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Analytical solution for the Zakharov-Kuznetsov equations by differential transform method

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Abstract—This paper presents the approximate analytical solution of a Zakharov-Kuznetsov ZK(m, n, k) equation with the help of the differential transform method (DTM). The DTM method is a powerful and efficient technique for finding solutions of nonlinear equations without the need of a linearization process. In this approach the solution is found in the form of a rapidly convergent series with easily computed components. The two special cases, ZK(2,2,2) and ZK(3,3,3), are chosen to illustrate the concrete scheme of the DTM method in ZK(m, n, k) equations. The results demonstrate reliability and efficiency of the proposed method.

Keywords—Zakharov-Kuznetsov equation, differential transform method, closed form solution.

I. Introduction

In this paper the applied DTM is used to solve the Zakharov-Kuznetsov ZK(m, n, k) equations of the form

$$u_t + a(u^m)_x + b(u^n)_{xxx} + c(u^k)_{yyx} = 0, \qquad m, n, k \neq 0, \quad (1)$$

where a, b, c are arbitrary constants and m, n, k are integers. This equation governs the behavior of weakly nonlinear ion-acoustic waves in plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [1]-[2]. The ZK equation was first derived for describing weakly nonlinear ion-acoustic waves in strongly magnetized lossless plasma in two dimensions [3].

Wazwaz [4] used extended tanh method for analytic treatment of the ZK equation, the modified ZK equation, and the generalized forms of these equations. Huang [5] applied the polynomial expansion method to solve the coupled ZK equations. Zhao et al. [6] obtained numbers of solitary waves, periodic waves and kink waves using the theory of bifurcations of dynamical systems for the modified ZK equation. Inc [7] solved nonlinear dispersive ZK equations using the Adomian decomposition method, and Biazar et al. [8] applied the homotopy perturbation method to solve the Zakharov-Kuznetsov ZK(m, n, k) equations.

In the present work, we are concerned with the application of the DTM for the ZK equations. The DTM is a numerical method based on a Taylor expansion. This method constructs an analytical solution in the form of a polynomial. The concept of DTM was first proposed and applied to solve linear and nonlinear initial value problems in electric circuit analysis by [9]. Unlike the traditional high order Taylor series method which requires a lot of symbolic computations, the DTM is an iterative procedure for obtaining Taylor series solutions. This method will not consume too much computer time when applying to nonlinear or parameter varying systems. This method gives an analytical solution in the form of a polynomial. But, it is different from Taylor series method that requires computation of the high order derivatives. The DTM is an iterative procedure that is described by the transformed equations of original functions for solution of differential equations. Recently, the application of DTM is successfully extended to obtain analytical approximate solutions to various linear and nonlinear problems. For instance see [10]-[16].

The paper is organized as follows. In Section 2, theoretical aspects of the method are discussed. In Section 3, several examples with analytical solutions will be given to show the impressiveness of the suggested method. A proof of solution is exhibited in section 4. Finally, conclusions are given in Section 5.

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II. Differential transform method

2.1 Two-dimensional differential transform

The basic definition and the fundamental theorems of the DTM and its applicability for various kinds of differential equations are given in [17]-[20]. For convenience of the reader, we present a review of the DTM.

The differential transform function of the function $w(x, y)$ is the following form:

$$W(k, h) = \frac{1}{k!h!}\left[\frac{\partial^{(k+h)} w(x, y)}{\partial x^k \partial y^h}\right]_{(x=x_0,\, y=y_0)}, \quad (2)$$

where $w(x, y)$ is the original function and $W(k, h)$ is the transformed function.

The inverse differential transform of $W(k, h)$ is defined as

$$w(x, y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} W(k, h)(x - x_0)^k (y - y_0)^h. \quad (3)$$

Combining Eq. (2) and Eq. (3), it can be obtained that

$$W(k, h) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{1}{k!h!}\left[\frac{\partial^{(k+h)} w(x, y)}{\partial x^k \partial y^h}\right]_{(x=x_0,\, y=y_0)} (x - x_0)^k (y - y_0)^h. \quad (4)$$

When (x_0, y_0) are taken as $(0, 0)$, the function $w(x, y)$ in Eq. (4) is expressed as the following

$$W(k, h) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{1}{k!h!}\left[\frac{\partial^{(k+h)} w(x, y)}{\partial x^k \partial y^h}\right]_{(x=x_0,\, y=y_0)} x^k y^h, \quad (5)$$

TABLE I
THE OPERATIONS FOR THE TWO-DIMENSIONAL DIFFERENTIAL TRANSFORM METHOD.

Original function	Transformed function
$w(x,y)=u(x,y)\mp v(x,y),$	$W(k,h)=U(k,h)\mp V(k,h)$
$w(x,y)=\alpha u(x,y)$	$W(k,h)=\alpha U(k,h)$
$w(x,y)=\frac{\partial u(x,y)}{\partial x}$	$W(k,h)=(k+1)U(k+1,h)$
$w(x,y)=\frac{\partial u(x,y)}{\partial y}$	$W(k,h)=(h+1)U(k,h+1)$
$w(x,y)=\frac{\partial^{(r+s)}u(x,y)}{\partial x^r \partial y^s}$	$W(k,h)=(k+1)(k+2)...(k+r)(h+1)(h+2)...(h+s)U(k+r,h+s)$
$w(x,y)=u(x,y)v(x,y)$	$W(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)V(k-r,s)$
$w(x,y)=x^m y^n$	$W(k,h)=\delta(k-m,h-n)=\delta(k-m)\delta(h-n)$, where $\delta(k-m)=\{ \begin{array}{ll} 1, & k=m,h=n \\ 0, & \text{otherwise} \end{array}$
$w(x,y)=\frac{\partial u(x,y)}{\partial x}\frac{\partial v(x,y)}{\partial y}$	$W(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)(h-s+1)U(k-r+1,s)V(r,h-s+1)$
$w(x,y)=u(x,y)v(x,y)z(x,y)$	$W(k,h)=\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}U(r,h-s-p)V(t,s)Z(k-r-t,p)$
$w(x,y)=u(x,y)\frac{\partial v(x,y)}{\partial x}\frac{\partial z(x,y)}{\partial x}$	$W(k,h)=\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}(t+1)(k-r-t+1)U(r,h-s-p)V(t+1,s)Z(k-r-t+1,p)$
$w(x,y)=u(x,y)\frac{\partial v^2(x,y)}{\partial x^2}$	$W(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+2)(k-r+1)U(r,h-s)V(k-r+2,s)$
$w(t)=t$	$W(k)=\delta(k-1)$
$w(x,y)=x^m e^{at}$	$W(k,h)=\frac{a^h}{h!}\delta(k-m).$
$w(x,y)=e^{y-x}$	$W(k,h)=\frac{1^h}{h!}\frac{(-1)^k}{k!}$

and Eq. (3) is shown as

$$w(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}W(k,h)x^k y^h. \tag{6}$$

In real applications, the function $w(x,y)$ by a finite series of Eq. (6) can be written as

$$w(x,y)=\sum_{k=0}^{n}\sum_{h=0}^{m}W(x,y)x^k y^h. \tag{7}$$

The fundamental mathematical operations performed by two dimensional differential transform method can readily be obtained and are listed in Table 1.

2.2 Three-dimensional differential transform

By using the same theory as in two-dimensional differential transform, we can reach the three-dimensional case. The basic definitions of the three-dimensional differential transform are shown as below.

Given a w function which has three components such as x,y,t. Three-dimensional differential transform function of the function $w(x,y,t)$ is defined

$$W(k,h,m)=\frac{1}{k!h!m!}[\frac{\partial^{(k+h+m)}W(x,y,t)}{\partial x^k \partial y^h \partial t^m}]_{(0,0,0)}, \tag{8}$$

where $w(x,y,t)$ is the original function and $W(k,h,m)$ is the transformed function.

The inverse differential transform of $W(k,h,m)$ is defined as

$$w(x,y,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{m=0}^{\infty}W(k,h,m)x^k y^h t^m, \tag{9}$$

and from Eqs. (8) and (9) can be concluded

$$w(x,y,t)=$$

$$\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{k!h!m!}[\frac{\partial^{(k+h+m)}W(x,y,t)}{\partial x^k \partial y^h \partial t^m}]_{(0,0,0)}x^k y^h t^m. \tag{10}$$

The fundamental mathematical operations performed by three dimensional differential transform method are listed in Table 2.

III. NUMERICAL RESULTS

In this part, DTM will be applied for solving two special equations, namely ZK(2,2,2) and ZK(3,3,3) with specific initial conditions. The results reveal that the method is very effective and simple.

Example 3.1: We consider the following ZK(2,2,2) equation:

$$u_t-(u^2)_x+\frac{1}{8}(u^2)_{xxx}+\frac{1}{8}(u^2)_{yyx}=0, \tag{11}$$

the exact solution to Eq. (11) subject to the initial condition

$$u(x,y,0)=\frac{4}{3}\lambda\sinh^2(\frac{1}{2}(x+y)), \tag{12}$$

where λ is an arbitrary constant.

Using the DTM, we obtain the following relations:

$$\begin{aligned}&(m+1)U(k,h,m+1)+\\&2\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}(k-r+1)U(r,h-s,m-p)U(k-r+1,s,p)+\\&\frac{3}{4}\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}(r+1)U(r+1,h-s,m-p)(k-r+1)\\&(k-r+2)U(k-r+2,s,p)+\\&\frac{1}{4}\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}U(r,h-s,m-p)\end{aligned}$$

TABLE II
THE OPERATIONS FOR THE THREE-DIMENSIONAL DIFFERENTIAL TRANSFORM METHOD.

Original function	Transformed function
$w(x,y,t)=u(x,y,t)\mp v(x,y,t)$	$W(k,h,m)=U(k,h,m)\mp V(k,h,m)$
$w(x,y,t)=\alpha u(x,y,t)$	$W(k,h,m)=\alpha U(k,h,m)$
$w(x,y,t)=\frac{\partial u(x,y,t)}{\partial x}$	$W(k,h,m)=(k+1)U(k+1,h,m)$
$w(x,y,t)=\frac{\partial u(x,y,t)}{\partial y}$	$W(k,h,m)=(h+1)U(k,h+1,m)$
$w(x,y,t)=\frac{\partial u(x,y,t)}{\partial t}$	$W(k,h,m)=(m+1)U(k,h,m+1)$
$w(x,y,t)=\frac{\partial^{(r+s+p)}u(x,y,t)}{\partial x^r\partial y^s\partial t^p}$	$W(k,h,m)=(k+1)(k+2)...(k+r)(h+1)(h+2)...(h+s)(m+1)(m+2)...(m+p)U(k+r,h+s,m+p)$
$w(x,y,t)=u(x,y,t)v(x,y,t)$	$W(k,h,m)=\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}U(r,h-s,m-p)V(k-r,s,p)$
$w(x,y,t)=u(x,y,t)v(x,y,t)q(x,y,t)$	$W(k,h,m)=\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}\sum_{q=0}^{m}\sum_{n=0}^{m-q}U(r,h-s-p,m-q-n)V(t,s,q)Q(k-r-t,p,n)$
$w(x,y,t)=\frac{\partial u(x,y,t)}{\partial x}\frac{\partial v(x,y,t)}{\partial y}$	$W(k,h,m)=\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}(k-r+1)(h-s+1)U(k-r+1,s,p)V(r,h-s+1,m-p)$

$$
\begin{aligned}
&(k-r+1)(k-r+2)(k-r+3)U(k-r+3,s,p)+\\
&\frac{1}{2}\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}(h-s+1)\\
&U(r,h-s+1,m-p)(s+1)(k-r+1)\\
&U(k-r+1,s+1,p)+\frac{1}{4}\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}(r+1)\\
&U(r+1,h-s,m-p)(s+1)(s+2)\\
&U(k-r,s+2,p)+\frac{1}{4}\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}U(r,h-s,m-p)
\end{aligned}
$$

$$
(k-r+1)(s+1)(s+2)U(k-r+1,s+2,p)=0, \quad (13)
$$

and

$$
U(k,h,0)=
$$

$$
-\frac{2}{3}\ \lambda\ \delta(k)\delta(h)+\frac{1}{3}\ \frac{(-1)^k(-1)^h}{k!h!}\ \lambda+\frac{1}{3}\ \frac{1}{k!h!}\ \lambda. \quad (14)
$$

Substituting Eq. (14) into Eq. (13) and by a recursive method, the results are listed as follows:

If $k+h+m$=odd, $U(k,h,m)=0$, except $U(0,0,0)=0$,
Otherwise

$$
\begin{aligned}
&U(2,0,0)=\frac{\lambda}{3},\ U(1,1,0)=\frac{2\lambda}{3},\ U(0,2,0)=\frac{\lambda}{3},\\
&U(2,2,0)=\frac{\lambda}{6},...\\
&U(1,0,1)=-\frac{2\lambda^2}{3},\ U(0,1,1)=-\frac{2\lambda^2}{3},\\
&U(2,1,1)=-\frac{\lambda^2}{3},\ U(1,2,1)=-\frac{\lambda^2}{3},...\\
&U(0,0,2)=\frac{\lambda^3}{3},\ U(2,0,2)=\frac{\lambda^3}{6},\ U(1,1,2)=\frac{\lambda^3}{3},\\
&U(0,2,2)=\frac{\lambda^3}{6},\ U(2,2,2)=\frac{\lambda^3}{12},...
\end{aligned}
$$

Consequently substituting all $U(k,h,m)$ into Eq. (9) and after some manipulations, we obtain the closed form series solution as

$$
u(x,y,t)=
$$

$$
\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{m=0}^{\infty}U(k,h,m)x^ky^ht^m=\frac{4}{3}\lambda\sinh^2(\frac{1}{2}(x+y-\lambda t)),
$$

which is the exact solution of this problem.

Example 3.2: Now we consider the ZK(2,2,2) equation:

$$
u_t-(u^2)_x+\frac{1}{8}(u^2)_{xxx}+\frac{1}{8}(u^2)_{yyx}=0, \quad (15)
$$

the exact solution to Eq. (15) subject to the initial condition

$$
u(x,y,0)=-\frac{4}{3}\ \lambda\ \cosh^2(\frac{1}{2}(x+y)), \quad (16)
$$

where λ is an arbitrary constant.

Employing the DTM, we obtain the following relations:

$$
\begin{aligned}
&(m+1)U(k,h,m+1)+\\
&2\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}(k-r+1)U(r,h-s,m-p)\\
&U(k-r+1,s,p)+\frac{3}{4}\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}(r+1)\\
&U(r+1,h-s,m-p)(k-r+1)(k-r+2)\\
&U(k-r+2,s,p)+\frac{1}{4}\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}U(r,h-s,m-p)\\
&(k-r+1)(k-r+2)(k-r+3)U(k-r+3,s,p)+\\
&\frac{1}{2}\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}(h-s+1)U(r,h-s+1,m-p)(s+1)\\
&(k-r+1)U(k-r+1,s+1,p)+\\
&\frac{1}{4}\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}(r+1)U(r+1,h-s,m-p)\\
&(s+1)(s+2)U(k-r,s+2,p)+\\
&\frac{1}{4}\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}U(r,h-s,m-p)(k-r+1)(s+1)
\end{aligned}
$$

$$(s+2)U(k-r+1,s+2,p)=0, \tag{17}$$

and

$$U(k,h,0)=$$

$$-\frac{2}{3}\ \lambda\ \delta(k)\delta(h)-\frac{1}{3}\ \frac{(-1)^k(-1)^h}{k!h!}\ \lambda-\frac{1}{3}\ \frac{1}{k!h!}\ \lambda. \tag{18}$$

Substituting Eq. (18) into Eq. (17) and by a recursive method, the results are listed as follows:

If $k+h+m$=odd, $\quad U(k,h,m)=0$,

Otherwise

$$\begin{aligned}
&U(0,0,0)=-\frac{4\lambda}{3},\ U(2,0,0)=-\frac{\lambda}{3},\ U(1,1,0)=-\frac{2\lambda}{3},\\
&U(0,2,0)=-\frac{\lambda}{3},\ U(2,2,0)=-\frac{\lambda}{6},...\\
&U(1,0,1)=\frac{2\lambda^2}{3},\ U(0,1,1)=\frac{2\lambda^2}{3},\ U(2,1,1)=\frac{\lambda^2}{3},\\
&U(1,2,1)=\frac{\lambda^2}{3},...\\
&U(0,0,2)=-\frac{\lambda^3}{3},\ U(2,0,2)=-\frac{\lambda^3}{6},\ U(1,1,2)=-\frac{\lambda^3}{3},\\
&U(0,2,2)=-\frac{\lambda^3}{6},\ U(2,2,2)=-\frac{\lambda^3}{12},...
\end{aligned}$$

Consequently substituting all $U(k,h,m)$ into Eq. (9) we achieve the closed form series solution as

$$u(x,y,t)=$$

$$\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{m=0}^{\infty}U(k,h,m)x^ky^ht^m=-\frac{4}{3}\lambda\cosh^2(\frac{1}{2}(x+y-\lambda t)),$$

which is the exact solution of the problem.

Example 3.3: Consider the ZK(3, 3, 3) equation in the following form:

$$u_t-(u^3)_x+2(u^3)_{xxx}+2(u^3)_{yyx}=0, \tag{19}$$

subject to the initial condition:

$$u(x,y,0)=\sqrt{\frac{3\lambda}{2}}\ \text{Sinh}\left[\frac{1}{6}(x+y)\right], \tag{20}$$

where λ is an arbitrary constant.

Utilizing the DTM, we attain

$$\begin{aligned}
&(m+1)U(k,h,m+1)+\\
&3\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}\sum_{q=0}^{m}\sum_{n=0}^{m-q}(k-r-t+1)\\
&U(r,h-s-p,m-q-n)U(t,s,q)\\
&U(k-r-t+1,p,n)-\\
&12\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}\sum_{q=0}^{m}\sum_{n=0}^{m-q}(r+1)\\
&U(r+1,h-s-p,m-q-n)(t+1)U(t+1,s,q)\\
&(k-r-t+1)U(k-r-t+1,p,n)-\\
&36\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}\sum_{q=0}^{m}\sum_{n=0}^{m-q}U(r,h-s-p,m-q-n)\\
&(t+1)U(t+1,s,q)(k-r-t+2)(k-r-t+1)\\
&U(k-r-t+2,p,n)-\\
&6\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}\sum_{q=0}^{m}\sum_{n=0}^{m-q}U(r,h-s-p,m-q-n)U(t,s,q)\\
&(k-r-t+3)(k-r-t+2)(k-r-t+1)\\
&U(k-r-t+3,p,n)-12\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}\sum_{q=0}^{m}\sum_{n=0}^{m-q}(r+1)\\
&U(r+1,h-s-p,m-q-n)(s+1)U(t,s+1,q)(p+1)\\
&U(k-r-t,p+1,n)-\\
&24\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}\sum_{q=0}^{m}\sum_{n=0}^{m-q}U(r,h-s-p,m-q-n)(s+1)\\
&U(t,s+1,q)(k-r-t+1)(p+1)\\
&U(k-r-t+1,p+1,n)-\\
&12\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}\sum_{q=0}^{m}\sum_{n=0}^{m-q}U(r,h-s-p,m-q-n)\\
&(t+1)U(t+1,s,q)(p+2)(p+1)U(k-r-t,p+2,n)-\\
&6\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}\sum_{q=0}^{m}\sum_{n=0}^{m-q}U(r,h-s-p,m-q-n)U(t,s,q)
\end{aligned}$$

$$(p+3)(p+2)(p+1)U(k-r-t,p+3,n)=0, \tag{21}$$

and

$$U(k,h,0)=$$

$$-\frac{1}{2}\sqrt{\frac{3}{2}}\sqrt{\lambda}\left(\frac{\left(\frac{-1}{6}\right)^k\left(\frac{-1}{6}\right)^h}{k!h!}-\frac{\left(\frac{1}{6}\right)^k\left(\frac{1}{6}\right)^h}{k!h!}\right). \tag{22}$$

Substituting Eq. (22) into Eq. (21), the results are summarized as follows:

If $k+h+m$=even, $\quad U(k,h,m)=0$,

Otherwise

$$U(1,0,0)=\frac{\sqrt{\lambda}}{2\sqrt{6}},\ U(0,1,0)=\frac{\sqrt{\lambda}}{2\sqrt{6}},$$
$$U(2,1,0)=\frac{\sqrt{\lambda}}{144\sqrt{6}},\ U(1,2,0)=\frac{\sqrt{\lambda}}{144\sqrt{6}},...$$
$$U(0,0,1)=-\frac{\lambda^{\frac{3}{2}}}{2\sqrt{6}},\ U(2,0,1)=-\frac{\lambda^{\frac{3}{2}}}{144\sqrt{6}},$$
$$U(1,1,1)=-\frac{\lambda^{\frac{3}{2}}}{72\sqrt{6}},\ U(0,2,1)=-\frac{\lambda^{\frac{3}{2}}}{144\sqrt{6}},$$
$$U(2,2,1)=-\frac{\lambda^{\frac{5}{2}}}{144\sqrt{6}},...$$
$$U(1,0,2)=\frac{\lambda^{\frac{5}{2}}}{144\sqrt{6}},\ U(0,1,2)=\frac{\lambda^{\frac{5}{2}}}{144\sqrt{6}},$$
$$U(2,1,2)=\frac{\lambda^{\frac{5}{2}}}{10368\sqrt{6}},\ U(1,2,2)=\frac{\lambda^{\frac{5}{2}}}{10368\sqrt{6}},...$$

Consequently substituting all $U(k,h,m)$ into Eq. (9) and after some manipulations, we obtain the closed form series solution as

$$u(x,y,t)=\sqrt{\frac{3\lambda}{2}}\text{Sinh}\left[\frac{1}{6}(x+y-\lambda t)\right],$$

which is the exact solution of the problem.

Example 3.4:

Finally, we exam the following $ZK(3,3,3)$ equation:

$$u_t-(u^3)_x+2(u^3)_{xxx}+2(u^3)_{yyx}=0, \tag{23}$$

subject to the initial condition:

$$u(x,y,0)=\sqrt{-\frac{3\lambda}{2}}\ \text{Cosh}\left[\frac{1}{6}(x+y)\right], \tag{24}$$

where λ is an arbitrary constant.

Using the DTM, we have

$$\begin{aligned}
&(m+1)U(k,h,m+1)+\\
&3\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}\sum_{q=0}^{m}\sum_{n=0}^{m-q}(k-r-t+1)\\
&U(r,h-s-p,m-q-n)U(t,s,q)\\
&U(k-r-t+1,p,n)-\\
&12\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}\sum_{q=0}^{m}\sum_{n=0}^{m-q}(r+1)\\
&U(r+1,h-s-p,m-q-n)(t+1)U(t+1,s,q)\\
&(k-r-t+1)U(k-r-t+1,p,n)-\\
&36\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}\sum_{q=0}^{m}\sum_{n=0}^{m-q}U(r,h-s-p,m-q-n)\\
&(t+1)U(t+1,s,q)(k-r-t+2)(k-r-t+1)\\
&U(k-r-t+2,p,n)-\\
&6\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}\sum_{q=0}^{m}\sum_{n=0}^{m-q}U(r,h-s-p,m-q-n)\\
&U(t,s,q)(k-r-t+3)(k-r-t+2)(k-r-t+1)\\
&U(k-r-t+3,p,n)-\\
&12\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}\sum_{q=0}^{m}\sum_{n=0}^{m-q}(r+1)\\
&U(r+1,h-s-p,m-q-n)(s+1)U(t,s+1,q)\\
&(p+1)U(k-r-t,p+1,n)-\\
&24\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}\sum_{q=0}^{m}\sum_{n=0}^{m-q}U(r,h-s-p,m-q-n)\\
&(s+1)U(t,s+1,q)(k-r-t+1)\\
&(p+1)U(k-r-t+1,p+1,n)-\\
&12\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}\sum_{q=0}^{m}\sum_{n=0}^{m-q}U(r,h-s-p,m-q-n)\\
&(t+1)U(t+1,s,q)(p+2)(p+1)\\
&U(k-r-t,p+2,n)-\\
&6\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}\sum_{q=0}^{m}\sum_{n=0}^{m-q}U(r,h-s-p,m-q-n)
\end{aligned}$$

$$U(t,s,q)(p+3)(p+2)(p+1)U(k-r-t,p+3,n)=0, \tag{25}$$

and

$$U(k,h,0)=\frac{1}{2}\sqrt{-\frac{3}{2}\ \lambda}\left(\frac{\left(\frac{-1}{6}\right)^k\left(\frac{-1}{6}\right)^h}{k!h!}+\frac{\left(\frac{1}{6}\right)^k\left(\frac{1}{6}\right)^h}{k!h!}\right). \tag{26}$$

Substituting Eq. (26) into Eq. (25) and by recursive method, the result is listed as follows: If k+h+m=odd, $U(k,h,m)=0$,

Otherwise

$$U(0,0,0)=\sqrt{\frac{-3\lambda}{2}},\ U(2,0,0)=\frac{\sqrt{-\lambda}}{24\sqrt{6}},$$
$$U(1,1,0)=-\frac{\sqrt{-\lambda}}{12\sqrt{6}},\ U(0,2,0)=\frac{\sqrt{-\lambda}}{24\sqrt{6}},$$
$$U(2,2,0)=\frac{\sqrt{-\lambda}}{1728\sqrt{6}},...$$
$$U(1,0,1)=\frac{(-\lambda)^{3/2}}{12\sqrt{6}},\ U(0,1,1)=\frac{(-\lambda)^{3/2}}{12\sqrt{6}},$$
$$U(2,1,1)=\frac{(-\lambda)^{3/2}}{864\sqrt{6}},U(1,2,1)=\frac{(-\lambda)^{3/2}}{864\sqrt{6}},...$$
$$U(0,0,2)=\frac{(-\lambda)^{5/2}}{24\sqrt{6}},\ U(2,0,2)=\frac{(-\lambda)^{5/2}}{1728\sqrt{6}},$$
$$U(1,1,2)=\frac{(-\lambda)^{5/2}}{864\sqrt{6}},\ U(0,2,2)=\frac{(-\lambda)^{5/2}}{1728\sqrt{6}},$$
$$U(2,2,2)=\frac{(-\lambda)^{5/2}}{124416\sqrt{6}},...$$

Consequently substituting all $U(k,h,m)$ into Eq. (9) and after some manipulations, we obtain the closed form series solution as

$$u(x,y,t)=\sqrt{-\frac{3\lambda}{2}}\text{Cosh}\left[\frac{1}{6}(x+y-\lambda t)\right],$$

which is the exact solution of the problem.

IV. Proof of Solution

A Mathematica program is given as an example to verify that $u(x, y, t)$ solutions of the Eq. (1), is as follows:

If $a = 1$ then

$$u = \left(\frac{2\lambda n}{a(n+1)} \text{Sin} \left[\frac{1}{2} \sqrt{\frac{a}{b+c}} \left(\frac{n-1}{n} \right) (x + y - \lambda t) \right]^2 \right)^{\frac{1}{n-1}},$$

Simplify$[D[u,t] - D\left[u^3, x\right] + 2\ D\left[u^3, \{y,2\}, \{x,1\}\right] + 2\ D\left[u^3, \{x,3\}\right]]$.

If $a = -1$ then

$$u = \left(\frac{2\lambda n}{a(n+1)} \text{Sinh} \left[\frac{1}{2} \sqrt{\frac{a}{b+c}} \left(\frac{n-1}{n} \right) (x + y - \lambda t) \right]^2 \right)^{\frac{1}{n-1}},$$

Simplify$[D[u,t] - D\left[u^3, x\right] + 2\ D\left[u^3, \{y,2\}, \{x,1\}\right] + 2\ D\left[u^3, \{x,3\}\right]]$.

V. Conclusion

In this work, we have successfully developed DTM to obtain an approximation to the solution of the Zakharov equation. It is apparent that this method is a very influential and efficient technique. There is no need for linearization or perturbations; large computational work and round-off errors are avoided. The results obtained demonstrate the reliability of the algorithm and its applicability to some partial differential equations. It provides more realistic series solutions that converge very rapidly in real physical problems. It may be also concluded that DTM is very powerful and reliable in finding analytical as well as numerical solutions for wide classes of nonlinear differential equations.

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