

# Approximation for Average Error Probability of BPSK in the Presence of Phase Error

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**Abstract**—Phase error in communications systems degrades error performance. In this paper, we present a simple approximation for the average error probability of the binary phase shift keying (BPSK) in the presence of phase error having a uniform distribution on arbitrary intervals. For the simple approximation, we use symmetry and periodicity of a sinusoidal function. Approximate result for the average error probability is derived, and the performance is verified through comparison with simulation result.

**Keywords**—Average error probability, Phase shift keying, Phase error

## I. INTRODUCTION

IN communications systems, error performance is degraded by various factors, and one of the most significant degradation factors is a phase error. A phase error is the random fluctuation in the phase of a waveform, caused by various physical phenomena. For example, the imperfect frequency recovery of the receiver oscillators disturbs the phase reference of the received symbol, and the Doppler effect in wireless channel can also lead to the residual phase error in the received signal [1], [2].

To analyze the performance of communications systems in the presence of phase error, the average error probability should be obtained by integrating an instantaneous error probability for a given phase error which has a specific probability density function (pdf). For an additive white Gaussian noise (AWGN) channel, since the instantaneous error probability in the presence of phase error is expressed as a complementary error function of a cosine function, the exact average error probability should be evaluated by a numerical method. However, the numerical method requires high computational complexity and much operation time. For analytical purposes, approximations are useful, and some approximations for noisy phase reference have been studied in many places in the literature [3]–[5].

In the case of uniformly distributed phase error, a simple approximation for integration is available. In this paper, we present an approximation for the average error probability of BPSK in the presence of phase error having a uniform distribution on arbitrary interval, using symmetry and periodicity of a sinusoidal function.

The organization of the rest of the paper is as follows: In section II, an average error probability of the received signal

with noisy phase reference is described. In section III, we present the simple approximation for integrating a complementary error function of a cosine function and derive approximated average error probability of BPSK in the presence of phase error. Then, in section IV, simulation results are shown. Finally, we conclude with a short discussion in section V.

## II. AVERAGE ERROR PROBABILITY WITH PHASE ERROR

A coherent detect receiver achieves high performance in communications systems. Fig. 1 shows the coherent receiver structure [6].

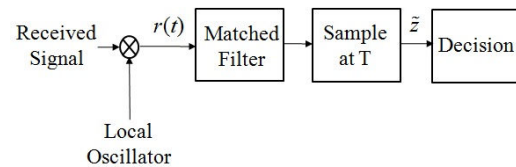


Fig. 1 Coherent receiver structure

In the coherent receiver, the matched filter provides the maximum signal-to-noise ratio (SNR) at its output. Then, the signal  $\tilde{z}$  sampled at each symbol duration  $T$  can be written as

$$\tilde{z} = \sqrt{E} \cdot e^{i(\phi_i + \theta)} + n \quad (1)$$

where  $\sqrt{E}$  is the signal power,  $\phi_i$  is modulated phase,  $\theta$  is phase error, and  $n$  is AWGN. In the case of BPSK, an instantaneous error probability in the presence of phase error and AWGN is given by

$$P_e(E|\theta) = \frac{1}{2} \operatorname{erfc}(\sqrt{\gamma} \cos \theta) \quad (2)$$

where  $\gamma = E/N_0$  is the SNR at the detector output. And  $\operatorname{erfc}(\bullet)$  is a complementary error function and defined as [7]

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du. \quad (3)$$

If we assume that the phase error is static, then (2) is directly used to evaluate the error probability. On the other hand, we have to integrate (2) against the specific pdf of the phase error when the phase error varies randomly throughout time. Considering random phase error, we would then expect the average error probability to be as follows:

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$$P_{e,avr}(E) = \frac{1}{2} \int P_e(E|\theta) \cdot p(\theta) d\theta \quad (4)$$

$$= \frac{1}{2} \int \text{erfc}(\sqrt{\gamma} \cos \theta) \cdot p(\theta) d\theta$$

where  $p(\theta)$  is the probability density function of the phase error. The exact value of the average error probability is obtained by a numerical method because a complementary error function in (4) includes a cosine function. In the next section, we present a simple approximation in the case of the phase error having a uniform distribution function on arbitrary interval.

### III. SIMPLE APPROXIMATION

In this section, we present a simple approximation for integrating a complementary error function of a cosine function and derive the approximated average error probability of BPSK in the presence of a phase error. When the phase error has a uniform probability density on  $-c < \theta \leq c$ ,  $p(\theta)$  is given by

$$p(\theta) = \begin{cases} \frac{1}{2c}, & -c < \theta \leq c \\ 0, & \text{otherwise} \end{cases} \quad (5)$$

where  $c$  has a arbitrary value for  $0 < c \leq \frac{\pi}{2}$ . Putting (5) into (4), we obtain

$$P_{e,avr}(E) = \frac{1}{2} \int_{-c}^c \text{erfc}(\sqrt{\gamma} \cos \theta) \cdot \frac{1}{2c} d\theta. \quad (6)$$

Since (6) has a complementary error function of a cosine function, the exact average error probability should be evaluated by numerical integration. As a first step of approximation for (6), the cosine function is replaced by a sine function, and the interval of integration should be shifted. Fig. 2 shows the symmetry and periodicity of a cosine and the sine function for the equivalent integration region. Fig. 3 represents a complementary error function of a cosine function.

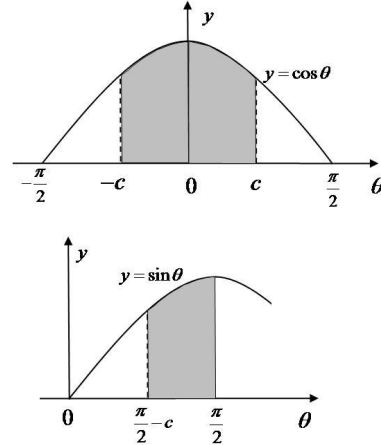


Fig. 2 Symmetry and periodicity of a cosine and sine function for the equivalent integration region

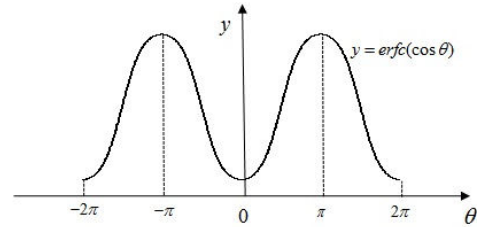


Fig. 3 Complementary error function of a cosine function

Because a cosine function is even-symmetric,  $\text{erfc}(\sqrt{\gamma} \cos \theta)$  is also even-symmetric, as shown in Figure 3. Thus, the integration region can be reduced, and (6) be rewritten as

$$P_{e,avr}(E) = \frac{1}{2c} \int_0^c \text{erfc}(\sqrt{\gamma} \cos \theta) d\theta. \quad (7)$$

Integration of  $\text{erfc}(\sqrt{\gamma} \cos \theta)$  for  $0 < \theta \leq c$  is equal to integration of  $\text{erfc}(\sqrt{\gamma} \sin \theta)$  for  $\frac{\pi}{2} - c < \theta \leq \frac{\pi}{2}$ ; thus, the cosine function in (7) can be replaced by a sine function. By the symmetry and periodicity of sinusoidal functions, (7) can be rewritten as

$$P_{e,avr}(E) = \frac{1}{2c} \int_{\pi/2-c}^{\pi/2} \text{erfc}(\sqrt{\gamma} \sin \theta) d\theta. \quad (8)$$

The cosine function was replaced by a sine function, yet (8) is not enough to evaluate the average error probability directly. Because a sine function is monotonically increasing for  $0 < \theta \leq \frac{\pi}{2}$ , we replace a sine function by a linear function as a last step of approximation. Then, we obtain

$$P_{e,avr}(E) \approx \frac{1}{2c} \int_{\pi/2-c}^{\pi/2} \text{erfc}(\sqrt{\gamma}\alpha \cdot \theta) d\theta \quad (9)$$

where  $\alpha$  is a coefficient of the linear function and its value is dependent on integration interval. The optimal values of  $\alpha$  are proposed by the simulation results in section IV. Because the sine function of (8) is replaced by an approximated linear function, (9) can be evaluated using [8, eq. (06.27.21.0002.01)]. Finally, the average error probability of BPSK in the presence of a phase error is obtained as

$$\begin{aligned} P_{e,avr}(E) &\approx \frac{1}{2c} \left[ \theta \cdot \text{erfc}(\sqrt{\gamma}\alpha \cdot \theta) - \frac{e^{-\gamma\alpha^2\theta^2}}{\sqrt{\pi}\gamma\alpha} \right]_{\pi/2-c}^{\pi/2} \\ &= \frac{\pi}{4c} \text{erfc}(\beta) + \left( \frac{2c-\pi}{4c} \right) \text{erfc} \left( \beta \left( 1 - \frac{2c}{\pi} \right) \right) \\ &\quad - \frac{\sqrt{\pi}e^{-\beta^2}}{4\beta c} \left( 1 - e^{c(\pi-c)\gamma\alpha^2} \right) \end{aligned} \quad (10)$$

where  $\beta = \frac{\pi}{2} \sqrt{\gamma}\alpha$ . Since (10) consists of complementary error functions and elementary functions, the average error probability for the arbitrary value of  $c$  can be easily evaluated.

#### IV. RESULT

In the previous section, the linear function was used for approximation of the sinusoidal function. Thus, accuracy of (10) is highly dependent on the coefficient of the linear function, and the optimal values of the coefficient can be determined in order to minimize the error of (10). Fig. 4 shows the simulation results for the optimal values of  $\alpha$ .

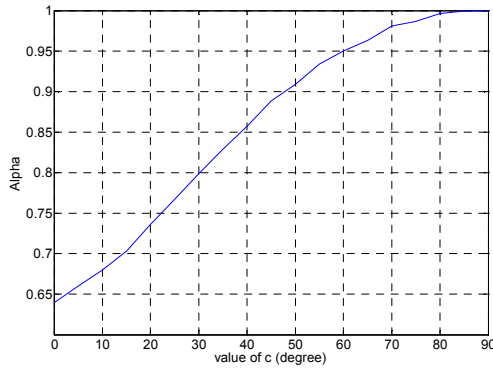


Fig. 4 Optimal values of alpha versus c

The optimal value of  $\alpha$  increases with increasing the upper bound  $c$  of phase error and converges on 1 at 90°. To verify the performance of the proposed approximation, we have simulated the average error probabilities for various phase errors and compared the simulation results with the approximated average error probability. Simulation and approximation results with a small phase error ( $c=10^\circ$ ) are shown in Fig. 5.

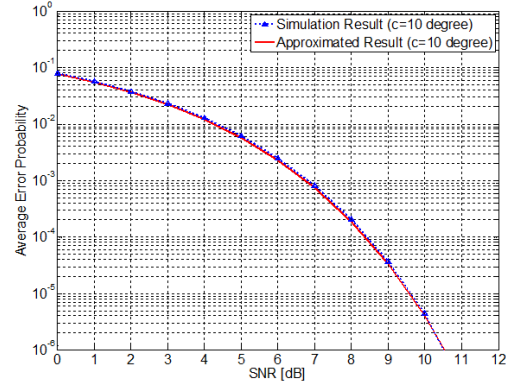


Fig. 5 Average error probability with small phase error ( $c=10^\circ$ )

As shown in Fig. 5, the average error probability curve of (10) is very close to the simulation result when the phase error is small. Simulation and approximation results with a large phase error ( $c=30^\circ, 50^\circ$ ) are shown in Fig. 6.

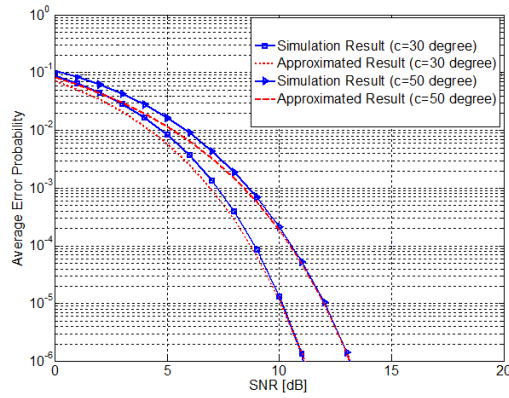


Fig. 6 Average error probability with large phase error ( $c=30^\circ, 50^\circ$ )

In the case of a large phase error, there is a little difference between the simulation and approximation results at low SNR, but the approximated average error probability shows accurate performance at high SNR.

#### V. CONCLUSION

In this paper, we present a simple approximation for the average error probability of BPSK in the presence of uniformly distributed phase error. This was achieved by using symmetry and periodicity of a sinusoidal function. The performance was verified by comparing the approximated average error probability with simulation results. The derived result should make the error probability analysis of wireless communication systems with phase error more efficient, and the presented approximation can be applied to other probability density cases.

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