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Analytical Solution for Free Vibration of Rectangular Kirchhoff Plate from Wave Approach

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***Abstract*—**In this paper, an analytical approach for free vibration analysis of four edges simply supported rectangular Kirchhoff plates is presented. The method is based on wave approach. From wave standpoint vibration propagate, reflect and transmit in a structure. Firstly, the propagation and reflection matrices for plate with simply supported boundary condition are derived. Then, these matrices are combined to provide a concise and systematic approach to free vibration analysis of a simply supported rectangular Kirchhoff plate. Subsequently, the eigenvalue problem for free vibration of plates is formulated and the equation of plate natural frequencies is constructed. Finally, the effectiveness of the approach is shown by comparison of the results with existing classical solution.

***Keywords*—**Kirchhoff plate, propagation matrix, reflection matrix, vibration analysis.

I. INTRODUCTION

IN classical method for solving the vibration of plate we apply the boundary conditions to the general solution of the differential equation of motion of plate. It yields to find the natural frequencies [1].

Various approaches have been applied for vibration analysis of plates. In free vibration analysis of plate except the classic method [1], Nomura and Wang [2] addressed a computer-aided implementation of classical analytical procedure to treat free vibration of rectangular plates supported at arbitrary interior points. Liew, Lam, and chaw [3] used orthogonal plate function to solve the free vibration analysis of rectangular plates. Yongqiang [4] presented a finite strip Fourier p-element method for the vibration analysis of plate.

Vibrations can be described as a linear combination of the modes of a structure. An alternative is to describe vibrations as propagating waves travelling in the structure. It was found that the two descriptions often give enlightening complementary perspectives. Wave propagation, transmission and reflection in solids have been studied by a number of researchers [5-9]. By these characteristics Mei solved Longitudinal Vibrations of bars [10]. Mei, Karpenko, Moody, and Allen found an analytical approach to free and forced vibrations of axially loaded cracked Timoshenko beams [11].

In this paper we present an analytical solution for free vibration of four edges simply supported rectangular Kirchhoff plate by wave propagation method which is organized as follows: In the next section, the equation of motion for Kirchhoff plate is presented and expressions for the propagation of waves derived. In Section III, propagation matrix and reflection matrix are derived. In Section IV, we consider the Vibration analysis using wave approach. In section V we compare this method by classical method and describe other applications of this approach.

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II. EQUATION OF MOTION AND WAVE PROPAGATION

The equation of motion for Kirchhoff plate is defined as [1]:

$$-D\nabla^4 w = M\ddot{w} \tag{1}$$

Where $\nabla^4 = \nabla^2\nabla^2$ is known as the biharmonic operator, M is the mass per unit area of plate, w is the transverse deflection of the plate in point (x,y), in which (x,y) is coordinate of an arbitrary point on plate and D is the plate flexural rigidity:

$$D = \frac{Eh^3}{12(1-\nu^2)} \tag{2}$$

In which E is Young's modulus, h is the plate thickness and ν is Poisson's ratio.

Assuming time harmonic motion and using separation of variables, the solution to (1) may be written as:

$$w(x,y,t) = w(x)w(y)\exp(i\omega t) \tag{3}$$

We can rewrite (3) as:

$$w(x,y,t) = x_0 y_0 \exp(-ik_1 x)\exp(-ik_2 y)\exp(i\omega t) \tag{4}$$

where x_0 and y_0 are arbitrary constant, ω is the frequency, k_1 and k_2 are the wave numbers. Substituting this expression into (1), we have:

$$\left[D(k_1^4 + k_2^4 + 2k_1^2 k_2^2) - M\omega^2\right]x_0 y_0 = 0 \tag{5}$$

We can rewrite it as:

$$(k_1^2 + k_2^2)^2 = \frac{M}{D}\omega^2 \tag{6}$$

or

$$\omega = (k_1^2 + k_2^2)\sqrt{\frac{D}{M}} \tag{7}$$

With the time dependence $\exp(i\omega t)$ suppressed, by considering (4) $w(x)$ becomes:

$$w(x) = a_1^+ e^{-ik_1x} + a_1^- e^{ik_1x} + a_2^+ e^{-k_1x} + a_2^- e^{k_1x} \quad (8)$$

In which a's are the amplitudes of $w(x)$ and we can write a similar expression for $w(y)$.

III. Propagation and Reflection Wave Matrices

A. Propagation Matrix

From wave standpoint, vibrations propagating in an object and reflecting at boundaries are governed by the so-called propagation and reflection matrices. Consider two points A and B on a flexurally vibrating plate along the X-direction at a distance x apart; denoting the positive- and negative-going wave vectors at points A and B as $\mathbf{a}^+$ and $\mathbf{a}^-$, and $\mathbf{b}^+$ and $\mathbf{b}^-$, respectively. $\mathbf{a}$ and $\mathbf{b}$ wave vectors are related by [6]:

$$\mathbf{b}^+ = \mathbf{f}(x)\mathbf{a}^+, \qquad \mathbf{a}^- = \mathbf{f}(x)\mathbf{b}^- \quad (9)$$

where from (8),

$$\mathbf{a}^+ = \begin{Bmatrix} a_1^+ \\ a_2^+ \end{Bmatrix}, \quad \mathbf{a}^- = \begin{Bmatrix} a_1^- \\ a_2^- \end{Bmatrix}, \quad \mathbf{b}^+ = \begin{Bmatrix} b_1^+ \\ b_2^+ \end{Bmatrix}, \quad \mathbf{b}^+ = \begin{Bmatrix} b_1^- \\ b_2^- \end{Bmatrix} \quad (10)$$

and,

$$\mathbf{f}(x) = \begin{bmatrix} e^{-ik_1x} & 0 \\ 0 & e^{-k_1x} \end{bmatrix} \quad (11)$$

$\mathbf{f}(x)$ is known as the propagation matrix for the distance x.

B. Reflection at Boundaries

The boundary conditions of a four edges simply supported plate are:

$$\begin{aligned} w(x,y) &= 0, \quad \frac{\partial^2 w}{\partial x^2} = 0, \quad x = 0, l \\ w(x,y) &= 0, \quad \frac{\partial^2 w}{\partial y^2} = 0, \quad y = 0, d \end{aligned} \quad (12a,b)$$

A simply supported boundary is shown in Fig. 1. The incident waves $\mathbf{a}^+$ give rise to reflected waves $\mathbf{a}^-$, which are related by:

$$\mathbf{a}^- = \mathbf{r}\mathbf{a}^+ \quad (13)$$

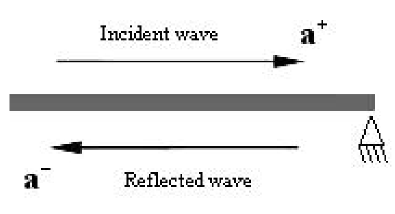

Fig. 1 Side view of a boundary

The reflection matrix $\mathbf{r}$ can be determined by considering boundary conditions, that is:

$$w(x) = 0 \quad (14)$$

$$\frac{\partial^2 w(x)}{\partial x^2} = 0 \quad (15)$$

For the boundary at $x = 0$, from (8) we have:

$$a_1^+ + a_1^- + a_2^+ + a_2^- = 0 \quad (16)$$

$$-a_1^+ - a_1^- + a_2^+ + a_2^- = 0 \quad (17)$$

We can write (16) and (17) as:

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} a_1^+ \\ a_2^+ \end{Bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} a_1^- \\ a_2^- \end{Bmatrix} = 0 \quad (18)$$

Or

$$\boldsymbol{\gamma}\mathbf{a}^+ + \boldsymbol{\eta}\mathbf{a}^- = 0$$

Where

$$\boldsymbol{\gamma} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \qquad \boldsymbol{\eta} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad (19a,b)$$

So

$$\mathbf{a}^- = -\boldsymbol{\eta}^{-1}\boldsymbol{\gamma}\mathbf{a}^+ \quad (20)$$

By comparing this equation with (12)

$$\mathbf{r} = -\boldsymbol{\eta}^{-1}\boldsymbol{\gamma} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad (21)$$

IV. Vibration Analysis using Wave Approach

In this section, the derived propagation and reflection matrices are combined to provide a concise and systematic approach for vibration analysis of rectangular Kirchhoff plate. The presented wave approach is illustrated through free vibration analysis of four edge simply support plate (SSSS).

In Fig. 2 a four edges simply supported plate shown along x axis is depicted with its two simply supported boundary conditions at A and B. Incident and reflected waves at the simply supported boundaries, A and B, are denoted by $\mathbf{a}^{\mp}, \mathbf{b}^{\pm}$ respectively. The relationship between the incident and the reflected waves at the boundaries are described as:

$$\mathbf{a}^+ = \mathbf{r}\mathbf{a}^-, \quad \mathbf{b}^- = \mathbf{r}\mathbf{b}^+ \quad (22a,b)$$

In which $\mathbf{r}$ is the reflection matrix for simply supported boundary condition.

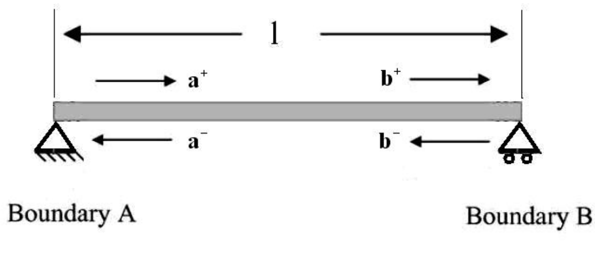

Fig. 2 Side view of plate, on boundaries A and B

The propagation relations are**:**

$$\mathbf{b}^+ = \mathbf{f}(l)\mathbf{a}^+, \quad \mathbf{a}^- = \mathbf{f}(l)\mathbf{b}^- \quad (23a,b)$$

Where $\mathbf{f}(l)$ is the propagation matrix between A and B. By rewriting (22), (23) in a matrix form equation**:**

$$\begin{bmatrix} -\mathbf{I} & \mathbf{r} & 0 & 0 \\ \mathbf{f}(l) & 0 & -\mathbf{I} & 0 \\ 0 & -\mathbf{I} & 0 & \mathbf{f}(l) \\ 0 & 0 & \mathbf{r} & -\mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{a}^+ \\ \mathbf{a}^- \\ \mathbf{b}^+ \\ \mathbf{b}^- \end{Bmatrix} = 0 \tag{24}$$

For a non-trivial solution, it follows that:

$$\begin{vmatrix} -\mathbf{I} & \mathbf{r} & 0 & 0 \\ \mathbf{f}(l) & 0 & -\mathbf{I} & 0 \\ 0 & -\mathbf{I} & 0 & \mathbf{f}(l) \\ 0 & 0 & \mathbf{r} & -\mathbf{I} \end{vmatrix} = 0 \tag{25}$$

This leads us to:

$$\begin{aligned} &(1-e^{-2ik_1 l})(1-e^{-2k_1 l}) = 0 \\ &\Rightarrow k_1 = \frac{n\pi}{l}, \quad n = 1,2,... \end{aligned} \tag{26}$$

By writing similar expression for the other side of the plate and assuming its length is d we have:

$$k_2 = \frac{m\pi}{d}, \quad m = 1,2,... \tag{27}$$

Substituting (22) and (23) in (5) yields

$$\omega = \omega_{nm} = \pi^2 \left(\frac{n^2}{l^2} + \frac{m^2}{d^2}\right)\sqrt{\frac{D}{M}}, \qquad m,n = 1,2,... \tag{28}$$

Equation (28) is in full agreement with the classical solution [1].

V. Comparison and Discussion

In this paper, we presented an analytical solution for find the natural frequencies of free vibration of four edges simply supported rectangular Kirchhoff plate. Classical method [1] lead us to apply eight boundary conditions to an equation with eight terms in which each term made by multiply two sinusoidal and hyperbolic functions; solving these eight equations is boring and complicated. Moreover, for other boundary conditions the classical method gets so complicated, but in the presented approach not only reflection matrices for different boundary conditions can be derived simply, but also the natural frequencies can be obtained easily. Furthermore this method can be applied to plates which have discontinuities such as cracks parallel to their edges or changes in cross-section.

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