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Asymptotic Properties of a Stochastic Predator-Prey Model with Bedding-DeAngelis Functional Response

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Abstract—In this paper, a stochastic predator-prey system with Bedding-DeAngelis functional response is studied. By constructing a suitable Lyapunov founction, sufficient conditions for species to be stochastically permanent is established. Meanwhile, we show that the species will become extinct with probability one if the noise is sufficiently large.

Keywords—Stochastically permanent, extinct, white noise, Bedding-DeAngelis functional response.

I. Introduction

IN mathematical biology, the predator's functional response which is the rate of prey consumption by an average predator is one of the significant elements of the predator-prey relationship. Generally, the functional response can be classified into two types: prey-dependent and predator-dependent. And Bedding-DeAngelis functional response belongs to predator-dependent functional response. As a matter of fact, the phenomenon that predators have to share or compete for food is common. Therefore, studying Bedding-DeAngelis functional response is meaningful.

A. The Model

However, we have no choice but to admit that all population systems are often subject to environmental noises. So, considering the corresponding stochastic population is necessary and important[1]-[13]. In [1], Liu and Wang introduced global stability of a nonlinear stochastic predator-prey system with Beddington-DeAngelis functional response. As we all know, stochastically permanent and extinct are also very important. There are two noise sources in [1], but their coupled mode is very simple. We know one noise source not only has influence on the growth rate of predator but also on the prey's. Therefore, from the argument above, we study the following form in this paper:

$$\begin{cases} dx = x\left[r_1 - b_1 x - \dfrac{a_1 y}{1+\beta x+\gamma y}\right]dt \\ \qquad +x[\sigma_1 dB_1(t) + \mu_2 dB_2(t)], \\ dy = y\left[r_2 - \dfrac{a_2 x}{1+\beta x+\gamma y} - b_2 y\right]dt \\ \qquad +y(t)[\sigma_2 dB_1(t) + \mu_1 dB_2(t)], \end{cases} \tag{1}$$

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where $x(t)$ and $y(t)$ stand for the population densities of prey and predator at time t, respectively; $r_i, b_i, a_i, \beta, \gamma$ are positive parameters, $i = 1, 2$. μ_i^2 and σ_i^2 represent the intensities of the white noises, $i = 1, 2$. Let $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{F_t\}_{t\geq 0}$ satisfying the usual conditions, i.e. it is right continuous and increasing while F_0 contains all $P-$null sets. We denote by R_+^2 the positive cone in R^2, and also denote by $X(t) = (x(t), y(t))$ and $|X(t)| = (x^2(t) + y^2(t))^{\frac{1}{2}}$.

1) The Preliminaries: In this section, we give some definitions, lemmas, assumptions and notations. The proof of Lemma 1, Lemma 2 and Lemma 3 are similar to [14]. Here, we omit them.

Definition 1 (see [14]) The solution $X(t) = (x(t), y(t))$ of (1) are said to be stochastically permanent, if for any $\epsilon \in (0, 1)$, there exists a pair of positive constants $\delta = \delta(\epsilon)$ and $\chi = \chi(\epsilon)$ such that for any initial value $X(0) = (x(0), y(0)) \in R_+^2$, the solution $X(t)$ to (1) has the properties that

$$\liminf_{t\to\infty} P\{|X(t)| \geq \delta\} \geq 1-\epsilon,\ \liminf_{t\to\infty} P\{|X(t)| \leq \chi\} \geq 1-\epsilon.$$

Lemma 1 For any initial value $x_0 > 0, y_0 > 0$, there is an unique positive local solution $(x(t), y(t))$ for $t \in [0, \tau_e)$ of model (1) almost surely (a.s.).

Lemma 2 For any given initial value $X_0 = (x_0, y_0) \in R_+^2$, there is an unique solution $X(t) = (x(t), y(t))$ to model (1) on $t \geq 0$ and the solution will remain in R_+^2 with probability 1.

Lemma 3 The solutions of model (1) are stochastically ultimately bounded for any initial value $X_0 = (x_0, y_0) \in R_+^2$.

Assumption (A_1):

$$\frac{1}{2}\max\Big\{\left(\sigma_1^2+\mu_2^2+\sigma_1\sigma_2+\mu_1\mu_2\right), \Big(\sigma_2^2+\mu_1^2 \\ +\sigma_1\sigma_2+\mu_1\mu_2\Big)\Big\} < \min\left\{r_1+\frac{a_1}{\gamma}, r_2\right\},$$

Assumption (A_2): $r_1 - \dfrac{\sigma_1^2+\mu_2^2}{2} < 0$,

Assumption (A_3): $r_2 + \dfrac{a_2}{\beta} - \dfrac{\sigma_2^2+\mu_1^2}{2} < 0$.

For convenience of statement, we introduce some notations: let

$$M(x,y) = \frac{a_1 y}{1+\beta x+\gamma y}, N(x,y) = \frac{a_2 y}{1+\beta x+\gamma y}.$$

II. Conclusion

Theorem 1 Under Assumption (A_1), for any initial value $X(0) = (x(0), y(0)) \in R^2_+$, the solution $X(t) = (x(t), y(t))$ satisfies that

$$\limsup_{t\to\infty} E(\frac{1}{|X(t)|^\alpha}) \leq K, \tag{2}$$

where α is an arbitrary positive constant satisfying

$$\frac{\alpha+1}{2}\max\Big\{(\sigma_1^2+\mu_2^2+\sigma_1\sigma_2+\mu_1\mu_2),\Big(\sigma_2^2+\mu_1^2+\sigma_1\sigma_2 +\mu_1\mu_2\Big)\Big\} < \min\Big\{r_1+\frac{a_1}{r}, r_2\Big\}. \tag{3}$$

there exist an arbitrary positive constant $s > 0$ satisfying

$$\alpha\min\Big\{r_1+\frac{a_1}{r}, r_2\Big\} - \frac{\alpha(\alpha+1)}{2}\max\Big\{\Big(\sigma_1^2+\mu_2^2+\sigma_1\sigma_2 +\mu_1\mu_2\Big),\Big(\sigma_2^2+\mu_1^2+\sigma_1\sigma_2+\mu_1\mu_2\Big)\Big\} - s > 0. \tag{4}$$

Theorem 2 Assume (A_1) hold, equation(1) is Stochastically permanent.

The proof is application of the well-known Chebyshev inequality, Lemma 3 and Theorem 1. Here, we omit it.

Theorem 3 Assume (A_2) and (A_3) hold. For any given initial value $(x_0, y_0) \in R^2_+$, the solution $(x(t), y(t))$ to (1) will be extinct exponentially with probability one.

Appendix A
Proof of Theorem 1

Proof: The proof is motivated by the method of [12]. Define $V_1(x,y) = x + y$, for $(x,y) \in R^2_+$, by the Itôs formula, we compute

$$\begin{aligned} dV_1(x,y) = &\Big\{x\Big[r_1 - b_1x - M(x,y)\Big] + y\Big[r_2 + N(x,y) -b_2y\Big]\Big\}dt + x\Big[\sigma_1 dB_1(t) + \mu_2 dB_2(t)\Big] \\ &+y\Big[\sigma_2 dB_1(t) + \mu_1 dB_2(t)\Big]. \end{aligned}$$

Then define $W(x,y) = \dfrac{1}{V_1(x,y)}$, dropping $x(t)$ from $U(x(t),y(t)), V_3(x(t),y(t))$ and t from $x(t), y(t)$, we have

$$\begin{aligned} dW = LWdt - W^2\Big\{&x\Big[\sigma_1 dB_1(t) + \mu_2 dB_2(t)\Big] \\ &+y\Big[\sigma_2 dB_1(t) + \mu_1 dB_2(t)\Big]\Big\}, \end{aligned}$$

where

$$\begin{aligned} LW = &-W^2\Big[x\Big(r_1 - b_1x - M(x,y)\Big) + y\Big(r_2 + N(x,y) -b_2y\Big)\Big] + W^3\Big[\Big(\sigma_1^2+\mu_2^2\Big)x^2 + \Big(\sigma_2^2+\mu_1^2\Big)y^2 \\ &+2xy\Big(\sigma_1\sigma_2+\mu_1\mu_2\Big)\Big]. \end{aligned}$$

Under Assumption (A_1), let us choose a positive constant α such that it obeys (3). By the Itô formula, we get

$$\begin{aligned} L(1+W)^\alpha = &\alpha(1+W)^{\alpha-1}LW + \frac{\alpha(\alpha-1)}{2}W^4 \\ &\times(1+W)^{\alpha-2}\Big[\Big(\sigma_1^2+\mu_2^2\Big)x^2 + \Big(\sigma_2^2 +\mu_1^2\Big)y^2 + 2xy\Big(\sigma_1\sigma_2+\mu_1\mu_2\Big)\Big]. \end{aligned}$$

Then we choose $s > 0$ sufficiently small such that it satisfies (4). Consequently,

$$\begin{aligned} Le^{st}(1+W)^\alpha &= se^{st}\Big(1+W\Big)^\gamma + e^{st}L\Big(1+W\Big)^\gamma \\ &= e^{st}\Big(1+W\Big)^{\gamma-2}\Big[s\Big(1+W\Big)^2 + H\Big], \end{aligned}$$

where

$$\begin{aligned} H = &-\alpha W^2\Big[x\Big(r_1 - b_1x - M(x,y)\Big) + y\Big(r_2 + N(x,y) -b_2y\Big)\Big] - \alpha W^3\Big[x\Big(r_1 - b_1x - M(x,y)\Big) + y\Big(r_2 +N(x,y) - b_2y\Big)\Big] \\ &+ \alpha W^3\Big[\Big(\sigma_1^2+\mu_2^2\Big)x^2 + \Big(\sigma_2^2 +\mu_1^2\Big)y^2 + 2xy\Big(\sigma_1\sigma_2+\mu_1\mu_2\Big)\Big] + \frac{\alpha(\alpha+1)}{2}W^4 \\ &\times\Big[\Big(\sigma_1^2+\mu_2^2\Big)x^2 + \Big(\sigma_2^2+\mu_1^2\Big)y^2 + 2xy\Big(\sigma_1\sigma_2+\mu_1\mu_2\Big)\Big]. \end{aligned}$$

In the following analysis, we will discuss the upper boundedness of the function $(1 + W)^{\alpha-2}[s(1 + W)^2 + H]$. It is easy to imply that

$$\begin{aligned} &W^3\Big[\Big(\sigma_1^2+\mu_2^2\Big)x^2 + \Big(\sigma_2^2+\mu_1^2\Big)y^2 + 2xy\Big(\sigma_1\sigma_2+\mu_1\mu_2\Big)\Big] \\ &\leq \Big(\max\Big\{\Big(\sigma_1^2+\mu_2^2\sigma_1\sigma_2+\mu_1\mu_2\Big),\Big(\sigma_2^2+\mu_1^2\sigma_1\sigma_2+\mu_1\mu_2\Big)\Big\}\Big)U \end{aligned}$$

and

$$\begin{aligned} &\frac{\alpha(\alpha+1)}{2}W^4\Big[\Big(\sigma_1^2+\mu_2^2\Big)x^2 + \Big(\sigma_2^2+\mu_1^2\Big)y^2 + 2xy\Big(\sigma_1\sigma_2 +\mu_1\mu_2\Big)\Big] \leq \frac{\alpha(\alpha+1)}{2}\Big(\max\Big\{\Big(\sigma_1^2+\mu_2^2+\sigma_1\sigma_2+\mu_1\mu_2\Big), \\ &\Big(\sigma_2^2+\mu_1^2+\sigma_1\sigma_2+\mu_1\mu_2\Big)\Big\}\Big)W^2. \end{aligned}$$

Hence,

$$\begin{aligned} Le^{st}(1+W)^\alpha &= e^{st}(1+W)^{\alpha-2}\Big[s(1+W)^2 + H\Big] \\ &\leq e^{st}(1+W)^{\alpha-2}\Big\{\Big[s + \alpha\max\Big\{b_1, b_2\Big\}\Big] \\ &\quad+\Big[2s - \alpha\min\Big\{r_1+\frac{a_1}{r}, r_2\Big\} + \alpha\max\{b_1,b_2\} \\ &\quad+\alpha\max\Big\{\Big(\sigma_1^2+\mu_2^2+\sigma_1\sigma_2+\mu_1\mu_2\Big),\Big(\sigma_2^2 +\mu_1^2+\sigma_1\sigma_2+\mu_1\mu_2\Big)\Big\}\Big]W - \Big[-\alpha\min\Big\{r_1 +\frac{a_1}{r}, r_2\Big\} - \frac{\alpha(\alpha+1)}{2}\max\Big\{\Big(\sigma_1^2+\mu_2^2 \\ &\quad+\sigma_1\sigma_2+\mu_1\mu_2\Big),\Big(\sigma_2^2+\mu_1^2+\sigma_1\sigma_2+\mu_1\mu_2\Big)\Big\} -s\Big]U^2\Big\}. \end{aligned}$$

From (4), we know that there exists a positive constant S such that $Le^{st}(1 + W)^\alpha \leq Se^{st}$.

Therefore,

$$\begin{aligned} E\Big[e^{st}\Big(1+W(t)\Big)^\alpha\Big] &\leq \Big(1+W(0)\Big)^\alpha + \frac{S}{s}e^{st} \\ &= \Big(1+W(0)\Big)^\alpha + K_1e^{st}, \end{aligned}$$

where $K_1 = \dfrac{S}{s}$.

So, we have

$$\limsup_{t\to\infty} EW^\alpha(t) \leq \limsup_{t\to\infty} E(1+W(t))^\alpha \leq K_1.$$

Note that $(x+y)^\alpha \leq 2^\alpha(x^2+y^2)^{\frac{\alpha}{2}} = 2^\alpha|X|^\alpha$, where $X = (x,y) \in R^2$. Now, we can obtain that

$$\limsup_{t\to\infty} E(\frac{1}{|X(t)|^\alpha}) \leq 2^\alpha \limsup_{t\to\infty} EW^\alpha(t) \leq 2^\alpha K_1 =: K.$$

Theorem 1 is proved.

Appendix B
Proof of Theorem 3

Proof: Define Lyapunov function $V_2 = \ln x$. Applying Itô formula leads to

$$\begin{aligned} dV_2 &= d(\ln x) \\ &= \left[\left(r_1 - \frac{\sigma_1^2+\mu_2^2}{2}\right) - b_1 x - M(x,y)\right] dt + \sigma_1 dB_1(t) \\ &\quad +\mu_2 dB_2(t). \end{aligned}$$

Integrating it from 0 to t, yields

$$\begin{aligned} \ln x(t) = \ln x_0 &+ \left(r_1 - \frac{\sigma_1^2+\mu_2^2}{2}\right) t - b_1 \int_0^t x(s)ds \\ &-c_1 \int_0^t M(x(s),y(s))ds \\ &+\sigma_1 \int_0^t dB_1(s) + \mu_2 \int_0^t dB_2(s). \end{aligned}$$

Consequently,

$$\ln x(t) \leq \ln x_0 + \left(r_1 - \frac{\sigma_1^2+\mu_2^2}{2}\right) t + \sigma_1 B_1(t) + \mu_2 B_2(t).$$

Dividing t on the both sides and letting $t \to \infty$, we can obtain

$$\limsup_{t\to\infty} \frac{\ln x(t)}{t} \leq r_1 - \frac{\sigma_1^2+\mu_2^2}{2} < 0 \ a.s.$$

Similarly, define Lyapunov function $V_3 = \ln y$, by the Itô formula, we have

$$\begin{aligned} \ln y(t) = \ln y_0 &+ \left(r_2 - \frac{\sigma_2^2+\mu_1^2}{2}\right) t - b_2 \int_0^t y(s)ds \\ &-c_2 \int_0^t N(x(s),y(s))ds \\ &+\sigma_2 \int_0^t dB_1(s) + \mu_1 \int_0^t dB_2(s). \end{aligned}$$

Therefore,

$$\frac{\ln y(t)}{t} \leq \frac{\ln y_0}{t} + r_2 - \frac{\sigma_2^2+\mu_1^2}{2} + \frac{a_2}{\beta} + \frac{\sigma_2 B_1(t)}{t} + \frac{\mu_1 B_2(t)}{t}.$$

Let $t \to \infty$, we have

$$\limsup_{t\to\infty} \frac{\ln y(t)}{t} \leq r_2 + \frac{a_2}{\beta} - \frac{\sigma_2^2+\mu_1^2}{2} < 0 \ a.s.$$

The desired assertion is derived.

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