Analytical Solutions of Kortweg-de Vries (KdV) Equation

Foad Saadi, M. Jalali Azizpour and S.A. Zahedi

Abstract—The objective of this paper is to present a comparative study of Homotopy Perturbation Method (HPM), Variational Iteration Method (VIM) and Homotopy Analysis Method (HAM) for the semi analytical solution of Kortweg-de Vries (KdV) type equation called KdV. The study have been highlighted the efficiency and capability of aforementioned methods in solving these nonlinear problems which has been arisen from a number of important physical phenomenon.

Keywords—Variational Iteration Method (VIM), Homotopy Perturbation Method (HPM), Homotopy Analysis Method (HAM), KdV Equation.

I. INTRODUCTION

IT was 1895 that Korteweg and Vries derived KdV equation to model Russell's phenomenon of solitons [1] like shallow water waves with small but finite amplitudes [2]. Solitons are localized waves that propagate without change of it's shape and velocity properties and stable against mutual collision [3]. It has also been used to describe a number of important physical phenomena such as magneto hydrodynamics waves in warm plasma, acoustic waves in an inharmonic crystal and ion-acoustic waves [4]. Consider three models of KdV equation called KdV, K (2,2) and modified KdV [1] as given respectively by:

$$u_t - 3(u^2)_x + u_{xxx} = 0 \tag{1}$$

$$u_t + (u^2)_x + (u^2)_{xxx} = 0 \tag{2}$$

and

$$u_t + \frac{1}{2}(u^2)_x - u_{xx} = 0. \tag{3}$$

Eq. (1) is the pioneering equation that gives rise to solitary wave solutions. Solitons, which are waves with infinite support, are generated as a result of the balance between the nonlinear convection and the linear dispersion in the above equations. Solitons are localized waves that propagate without change of their shape and velocity properties and stable against mutual collisions [5]. The equation of K (n,n):

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$$u_t + (u^n)_x + (u^n)_{xxx} = 0 \tag{4}$$

is the pioneering equation for compactions. In solitary waves theory, compactons are defined as is the pioneering equation for compactons. In solitary waves theory, compactons are defined as solitons with finite wave lengths or solitons free of exponential tails [6]. Compactons are generated as a result of the delicate interaction between nonlinear convection with the genuine nonlinear dispersion in (4).

Finally, the modified KdV equation appears in fluid mechanics. This equation incorporates both convection and diffusion in fluid dynamics and is used to describe the structure of shock waves [5]. Hence, KdV type equations have significant roles in engineering and physics. Besides, the analytical solutions of these governing equations may guide authors to know the described process deeply and sometimes leads them to know some facts that are not simply understood through common observations, but it is quite difficult to obtain the analytical solution of these problems as these are functioning highly nonlinear.

Many researchers studied the similar kinds of problems in other applications [7]-[11] and many powerful methods have been proposed to seek the exact solutions of nonlinear differential equations; for instance, Backlund transformation [12],[13], Darboux transformation [14] and the inverse scattering method [15].

In this paper, He's variational iteration method (VIM) and homotopy perturbation method (HPM) [7]-[9] and Liao's homotopy analysis method (HAM) [10] are used to conduct an analytic study on the KdV in order to show all the methods above, are capable in solving a large number of linear or nonlinear differential equations, also all the aforementioned methods give rapidly convergent successive approximations of the exact solution if such solution exists, otherwise approximations can be used for numerical purposes.

II. VARIATIONAL ITERATION METHOD

To illustrate the basic concepts of variational iteration method, we consider the following deferential equation [5]:

$$Lu + Nu = g(x) \tag{5}$$

Where is a linear operator, a nonlinear operator, and a heterogeneous term. According to VIM, we can construct a correction functional as follows [5]:

$$u_{n+1}(x) = u_n(x) +$$
$$\int_0^x \lambda \{ L\, u_n(\tau) + N\, \tilde{u}_n(\tau) - g(\tau) \} d\tau \qquad (6)$$

Where λ is a general Lagrangian multiplier [8], [9], which can be identified optimally via the variational theory [7], the subscript n indicates the nth order approximation u_n which is considered as a restricted variation, i.e. $\delta u_n = 0$.

III. MATH HOMOTOPY PERTURBATION METHOD

To illustrate the basic ideas of this method, we consider the following nonlinear differential Equation:

$$A(u) - f(r) = 0. \quad r \in \Omega \qquad (7)$$

Considering the boundary conditions of:

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma \qquad (8)$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω. The operator A can be divided into two parts of L and N, where L is the linear part, while N is a nonlinear one. Eq. (7) can, therefore, be rewritten as:

$$L(u) + N(u) - f(r) = 0 \qquad (9)$$

By the homotopy technique, we construct a homotopy as $v(r,p): \Omega \times [0,1] \to \Re$ which satisfies [16]:

$$H(v,p) = (1-p)[L(v) - L(u_0)] +$$
$$p[A(v) - f(r)] = 0 \; P \in [0,1] \; r \in \Omega \qquad (10)$$

Where $p \in [0,1]$ is an embedding parameter and u_0 is an initial approximation of Eq. (10) which satisfies the boundary conditions. Obviously, considering Eq. (10) we will have:

$$H(v,0) = L(v) - l(u_0) = 0 \qquad (11)$$
$$H(v,1) = A(v) - f(r) = 0 \qquad (12)$$

The changing process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation, and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopy. According to HPM, we can first use the embedding parameter p as a "small parameter", and assume that the solution of Eq. (10) can be written as a power series in p:

$$v = v_0 + p v_1 + p^2 v_2 + \cdots \qquad (13)$$

Setting $p = 1$ results in the approximate solution of Eq. (10):

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \ldots \qquad (14)$$

The combination of the perturbation method and the homotopy method is called the HPM, which lacks the limitations of the conventional perturbation methods, although this technique can have full advantages of the conventional perturbation techniques. The series (14) is convergent for most cases. However, the convergence rate depends on the nonlinear operator $A(v)$.

IV. HOMOTOPY ANALYSIS METHOD

Consider the following differential equation [17, 18]:

$$N[u(\tau)] = 0 \qquad (15)$$

where, N is a nonlinear operator, τ denotes an independent variable, $u(\tau)$ is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the conventional homotopy method, Liao [17] constructed the so-called zero-order deformation equation as:
where $p \in [0,1]$ is the embedding parameter, $h \neq 0$ a nonzero auxiliary parameter, $H(\tau) \neq 0$ an auxiliary function, L an auxiliary linear operator, $u_0(\tau)$ an initial guess of

$$(1-p)L[\phi(\tau;p) - u_0(\tau)] = phH(\tau)N[\phi(\tau;p)] \qquad (16)$$

$u(\tau)$ and $\phi(\tau; p)$ is an unknown function. It is important to have enough freedom to choose auxiliary unknowns in HAM. Obviously, when $p = 0$, and $p = 1$, it holds:
$\phi(\tau;0) = u_0(\tau)$ and $\phi(\tau;1) = u(\tau)$.

Thus, as p increases from 0 to 1, the solution $\phi(\tau;p)$ varies from the initial guess, $u_0(\tau)$ to the solution $u(\tau)$. Expanding $\phi(\tau;p)$ in Taylor series with respect to p, we have:

$$\phi(\tau;p) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau) p^m \qquad (17)$$

Where,

$$u_m(\tau) = \frac{1}{m!} \frac{\partial^m \phi(\tau;p)}{\partial p^m} \bigg|_{p=0} \qquad (18)$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are quite properly chosen, the series (17) converges at $p = 1$ then we have:

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau) \qquad (19)$$

This must be one of the solutions of the original nonlinear equation, as proved by Liao [17]. As $h = -1$ and $H(\tau) = 1$, Eq. (16) becomes:

$$(1-p)L[\phi(\tau;p) - ul0(\tau)] + pN[\phi(\tau;p)] = 0 \qquad (20)$$

$$u_{n+1}(x,t) = u_n(x,t) -$$
$$\int_0^t \{ u_\tau - 3(u)^2_x + u_{xxx} \} d\tau \qquad (31)$$

which is mostly used in HPM, whereas the solution can be obtained directly without using Taylor series. According to

the equation (19), the governing equation can be deduced from the zero-order deformation Eq. (17). The vector is defined as:

$$\vec{u}_n = \{u_0(\tau), u_1(\tau), \ldots, u_n(\tau)\} \qquad (21)$$

Differentiating Eq. (17) for m times with respect to the embedding parameter p, and then setting $p = 0$ and finally dividing them by $m!$, we will have the so-called m^{th}-order deformation equation as:

$$L[u_m(\tau) - \chi_m u_{m-1}(\tau)] = h H(\tau) R_m(\vec{u}_{m-1}) \qquad (22)$$

where,

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(\tau; p)]}{\partial p^{m-1}} \qquad (23)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \qquad (24)$$

It should be emphasized that $u_m(\tau)$ for $m \geq 1$ is governed by the linear Eq. (23) with the linear boundary conditions coming from the original problem, which can be easily solved using symbolic computation software.

V. SOLUTION KDV BY VARIATIONAL ITERATION METHOD

Considering the KdV equation as:

$$u_t - 3(u)^2_x + u_{xxx} = 0, \begin{cases} -\infty < x < +\infty \\ t > 0 \end{cases} \qquad (25)$$

with the following initial condition:

$$u(x,0) = 6x \qquad (26)$$

To solve Eqs. (25) and (26) using VIM, we have the correction functional as:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \{u_t - u_{xxt} + u u_x\} d\tau \qquad (27)$$

where $u u_x$ indicates the restricted variations; i.e. $\delta(u u_x) = 0$. Making the above correction functional stationary, we obtain the following stationary conditions:

$$1 + \lambda\big|_{\tau=1} = 0 \qquad (28)$$

$$\lambda' = 0 \qquad (29)$$

The Lagrangian multiplier can therefore be identified as:

$$\lambda = -1 \qquad (30)$$

Substituting Eq. (30) into the correction functional equation system (27) results in the following iteration formula:

Each result obtained from Eq. (31) is $u(x,t)$ with its own error relative to the exact solution, but higher number iterations leads us to obtain results closer to the exact solution. Using the iteration formula (31) and the initial condition, first iterations were mad as follow
The second iteration results in:

And finally, the fifth iteration results in:

$$u_5(x,t) = 6x(1 + 36t + 1296t^2 + 15552t^3 + 1679616t^4 + 60466176t^5) + small\ terms \qquad (34)$$

It is obvious that $u_n(x,t)$ converges to $\dfrac{6x}{1 - 36t}$ as an exact solution for Eqs. (25, 26).

$$u_1(x,t) = 6x(1 + 36t) \qquad (32)$$

$$u_2(x,t) = 6x(1 + 36t + 1296t^2 + 15552t^3) \qquad (33)$$

VI. SOLUTION KDV BY HOMOTOPY PERTURBATION METHOD

With the same as mentioned previously, the equation is as:

$$u_t - 3(u)^2_x + u_{xxx} = 0, \begin{cases} -\infty < x < +\infty \\ t > 0 \end{cases} \qquad (35)$$

with the initial condition of:

$$u(x,0) = 6x. \qquad (36)$$

Substituting Eq. (35) into (36) and then substituting v from (14) and rearranging it as a power series in p, we have an equation system including $n+1$ equations to be simultaneously solved; n is the order of p in Eq. (14). Assuming $n = 5$, the system is as follows:

$$\begin{aligned}
&u_{0t} = 0 \quad \rightarrow \quad u_0(x,0) = 6x \\
&u_{1t} - 6u_0 u_{0x} + u_{0xxx} = 0 \rightarrow \quad u_1(x,0) = 0 \\
&\begin{cases} u_{2t} - 6u_1 u_{0x} - \\ 6u_0 u_{1x} + u_{1xxx} \end{cases} = 0 \quad \rightarrow \quad u_2(x,0) = 0 \\
&\begin{cases} u_{3t} - 6u_0 u_{2x} - 6u_1 u_{1x} \\ -6u_2 u_{0x} + u_{2xxx} \end{cases} = 0 \rightarrow u_3(x,0) = 0 \\
&\begin{cases} u_{4t} - 6u_3 u_{0x} - 6u_0 u_{3x} - \\ 6u_2 u_{1x} - 6u_1 u_{2x} + u_{3xxx} \end{cases} = 0 \rightarrow u_4(x,t) = 0 \\
&\begin{cases} u_{5t} - 6u_3 u_{1x} - 6u_4 u_{0x} - 6u_1 u_{3x} \\ -6u_2 u_{2x} - 6u_0 u_{4x} + u_{4xxx} \end{cases} = 0 \rightarrow u_5(x,t) = 0
\end{aligned} \qquad (37)$$

One can now try to obtain a solution for equation system (37), in the form of:

$$\begin{aligned}
&u_0(x,t) = 6x \\
&u_1(x,t) = 6x(36t) \\
&u_2(x,t) = 6x(1296t^2) \\
&u_3(x,t) = 6x(46656t^3) \\
&u_4(x,t) = 6x(1679616t^4) \\
&u_5(x,t) = 6x(60466176t^5)
\end{aligned} \qquad (38)$$

Having $u_i, i = 0,1,...5$, the solution $u(x,t)$ is as:

$$u(x,t) = \sum_{i=0}^{5} u_i(x,t) = 6x$$
$$\left[1 + 36t + (36t)^2 + (36t)^3 + (36t)^4 + (36t)^5\right] \qquad (39)$$

Trying higher iterations, we can obtain the exact solution of Eqs. (35,36) in the form of : $u(x,t) = \dfrac{6x}{1 - 36t}$

VII. SOLUTION KDV BY HOMOTOPY ANALYSIS METHOD

Consider Eqs. (25, 26) and let us solve them through HAM with proper assignment of $H(\tau) = 1$ subject to the initial condition and assuming $m = 4$:

$$u_0 = 6x$$
$$u_1 = 6x(-36ht)$$
$$u_2 = 6x(-36ht - 36h^2t + 1296h^2t^2) \qquad (40)$$
$$u_3 = 6x(-36ht - 36h^3t - 72h^2t + 2592h^3t^2 +$$
$$2592h^2t^2 - 46656h^2t^2)$$
$$u_4 = 6x(-36ht - 36h^4t - 108h^2t - 108h^3t + 7776h^3t^2$$
$$+ 3888h^2t^2 + 3888h^4t^2 - 46656h^2t^2 - 139968h^4t^3$$
$$- 139968h^3t^3 + 1679616h^4t^4)$$

The obtained h curve is drawn in Fig.1 As demonstrated, when h varies from -2.8 to -1 the results are independent from this parameter.
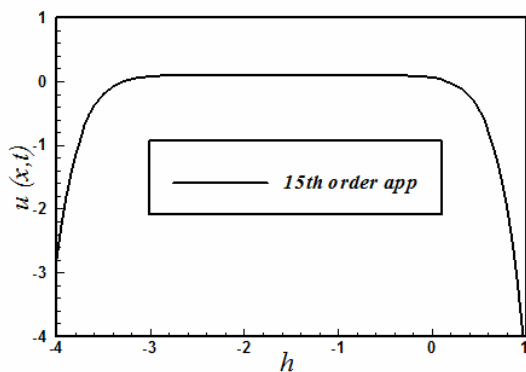

Fig.1. h curve in $x=0.01$ and $t=0.01$

Trying higher iterations with the unique and proper assignment of $h = -1$, (resulting in HPM), the results as

$\sum_{m=0}^{\infty} u_m(x,t)$ converge to the exact solution as mentioned

before in the form of $\dfrac{6x}{1 - 36t}$.

VIII. CONCLUSION

The main goals of this study were the assessment of capability of the He's Variational Iteration Method (VIM) and Homotopy Perturbation Method (HPM) and the Liao's Homotopy Analysis Method (HAM) to solve the KdV type equationds. The KdV equations that arise from many important and practical physical phenomena were examined for rational solutions.

In clear conclusion, all the above-mentioned methods were capable to solve this set of problems with successive rapidly convergent approximations without any restrictive assumptions or transformations causing changes in the physical properties of the problems. Also, adding up the number of iterations leads to the explicit solutions for the problems. Among these three methods, VIM is very comprehensible as it reduces the size of calculations and also it's iterations are direct and straightforward, however in HAM the auxiliary parameter h provides a convenient way to obtain the solution for fewer approximations. On the other hand, both HPM and HAM do not require small parameters in the equation so that the limitations of the conventional perturbation methods can be eliminated and thereby the calculations are simple and straight forward, though HPM can be more convenient.

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