude that $f_n(T(x_n)) \longrightarrow 0$. This shows that T is weak Dunford-Pettis. ■

The preceding Corollary, gives a sufficient conditions under which the weak Dunford-Pettis property and the Dunford-Pettis property coincide.

Corollary 2.8: Let E be a Banach lattice. Then E has the Dunford-Pettis property if and only if it has the weak Dunford-Pettis property, if one of the following assertions is valid:

1) The lattice operation of E are weak sequentially continuous;
2) The lattice operation of E' are weak sequentially continuous.

Our consequence of Theorem 2.5 we obtain the domination property for weak almost Dunford-Pettis operators.

Corollary 2.9: Let E and F be two Banach lattices. If S and T are two positive operators from E into F such that $0 \leq S \leq T$ and T is weak almost Dunford-Pettis operator, then S is also weak almost Dunford-Pettis operator.

Proof: Let $(x_n)_n$ be a weakly null sequence in E^+ and (f_n) be a weakly null sequence in $(F')^+$. According to (11) of Theorem 2.5, it suffices to show that $f_n(S(x_n)) \longrightarrow 0$. Since T is weak almost Dunford-Pettis, then Theorem 2.5 implies that $f_n(T(x_n)) \longrightarrow 0$. Now, by using the inequalities $0 \leq f_n(S(x_n)) \leq f_n(T(x_n))$ for each n, we see that $f_n(S(x_n)) \longrightarrow 0$. ■

Now, we look at the duality property of the class of positive weak almost Dunford-Pettis operators.

Theorem 2.10: Let E and F be two Banach lattices and let T be a positive operator from E into F. If the adjoint T' is weak almost Dunford-Pettis from F' into E', then T itself is weak almost Dunford-Pettis.

Proof: Let (x_n) be a weakly null sequence in E^+, and let (f_n) be a weakly null sequence in $(F')^+$. We have to prove that $f_n(T(x_n)) \longrightarrow 0$.

Let $\tau : E \longrightarrow E''$ be the canonical injection of E into its topological bidual E''. Since τ is a lattice homomorphism, the sequence $(\tau(x_n))$ is weakly null in $(E'')^+$. And as the adjoint T' is weak almost Dunford-Pettis from F' into E', we deduce by Theorem 2.1 that $\tau(x_n)(T'(f_n)) \longrightarrow 0$. But $\tau(x_n)(T'(f_n)) = T'(f_n)(x_n) = f_n(T(x_n))$ for each n. Hence $f_n(T(x_n)) \longrightarrow 0$ and this ends the proof. ■

We end this paper by a consequence of Theorem 2.10, we obtain Proposition 2 of Wnuk [5].

Corollary 2.11: Let E be a Banach lattice. If E' has the weak Dunford-Pettis property, then E itself has the weak Dunford-Pettis.

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