Analytical Solution of Stress Distribution on a Hollow Cylindrical Fiber of a Composite with Cylindrical Volume Element under Axial Loading

M. H. Kargarnovin, K. Momeni

Abstract**—** The study of the stress distribution on a hollow cylindrical fiber placed in a composite material is considered in this work and an analytical solution for this stress distribution has been constructed. Finally some parameters such as fiber's thickness and fiber's length are considered and their effects on the distribution of stress have been investigated. For finding the governing relations, continuity equations for the axisymmetric problem in cylindrical coordinate (r,θ,z) are considered. Then by assuming some conditions and solving the governing equations and applying the boundary conditions, an equation relates the stress applied to the representative volume element with the stress distribution on the fiber has been found.

Keywords**—**Axial Loading, Composite, Hollow Cylindrical Fiber, Stress Distribution.

I. Introduction

NOWADAYS as a new generation of advanced materials, composite materials have been used widely in different manufacturing and industrialized applications. The basic constituent part of these types of compound structures is related to its reinforcement part, which causes rather pronounced improvement in the mechanical properties of such compound material. Of great importance is the configuration of employed reinforcement. There are many different reinforcement topologies such as rod, pyramid, spherical and so on.

Primary analysis on the effect of orientation of the fibers on the stiffness and strength of materials is reported as early as 1952 [1]. Later on the superior mechanical properties of fibrous composites and the properties of their constituents were reported in a research dated back to 1964 [2]. The first research work on the estimation of strength distribution of a fiber embedded in a single-fiber composite using elasto-plastic approach is reported at 2001, [3]. Other works have been done since then, [4-10].

It has been noticed that no analytical work on the reinforcement effect of a hollow fiber embedded in a composite medium was reported until now. Therefore, in this paper a hollow cylindrical shape fiber has been considered as a reinforcement phase embedded in cylindrical matrix where an axial load is applied at the ends of compound medium.

II. Geometry and Assumptions

Fig. 1 illustrates the geometry of the compound structure under consideration for further analysis. It comprises of an isotropic external cylinder with length of 2L and radius R, plus another isotropic hollow cylindrical fiber with length of $2L_f$ and inner and outer radii of r_i and r_o, respectively. As it is seen, the hollow fiber is embedded within the external cylinder and a perfect bounding is established between them. For analysis of this problem a representative volume element (RVE) of such structure will be considered.

In order to analyze this problem following assumptions are made:

1. Both matrix and fiber are made of isotropic but different materials.
2. A perfect bonding between fiber and matrix exists.
3. Radial strain is much smaller than the axial strain. $\left|\frac{\partial u}{\partial z}\right| \prec\prec \left|\frac{\partial w}{\partial r}\right|$.
4. The applied axial force will be transferred to the hollow fiber via surrounding matrix.
5. The cross section area of the fiber is smaller than the RVE's or in other words the volume fraction of the fiber is much less than the volume fraction of the matrix.
6. The induced radial and hoop stresses are much smaller than the axial stress $\sigma_{\theta\theta} + \sigma_{rr} \prec\prec \sigma_{zz}$ which is an acceptable assumption especially for the case of materials with small Poisson ratio.
7. The effect of body forces compared to other applied forces can be ignored.

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A great amount of efforts have been done for finding the stress distribution function, but all of them have been carried out for the case of rod shape fibers. In some of these works for their analysis they have used models such as Rule of Mixture, Cox Model and Kelly-Tyson Model [11-15].

In solving this problem, three-dimensional theory of elasticity is employed for further analysis. The governing equilibrium equations of an axisymmetric problem in cylindrical coordinate (r,θ,z) is considered first, then by using kinematical relations, constitutive equations and finally applying boundary conditions in both reinforced and pure matrix parts in the RVE, the governing differential equations of stress distribution on the fiber is obtained. Solution of such differential equations yields the stress distribution on the hollow cylindrical shape fiber.

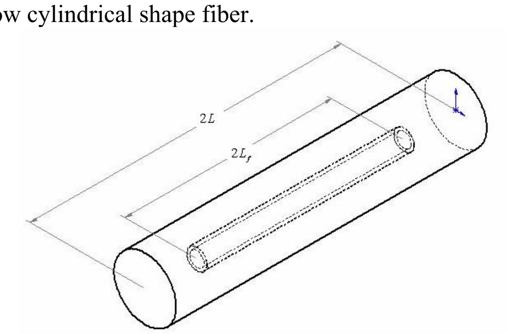

Fig. 1 Representative Volume Element of the Composite

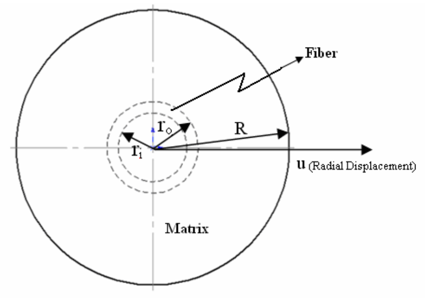

Fig. 2 Front view of the RVE and specifications of the problem's parameters

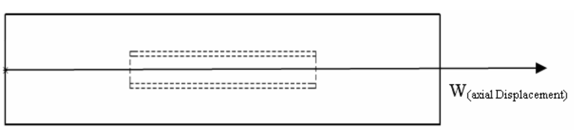

Fig. 3 Side view of RVE and specification of the axial displacement direction

III. Modeling and Simulation

The governing equilibrium equations of an axisymmetric problem in cylindrical coordinate by neglecting the body forces are as follows [16]:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \qquad \text{(1-a)}$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0 \qquad \text{(1-b)}$$

in which referred to Fig.(1) the σ_{ij}'s are the components of stress tensor in the cylindrical coordinate. The kinematical relations in such coordinate system are as follows [16]:

$$\varepsilon_{rr} = \frac{\partial u}{\partial r} \qquad \text{(2-a)}$$

$$\varepsilon_{\theta\theta} = \frac{u}{r} \qquad \text{(2-b)}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \qquad \text{(2-c)}$$

$$\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \qquad \text{(2-d)}$$

in which u and w represent the axial and radial displacement of any point within the media, respectively. Moreover, ε_{rr} , $\varepsilon_{\theta\theta}$, ε_{zz} and γ_{rz} are the only non-zero components of the strain tensor.

The constitutive equations, i.e. Hooke's Law are [16]:

$$\varepsilon_{rr} = \frac{1}{E}\left[\sigma_{rr} - \nu\left(\sigma_{\theta\theta} + \sigma_{zz}\right)\right] \qquad \text{(3-a)}$$

$$\varepsilon_{\theta\theta} = \frac{1}{E}\left[\sigma_{\theta\theta} - \nu\left(\sigma_{zz} + \sigma_{rr}\right)\right] \qquad \text{(3-b)}$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{rr} + \sigma_{\theta\theta}\right)\right] \qquad \text{(3-c)}$$

$$\gamma_{rz} = \frac{\sigma_{rz}}{G} \qquad \text{(3-d)}$$

in which E, v and G are the Young's modulus, Poisson's ratio and shear modulus, respectively.

Referred to Fig. (1), the governing boundary conditions for the reinforced and pure matrix are as follows:

$$\vec{t}^{\,m}\,|_{r=R} = \vec{0}, \qquad \text{(4-a)}$$

$$\vec{t}^{\,m}\,|_{z=\pm L} = \pm\sigma \hat{e}_3 \qquad \text{(4-b)}$$

where the interfacial traction continuity conditions are:

$$\vec{t}^{\,f}\,|_{-L_f<z<L_f,\, r=r_o} = \vec{t}^{\,m}\,|_{-L_f<z<L_f,\, r=r_o} \qquad \text{(5-a)}$$

$$\vec{t}^{\,f}\,|_{z=\pm L_f,\, r_i<r<r_o} = \vec{t}^{\,m}\,|_{z=\pm L_f,\, r_i<r<r_o} \qquad \text{(5-b)}$$

in which $\vec{t}$ is the traction vector, σ is the axial normal stress uniformly applied at the RVE's ends and superscripts f and m denote the fiber and matrix mediums, respectively.

A. Solution in the Reinforced Region

By integrating the Eq. (1-b) over the cross sectional area with respect to r from r_i to r_o for the reinforcing fiber one would get:

$$\frac{1}{\pi\left(r_o^2-r_i^2\right)}\int_{r_i}^{r_o}\frac{\partial\sigma_{zz}^f}{\partial z}(2\pi r)\,dr+\frac{1}{\pi\left(r_o^2-r_i^2\right)}\int_{r_i}^{r_o}\frac{1}{r}\frac{\partial}{\partial r}\left(r\sigma_{rz}^f\right)(2\pi r)\,dr=0 \tag{6}$$

The average axial normal stress over the cross section of the effective fiber can be defined as:

$$\bar{\sigma}_{zz}^f=\frac{1}{\pi\left(r_o^2-r_i^2\right)}\int_{r_i}^{r_o}\sigma_{zz}^f(r,z)\cdot(2\pi r)\,dr \tag{7}$$

Differentiation of Eq. (7) with respect to z from ($-L_f$ to L_f) and using Eq. (6) leads to:

$$\frac{d\bar{\sigma}_{zz}^f}{dz}=-\frac{2}{r_o^2-r_i^2}\left[r_o\sigma_o^f-r_i\sigma_i^f\right] \tag{8}$$

In this relation σ_o^f and σ_i^f represent the shear stresses in the interfaces between the matrix and fiber at r_i and r_o, respectively. By assuming that the matrix material will not penetrate into the hollow part of the fiber σ_i^f will become zero and the Eq. (8) will reduce to:

$$\frac{d\bar{\sigma}_{zz}^f}{dz}=-\frac{2}{r_o^2-r_i^2}r_o\sigma_o^f \tag{9}$$

Eq.(9) indirectly indicates that the σ_o^f is a function of z.
Hence we set:

$$\frac{\partial\sigma_{zz}^f}{\partial z}=f(z) \tag{10}$$

and by using Eq. (1-b) we have:

$$\frac{\partial\sigma_{rz}^f}{\partial r}+f(z)+\frac{\sigma_{rz}^f}{r}=0 \tag{11}$$

Eq. (11) is a first order linear differential equation in terms of σ_{rz}^f which its solution leads to:

$$\sigma_{rz}^f=-\frac{1}{2}f(z)\cdot r+\frac{c_1}{r} \tag{12}$$

Noting that $\sigma_{rz}^f\big|_{r=r_i}=0$ due to no matrix penetration into the hollow part of the fiber we have:

$$\sigma_{rz}^f=\frac{1}{2}f(z)\cdot r\left[\frac{r_i^2}{r^2}-1\right] \tag{13}$$

By applying the boundary condition $\sigma_{rz}^f|_{r=r_o}=\sigma_o^f$ to the Eq. (13), one would get:

$$f(z)=\frac{2}{r_o}\left[\frac{r_o}{r_i^2-r_o^2}\right]\sigma_o^f=\frac{2r_o}{r_i^2-r_o^2}\sigma_o^f(z) \tag{14-a}$$

$$\sigma_{rz}^f=\frac{r_o\cdot r}{r_i^2-r_o^2}\left[\frac{r_i^2}{r^2}-1\right]\sigma_o^f \tag{14-b}$$

In component form the boundary conditions of Eq's. (4 a,b) are:

$$\sigma_{rr}^m|_{r=R}=0, \tag{15-a}$$

$$\sigma_{zr}^m|_{r=R}=0 \tag{15-b}$$

Similarly from Eq. (5 a, b) we will have:

$$\sigma_{rr}^f|_{-L_f<z<L_f,r=r_o}=\sigma_{rr}^m|_{-L_f<z<L_f,r=r_o} \tag{16-a}$$

$$\sigma_{zr}^f|_{-L_f<z<L_f,r=r_o}=\sigma_{zr}^m|_{-L_f<z<L_f,r=r_o} \tag{16-b}$$

Now by integrating the Eq. (1-b) over the cross sectional area with respect to r from r_o to R and using Eq. (15-b) one would get:

$$\frac{d\bar{\sigma}_{zz}^m}{dz}=\frac{2r_o}{R^2-r_o^2}\tau_o^f \tag{17}$$

where,

$$\bar{\sigma}_{zz}^m(z)=\frac{1}{\pi\left(R^2-r_o^2\right)}\int_{r_o}^{R}\sigma_{zz}^m(r,z)\cdot(2\pi r)\,dr \tag{18}$$

A close inspection of Eq.(18) indicates that the σ_{zz}^m will be a function of z, moreover we have;

$$\frac{\partial\sigma_{zz}^m}{\partial z}=g(z) \tag{19}$$

Were $g(z)$ is an unknown function that must be determined. By substituting Eq. (19) in (1-b) and integrating over the cross-section with respect to r form r_o to R leads to:

$$g(z)=\frac{2}{R^2-r^2}r\cdot\sigma_{rz}^m(r)\Rightarrow\sigma_{rz}^m=\frac{g(z)}{2}\cdot\frac{R^2-r^2}{r} \tag{20}$$

After combining Eq. (20) and Eq. (5-b), we have:

$$g(z) = \frac{2 \cdot r_o}{R^2 - r_o^2} \sigma_o^f \tag{21}$$

After back substitution of *g(z)* in Eq. (20),

$$\sigma_{rz}^m(r) = \frac{r_o}{R^2 - r_o^2} \left(\frac{R^2 - r^2}{r} \right) \cdot \sigma_o^f \tag{22}$$

In view of assumption III i.e. $\left|\frac{\partial u}{\partial z}\right| \prec\prec \left|\frac{\partial w}{\partial r}\right|$ and Eqs. (2-d) and (3-d),

$$\sigma_{rz}^f = G^f \frac{\partial w^f}{\partial r}, \tag{23-a}$$

$$\sigma_{rz}^m = G^m \frac{\partial w^m}{\partial r} \tag{23-b}$$

Using Eqs. (23) and Eq. (22) we have,

$$\sigma_o^f = G^m \frac{R^2 - r_o^2}{R^2 - r^2} \cdot \frac{r}{r_o} \cdot \frac{\partial w^m}{\partial r} \tag{24}$$

After integrating Eq. (24) over the cross-section with respect to r from r_o to R, one would get:

$$\sigma_o^f = G^m \frac{R^2 - r_o^2}{r_o} \frac{w_R^m - w_{r_o}^m}{R^2 \ln(R/r_o) - 1/2(R^2 - r_o^2)} \tag{25}$$

Now we substitute σ_o^f from Eq.(25) into Eq. (22) which leads to,

$$\sigma_{rz}^m(r) = G^m \frac{R^2 - r^2}{r} \left[\frac{w_R^m - w_{r_o}^m}{R^2 \ln(R/r_o) - 1/2(R^2 - r_o^2)} \right] \tag{26}$$

By back substitution of σ_{rz}^m from Eq. (26) into Eq. (23-b) and carrying out the integration, it results in:

$$w_r^m(r,z) = w_{r_o}^m + \left[\frac{(w_R^m - w_{r_o}^m) \cdot \left(R^2 \ln(r/r_o) - 1/2(r^2 - r_o^2)\right)}{R^2 \ln(R/r_o) - 1/2(R^2 - r_o^2)} \right] \tag{27}$$

By recalling assumption VI, i.e. $\sigma_{\theta\theta} + \sigma_{rr} \prec\prec \sigma_{zz}$ for both matrix and the fiber, then by using Eqs. (2-c) and (3-c) we have:

$$\sigma_{zz}^f = E^f \frac{\partial w^f}{\partial z}, \tag{28-a}$$

$$\sigma_{zz}^m = E^m \frac{\partial w^m}{\partial z} \tag{28-b}$$

After substituting Eqs. (27) into Eg. (28-b) it results in:

$$\sigma_{zz}^m = \sigma_{zz}^m|_{r=r_o} + \frac{R^2 \ln(r/r_o) - 1/2(r^2 - r_o^2)}{R^2 \ln(R/r_o) - 1/2(R^2 - r_o^2)} \left[\sigma_{zz}^m|_{r=R} - \sigma_{zz}^m|_{r=r_o} \right] \tag{29}$$

Now, consider the force balance over the composite cross-section along z axis:

$$\pi R^2 \sigma = \int_i^o \sigma_{zz}^f \cdot (2\pi r)dr + \int_o^R \sigma_{zz}^m (2\pi r)dr \tag{30}$$

By using Eqs. (7), (29) and (30) we have:

$$\sigma_{zz}^m|_{r=R} = \sigma_{zz}^m|_{r=r_o} + \frac{R^2 \ln(R/r_o) - 1/2(R^2 - r_o^2)}{R^2 \ln(R/r_o) - 1/4(R^2 - r_o^2) \cdot (3R^2 - r_o^2)} \times \left[\overline{\sigma}_{zz}^f (r_i^2 - r_o^2) + R^2 \sigma + \sigma_{zz}^m|_{r=r_o} (r_o^2 - R^2) \right] \tag{31}$$

From Eqs. (9), (25), (28) and (31) it follows that,

$$\frac{d^2 \overline{\sigma}_{zz}^f}{dz^2} = \frac{R^2 - r_o^2}{r_i^2 - r_o^2} \frac{1}{1 + \nu_m} \times \frac{\overline{\sigma}_{zz}^f (r_i^2 - r_o^2) + R^2 \sigma + \sigma_{zz}^m|_{r_o} (r_o^2 - R^2)}{R^4 \ln(R/r_o) - 1/4(R^2 - r_o^2)(3R^2 - r_o^2)} \tag{32}$$

In view of assumption II, i.e. perfect bounding between two media:

$$\varepsilon_{zz}^m|_{r=r_o} = \varepsilon_{zz}^f|_{r=r_o} \Rightarrow \sigma_z^m|_{r=r_o} = \frac{E^m}{E^f} \sigma_{zz}^f|_{r=r_o} \tag{33}$$

In view of assumption V, i.e. low volume fraction of the fiber,

$$\sigma_{zz}^f \approx \overline{\sigma}_{zz}^f \Rightarrow \sigma_{zz}^m|_{r=r_o} = \frac{E^m}{E^f} \overline{\sigma}_{zz}^f \tag{34}$$

From Eq. (34) and Eq. (32) it follows that:

$$\frac{d^2 \overline{\sigma}_{zz}^f}{dz^2} = \frac{R^2 - r_o^2}{r_i^2 - r_o^2} \frac{1}{1 + \nu_m} \times \frac{\overline{\sigma}_{zz}^f \left[(r_i^2 - r_o^2) + \frac{E^m}{E^f}(r_o^2 - R^2) \right] + R^2 \sigma}{R^4 \ln(R/r_o) - 1/4(R^2 - r_o^2)(3R^2 - r_o^2)} \tag{35}$$

By considering η as follows,

$$\eta = \frac{R^2 - r_o^2}{r_i^2 - r_o^2}\frac{1}{1+\nu_m}\times \frac{1}{R^4 \ln(R/r_o) - 1/4\left(R^2 - r_o^2\right)\left(3R^2 - r_o^2\right)} \tag{36}$$

Eq. (35) becomes:

$$\frac{d^2\overline{\sigma}_{zz}^f}{dz^2} - \eta\left[\left(r_i^2 - r_o^2\right) + \frac{E^m}{E^f}\left(r_o^2 - R^2\right)\right]\overline{\sigma}_{zz}^f = R^2\eta\sigma \tag{37}$$

Eq.(37) represents an ordinary differential equation with constant coefficients and has a solution as follows:

$$\overline{\sigma}_{zz}^f = A\cdot sh\left(\sqrt{\eta\left[\left(r_i^2 - r_o^2\right) + \frac{E^m}{E^f}\left(r_o^2 - R^2\right)\right]}z\right) + B\cdot ch\left(\sqrt{\eta\left[\left(r_i^2 - r_o^2\right) + \frac{E^m}{E^f}\left(r_o^2 - R^2\right)\right]}z\right) + \frac{R^2\sigma}{\left(r_i^2 - r_o^2\right) + \frac{E^m}{E^f}\left(r_o^2 - R^2\right)} \tag{38}$$

Putting Eq. (38) into Eq. (9) results in,

$$\overline{\sigma}_{zz}^f = A\cdot\sqrt{\eta\left[\left(r_i^2 - r_o^2\right) + \frac{E^m}{E^f}\left(r_o^2 - R^2\right)\right]}\cdot ch\left(\sqrt{\eta\left[\left(r_i^2 - r_o^2\right) + \frac{E^m}{E^f}\left(r_o^2 - R^2\right)\right]}z\right) + B\cdot\sqrt{\eta\left[\left(r_i^2 - r_o^2\right) + \frac{E^m}{E^f}\left(r_o^2 - R^2\right)\right]}\cdot sh\left(\sqrt{\eta\left[\left(r_i^2 - r_o^2\right) + \frac{E^m}{E^f}\left(r_o^2 - R^2\right)\right]}z\right) = \frac{2r_o\sigma_o^f}{\left(r_i^2 - r_o^2\right)} \tag{39}$$

Eq. (39) can be put in the following form as:

$$\frac{2r_o\sigma_o^f}{r_i^2 - r_o^2} = \frac{1}{2}\frac{r_i^2 - r_o^2}{r_o}\cdot\alpha\cdot\left[A\cdot ch(\alpha\cdot z) + B\cdot sh(\alpha\cdot z)\right] \tag{40}$$

in which α is,

$$\alpha = \sqrt{\eta\left[\left(r_i^2 - r_o^2\right) + \frac{E^m}{E^f}\left(r_o^2 - R^2\right)\right]} \tag{41}$$

Using Eqs. (31), (34) and (38) in (29) result in,

$$\sigma_{zz}^f = \frac{R^2\ln(r/r_o) - 1/2\left(r^2 - r_o^2\right)}{R^2\ln(R/r_o) - 1/4\left(R^2 - r_o^2\right)\cdot\left(3R^2 - r_o^2\right)}R^2\sigma + \left\{\frac{E^m}{E^f} - \frac{\left[R^2\ln(R/r_o) - 1/2\left(R^2 - r_o^2\right)\right]\cdot\left[\left(r_o^2 - r_i^2\right) + \frac{E^m}{E^f}\left(R^2 - r_o^2\right)\right]}{R^2\ln(R/r_o) - 1/4\left(R^2 - r_o^2\right)\cdot\left(3R^2 - r_o^2\right)}\right\}\times \left[A\cdot sh(\alpha\cdot z) + B\cdot ch(\alpha\cdot z) + \frac{R^2\sigma}{\left[\left(r_o^2 - r_i^2\right) + \frac{E^m}{E^f}\left(R^2 - r_o^2\right)\right]}\right] \tag{42}$$

Putting Eq. (40) in Eq. (14-b) results in,

$$\sigma_{rz}^f = \frac{r}{2}\left[\frac{r_i^2}{r^2} - 1\right]\left\{A\cdot\alpha\cdot ch(\alpha\cdot z) + B\cdot\alpha\cdot sh(\alpha\cdot z)\right\} \tag{43}$$

Finally, the use of Eq. (40) in (22) gives

$$\sigma_{rz}^m = \frac{r_i^2 - r_o^2}{2\left(R^2 - r_o^2\right)}\left(\frac{R^2}{r} - r\right)\left[A\cdot\alpha\cdot ch(\alpha\cdot z) + B\cdot\alpha\cdot sh(\alpha\cdot z)\right] \tag{44}$$

Eqs. (38), (40) and (42) to (44) represent some expressions for unknowns stresses of $\overline{\sigma}_{zz}^f, \sigma_{rz}^f, \sigma_{zz}^m$ and σ_{rz}^m. The only remaining thing is specifying the constants A and B in these relations. To calculate the value of these two constant coefficients the pure matrix region must be considered. That is a matrix with an embedded solid virtual fiber with zero inner radius and outer radius equal to the outer radius of the actual hollow fiber in the reinforced region.

B. Solution in Pure Matrix Region

Equation (38) for this region will be as follows:

$$\overline{\sigma}_{zz}^m = A'\cdot sh\left(\sqrt{\frac{R^2 - r_o^2}{r_o^2}\frac{1}{1+\nu_m}\frac{R^2}{R^4\ln(R/r_o) - 1/4\left(R^2 - r_o^2\right)\left(3R^2 - r_o^2\right)}}z\right) + B'\cdot ch\left(\sqrt{\frac{R^2 - r_o^2}{r_o^2}\frac{1}{1+\nu_m}\frac{R^2}{R^4\ln(R/r_o) - 1/4\left(R^2 - r_o^2\right)\left(3R^2 - r_o^2\right)}}z\right) + \sigma \tag{45}$$

A' and B' are constants, the same as A and B in Eq. (38) but in the pure matrix region.
By substituting Eq. (45) into Eq. (9) and the result into Eq.(14-b) one would get:

$$\sigma_{rz}^{fm} = -\frac{r}{2}\left(A'\cdot\beta\cdot ch(\beta\cdot z) + B'\cdot\beta\cdot sh(\beta\cdot z)\right) \tag{46}$$

in which:

$$\beta = \sqrt{\frac{R^2 - r_o^2}{r_o^2}\frac{1}{1+\nu_m}\frac{R^2}{R^4\ln(R/r_o) - 1/4\left(R^2 - r_o^2\right)\left(3R^2 - r_o^2\right)}} \tag{47}$$

In order to calculate the unknown coefficients A' and B'

boundary conditions listed in Eqs. (4-b) and (5-b) have to be imposed. After applying the boundary conditions and doing some further calculations, A' and B' will be obtained as followings:

$$A'=0, \qquad B'=\frac{\sigma}{ch\left(\alpha\cdot L_f\right)}\left[1-\frac{R^2}{\left(r_o^2-r_i^2\right)+\frac{E^m}{E^f}\left(R^2-r_o^2\right)}\right] \tag{48}$$

Substituting the values of A' and B' in equations (38), (40) and (42-44), the following relations will result in:

$$\begin{aligned}
\bar{\sigma}_{zz}^{f} &= \sigma\left(1-\frac{R^2}{\left(r_o^2-r_i^2\right)+\frac{E^m}{E^f}\left(R^2-r_o^2\right)}\right)\frac{ch\left(\alpha\cdot z\right)}{ch\left(\alpha\cdot L_f\right)}+\frac{\sigma R^2}{\left(r_o^2-r_i^2\right)+\frac{E^m}{E^f}\left(R^2-r_o^2\right)}\\
\sigma_{o}^{f} &= \frac{r_i^2-r_o^2}{2r_o}\left[\frac{\alpha}{ch\left(\alpha\cdot L_f\right)}\left(1-\frac{R^2}{\left(r_o^2-r_i^2\right)+\frac{E^m}{E^f}\left(R^2-r_o^2\right)}\right)\cdot sh\left(\alpha\cdot z\right)\right]\cdot\sigma\\
\sigma_{zz}^{m} &= \frac{R^2\ln\left(r/r_o\right)-1/2\left(r^2-r_o^2\right)}{R^2\ln\left(R/r_o\right)-1/4\left(R^2-r_o^2\right)\cdot\left(3R^2-r_o^2\right)}R^2\sigma+\\
&\left\{\frac{E^m}{E^f}-\frac{\left[R^2\ln\left(R/r_o\right)-1/2\left(R^2-r_o^2\right)\right]\cdot\left[\left(r_o^2-r_i^2\right)+\frac{E^m}{E^f}\left(R^2-r_o^2\right)\right]}{R^2\ln\left(R/r_o\right)-1/4\left(R^2-r_o^2\right)\cdot\left(3R^2-r_o^2\right)}\right\}\times\\
&\left[\frac{ch\left(\alpha\cdot z\right)}{ch\left(\alpha\cdot L_f\right)}\left(1-\frac{R^2}{\left(r_o^2-r_i^2\right)+\frac{E^m}{E^f}\left(R^2-r_o^2\right)}\right)+\frac{R^2}{\left[\left(r_o^2-r_i^2\right)+\frac{E^m}{E^f}\left(R^2-r_o^2\right)\right]}\right]\cdot\sigma\\
\sigma_{rz}^{f} &= \frac{r}{2}\left[\frac{r_i^2}{r^2}-1\right]\left[\frac{\alpha\cdot sh\left(\alpha\cdot z\right)}{ch\left(\alpha\cdot L_f\right)}\left(1-\frac{R^2}{\left(r_o^2-r_i^2\right)+\frac{E^m}{E^f}\left(R^2-r_o^2\right)}\right)\right]\cdot\sigma\\
\sigma_{rz}^{m} &= \frac{r_i^2-r_o^2}{2\left(R^2-r_o^2\right)}\left(\frac{R^2}{r}-r\right)\left[\frac{\alpha\cdot sh\left(\alpha\cdot z\right)}{ch\left(\alpha\cdot L_f\right)}\left(1-\frac{R^2}{\left(r_o^2-r_i^2\right)+\frac{E^m}{E^f}\left(R^2-r_o^2\right)}\right)\right]\cdot\sigma
\end{aligned} \tag{49}$$

In order to verify the validity of the above relations if the inner radius of the hollow fiber i.e. r_i is approached to zero, which corresponds to a rod shape fiber, the formulas will be the same as the ones for the case of rod shape cylinder obtained by Cox [1].

IV. NUMERICAL SOLUTION AND CASE STUDY

Based on derived relations, for a hollow carbon fiber placed in the epoxy matrix and under an axial constant traction, the shear stress distribution in the medium is calculated and the results are represented in different graphs. Following data represent the geometry and material constants [11]:
E_f = 3800MPa, E_m = 38MPa, ν_m = 0.3, R = 1cm, r_i = 0.01cm, r_o=0.02cm and L_m=$2L_f$.

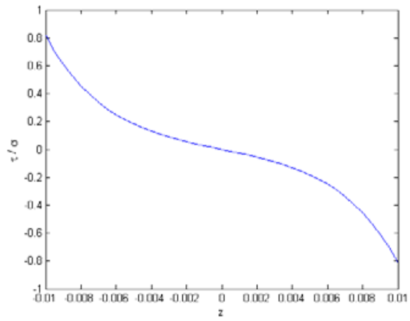

Fig. 4 Non-dimensional shear stress distribution vs. fiber length on the hollow carbon fiber (Lf = 0.01m, ro = 0.02cm)

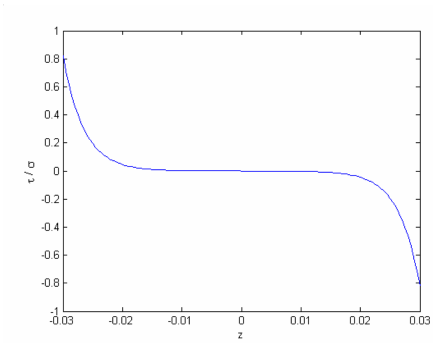

Fig. 5 Non-dimensional shear stress distribution vs. fiber length on the hollow carbon fiber (Lf = 0.03m, ro = 0.02cm)

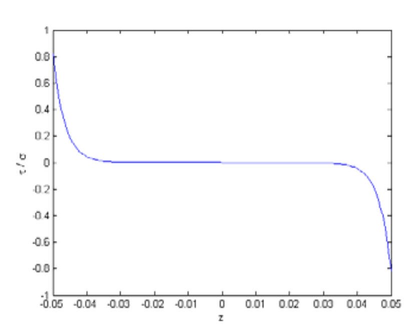

Fig. 6 Non-dimensional shear stress distribution vs. fiber length on the hollow carbon fiber (L_f = 0.05m, r_o = 0.02cm,)

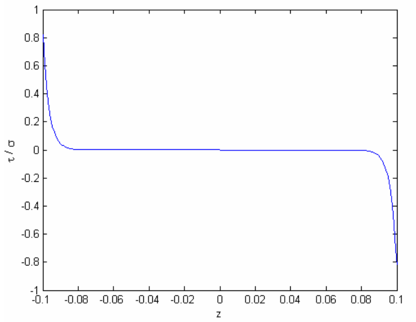

Fig. 7 Non-dimensional shear stress distribution vs. fiber length on the hollow carbon fiber (Lf = 0.1m, ro = 0.02cm)

In order to investigate the effect of fiber length in the shear stress distribution, four different fiber lengths i.e. L_f = 0.01, 0.03, 0.05, 0.1 meter are considered. The non-dimensionalized shear stress distributions for different above-mentioned fiber lengths are illustrated in Figs. (4-7). As it is seen in these figures, increasing the length of fiber will induce the shear stress distribution becomes a local phenomenon i.e. as the fiber length increases the shear jump will represent itself more locally towards the end of fiber length. Moreover, the non-dimensionalized shear stress ratio tends to value of 0.8.

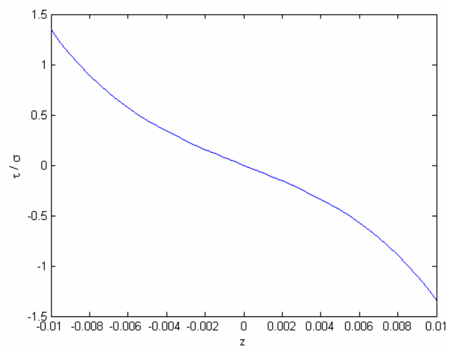

Fig. 8 Non-dimensional shear stress distribution vs. fiber length on the hollow carbon fiber (Lf = 0.01m, ro = 0.03cm)

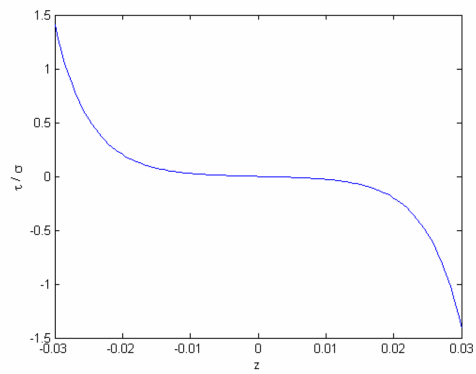

Fig. 9 Non-dimensional shear stress distribution vs. fiber length on the hollow carbon fiber (Lf = 0.03m, ro = 0.03cm)

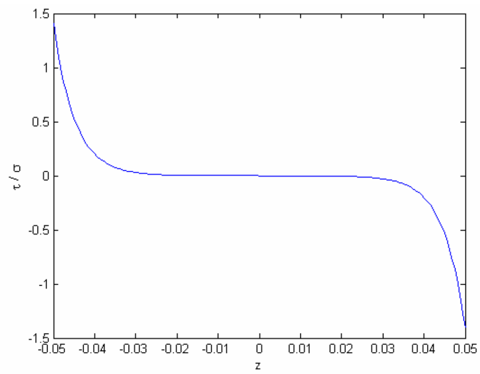

Fig. 10 Non-dimensional shear stress distribution vs. fiber length on the hollow carbon fiber (Lf = 0.05m, ro = 0.03cm)

A close inspection of Figures (6-9) reveals that, the trend of shear stress distributions is similar as those in Figs of (2-5), however we will see since the outer radius of the fiber in these cases is larger, and the shear stress ratio tends to a higher value i.e. 1.5.

V. Conclusion

As it can be seen in the presented results the ability of the fiber in stress transfer will be improved by increasing its aspect ratio. It is also possible to define a characteristic length for this type of fiber too, which its ability to stress transfer between the matrix and the fiber gets maximum value and remains constant for higher aspect ratios.

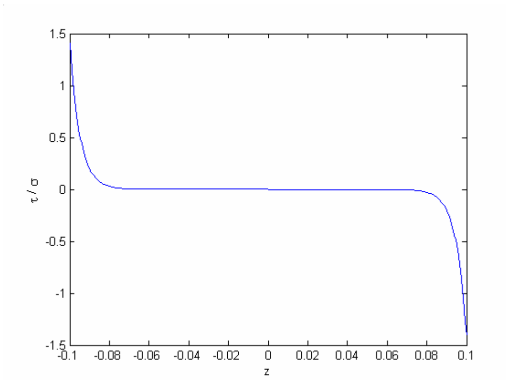

Fig. 11 Non-dimensional shear stress distribution vs. fiber length on the hollow carbon fiber (Lf = 0.1m, ro = 0.03cm)

The shear stress transfer will also increase by increasing the outer diameter of the fiber as it has been represented in the above. It is also expected that the stress transfer increase by reducing the inner diameter of the hollow fiber because the shear stress is proportional to the $\left(r_i^2 - r_o^2\right)/r_o$. It is also expected that the reduction in the inner radius of the fiber to have higher effect compared with increasing the outer radius due to the proportionality of the transferred stress to the r_i^2 compared with its proportionality to r_o.

Due to the above statements, it is expected that a rod-shaped fiber has higher stress transferability compared with a hollow cylindrical shaped fiber. Although it might be true for the case of which the matrix does not penetrate inside the hollow part but if it does it can cause increasing in the stress transferability of the fiber due to increasing in the area of the contact between the fiber and matrix which in this case can be nearly twice the case of no matrix penetration. Therefore to have a rational comparison between a rod-shaped fiber and a hollow cylindrical fiber which the matrix is penetrated inside its hollow part, further investigations needs to be done.

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